10.1 Power Series: $\sum_{k=0}^{\infty} a_k x^k$

The following functions are examples of **power series**:

$$G(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k$$
$$E(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$
$$S(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

We can also approximate these functions using partial sums of the power series, which turn out to be the MacLaurin polynomials discussed in Section 8.7.

A power series looks like an "infinite polynomial." Power series play particularly important roles in mathematics and applications because we can represent many important functions, such as $\sin(x)$, $\cos(x)$, e^x and $\ln(1+x)$, using power series.

For $k = 0$ we use the convention for power series that $x^0 = 1$ even when $x = 0$. This convention—which ignores the fact that 0^0 is an indeterminate form (as discussed in Section 3.7)—simply makes it easier for us to represent the series using summation notation.

Definition:

A **power series** is an expression of the form:

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where a_0, a_1, a_2, a_3, ... are constants, called the **coefficients** of the series, and x is a variable.

Replacing x with any number, the power series simply becomes a numerical series that may converge or diverge. If the power series does converge, the value of the function is the sum of the series. The domain of the function is the set of x-values for which the series converges.

Finding Where a Power Series Converges

Any power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ must converge at $x = 0$:

$$f(0) = \sum_{k=0}^{\infty} a_k \cdot 0^k = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + \cdots = a_0$$

To find which other values of x allow a power series to converge, you could plug in x-values one at a time, but that would be very inefficient. Instead, the Ratio Test allows you to determine whether the series converges or diverges for many values of x all at once.

Example 1. Find all values of x for which $\sum_{k=0}^{\infty} (2k+1)x^k$ converges.

Solution. Applying the Ratio Test,

$$\left|\frac{(2(k+1)+1)x^{k+1}}{(2k+1)x^k}\right| = \left|\frac{(2k+3)x}{(2k+1)}\right| = \frac{(2k+3)}{(2k+1)} \cdot |x| \longrightarrow |x|$$

The Ratio Test tells us that the series converges if this limit, $|x|$, is less than 1: $|x| < 1 \Rightarrow -1 < x < 1$. The Ratio Test also says that the series diverges if the limit is great than 1: $|x| > 1 \Rightarrow x < -1$ or $x > 1$. Finally, the Ratio Test provides no information if $|x| = 1$, so we need to check the two remaining values of x: the endpoints $x = -1$ and $x = 1$.

When $x = -1$, $\sum_{k=0}^{\infty} (2k+1)x^k = \sum_{k=0}^{\infty} (2k+1) \cdot (-1)^k$, and when $x = 1$, $\sum_{k=0}^{\infty} (2k+1)x^k = \sum_{k=0}^{\infty} (2k+1)$. Both of these series diverge (by the Test for Divergence), because in both series the terms do not approach 0.

The power series $\sum_{k=0}^{\infty} (2k+1)x^k$ converges if and only if $-1 < x < 1$. In other words, the series converges when x is in the interval $(-1, 1)$ and it diverges on the intervals $(-\infty, -1]$ and $[1, \infty)$. ◀

Example 2. Find all values of x for which $\sum_{k=0}^{\infty} \frac{x^k}{k \cdot 3^k}$ converges.

Solution. Applying the Ratio Test:

$$\left|\frac{\frac{x^{k+1}}{(k+1)\cdot 3^{k+1}}}{\frac{x^k}{k\cdot 3^k}}\right| = \left|\frac{x^{k+1}}{(k+1)\cdot 3^{k+1}} \cdot \frac{k\cdot 3^k}{x^k}\right| = \frac{k}{k+1} \cdot \frac{|x|}{3} \longrightarrow \frac{|x|}{3}$$

The Ratio Test tells us that the series converges absolutely when:

$$\frac{|x|}{3} < 1 \quad \Rightarrow \quad |x| < 3 \quad \Rightarrow \quad -3 < x < 3$$

The Ratio Test also tells us that the series diverges when:

$$\frac{|x|}{3} > 1 \quad \Rightarrow \quad |x| > 3 \quad \Rightarrow \quad x < -3 \text{ or } x > 3$$

The Ratio Test provides no information if $|x| = 3$, so we need to check the endpoints $x = -3$ and $x = 3$ separately. When $x = -3$, the series becomes:

$$\sum_{k=0}^{\infty} \frac{(-3)^k}{k \cdot 3^k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k}$$

This is the alternating harmonic series, which converges conditionally (by the Alternating Series Test). When $x = 3$, the series becomes:

$$\sum_{k=0}^{\infty} \frac{(3)^k}{k \cdot 3^k} = \sum_{k=0}^{\infty} \frac{1}{k}$$

This is the harmonic series, which diverges. In summary, the power series converges if $-3 \le x < 3$; that is, on the interval $[-3, 3)$. ◀

This power series converges *absolutely* on the interval $(-3, 3)$.

The Ratio Test is a powerful tool for determining where a power series converges. Typically, you also need to check the endpoints of an interval by replacing x with the endpoint values and then determining if the resulting numerical series converge or diverge at these values.

The Ratio Test does not help with the endpoints.

Practice 1. Find all values of x for which $\sum_{k=1}^{\infty} \frac{5^k \cdot x^k}{k}$ converges.

Interval of Convergence

In the preceding examples, the values of x for which the power series converged formed an interval. The next theorem and its corollary say that this *always* happens for *any* power series.

Interval of Convergence Theorem for Power Series

- If $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = c$, it converges when $|x| < |c|$.
- If $\sum_{k=0}^{\infty} a_k x^k$ diverges for $x = d$, it diverges when $|x| > |d|$.

Proof. If $\sum_{k=0}^{\infty} a_k c^k$ converges, $\lim_{k \to \infty} a_k c^k = 0$, so there is an N such that:

$$k \geq N \quad \Rightarrow \quad \left| a_k c^k \right| < 1 \quad \Rightarrow \quad |a_k| < \frac{1}{|c^k|}$$

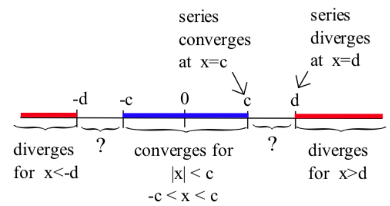
If $|x| < |c|$, then $\left| \frac{x}{c} \right| < 1$ and we can write:

$$\sum_{k=0}^{\infty} \left| a_k x^k \right| = \sum_{k=0}^{N-1} \left| a_k x^k \right| + \sum_{k=N}^{\infty} \left| a_k x^k \right| \leq \sum_{k=0}^{N-1} \left| a_k x^k \right| + \sum_{k=N}^{\infty} \left| \frac{x^k}{c^k} \right|$$

which is the sum of a finite number of terms and a (convergent) geometric series with ratio $\left| \frac{x}{c} \right| < 1$, hence $\sum_{k=0}^{\infty} \left| a_k x^k \right|$ converges by the Basic Comparison Test, so that $\sum_{k=0}^{\infty} a_k x^k$ converges (absolutely).

Now suppose that $\sum_{k=0}^{\infty} a_k d^k$ diverges. If $\sum_{k=0}^{\infty} a_k x^k$ were to converge with $|x| > |d|$, this would contradict the first part of the theorem. $\square$

If $\sum_{k=0}^{\infty} a_k x^k$ for a value $x = c$, then the series also converges for all values of x closer to the origin than c. If the power series diverges for a value $x = d$, then the power series diverges for all values of x farther from the origin than d. The Interval of Convergence Theorem does *not* tell us about the convergence of the power series for values of x with $|c| < |x| < |d|$ (see margin figure).

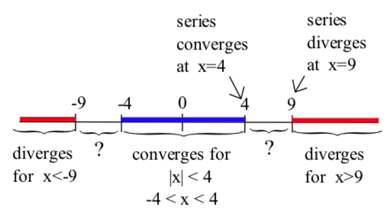


Example 3. If $\sum_{k=0}^{\infty} a_k x^k$ converges at $x = 4$ and diverges at $x = 9$, what can you conclude ("converge" or "diverge" or "no information") about the series when $x = 2, -3, -4, 5, -6, 8, -9, 10$ and -11?

Solution. We know the power series converges at $x = 4$, so with $c = 4$ in the Interval of Convergence Theorem we can conclude that the series converges for $x = 2$ and $x = -3$, because $|2| < |4|$ and $|-3| < |4|$.

We know the power series diverges at $x = 9$, so with $d = 9$ in the Interval of Convergence Theorem we can conclude that the series diverges for $x = 10$ and $x = -11$, because $|10| > |9|$ and $|-11| > |9|$. The remaining values of x ($-4, 5, -6, 8$ and -9) do not satisfy $|x| < 4$ or $|x| > 9$, so the series may converge or may diverge at those values—we don't have enough information.

The margin figure shows the regions where the Interval of Convergence Theorem guarantees convergence of this power series, guarantees divergence, and where it provides us with no information. ◀



Practice 2. If a power series $\sum_{k=0}^{\infty} a_k x^k$ converges at $x = 3$ and diverges at $x = -7$, what can you say about the convergence of this series for $x = -1, 2, -3, 4, -6, 7, -8$ and 17? Sketch the regions of known convergence and known divergence.

The following corollary of the Interval of Convergence Theorem guarantees that the set of values of x where the power series converges form an interval.

> **Corollary**
>
> The values of x for which the power series $\sum_{k=0}^{\infty} a_k x^k$ converges form an interval of the form $(-R, R)$, $[-R, R)$, $(-R, R]$ or $[-R, R]$.

Idea for a Proof. The extreme cases are $R = 0$, so that the power series converges only at $x = 0$, and $R = \infty$, so that the power series converges for all values of x. Otherwise, there is some number $c > 0$ so that the series converges for $x = c$ (and, by the Interval of Convergence Theorem, for all x with $|x| < |c|$) and another number $d > c$ so that the series converges for $x = d$ (and, by the Interval of Convergence Theorem, for all x with $|x| > |d|$). Consider the set of all such values c: these are bounded above by d, so there is a *least* upper bound of all such values c. Let R be the that least upper bound.

The existence of such a least upper bound is guaranteed by the Completeness Axiom of the real numbers (discussed in more advanced mathematics courses).

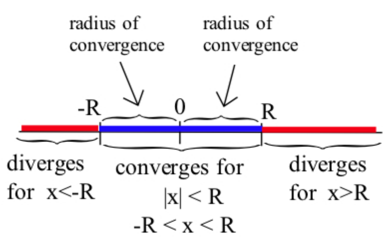
The Interval of Convergence Theorem and its Corollary do *not* tell us about the convergence of the power series at the endpoints of this interval: we need to check those two points individually.

Definition:
The **interval of convergence** of a power series $\sum_{k=0}^{\infty} a_k x^k$ is the interval of values of x for which the series converges.

From Examples 1 and 2, we know the interval of convergence of the power series $\sum_{k=0}^{\infty} (2k+1)x^k$ is $(-1, 1)$ and the interval of convergence of the power series $\sum_{k=0}^{\infty} \frac{x^k}{k \cdot 3^k}$ is $[-3, 3)$.

Radius of Convergence

More advanced mathematics courses study power series of the form $\sum_{k=0}^{\infty} a_k z^k$, where z is allowed to be a *complex* number (such as i or $3+4i$). Instead of an interval of convergence, these power series have a disk of convergence in the complex plane. The radius of this disk is called the **radius of convergence** of the power series. We use this same terminology for power series involving a real variable x.



Definition:
The **radius of convergence** of a power series $\sum_{k=0}^{\infty} a_k x^k$ is the number R so that the power series converges for $|x| < R$ and diverges for $|x| > R$. (The series may converge or may diverge if $|x| = R$.)

The radius of convergence of a power series is half of the length of its interval of convergence.

Example 4. What is the radius of convergence of the series from Example 1? From Example 2?

Solution. The series $\sum_{k=0}^{\infty} (2k+1)x^k$ converges if $-1 < x < 1$, so $R = 1$. The series $\sum_{k=0}^{\infty} \frac{x^k}{k \cdot 3^k}$ converges if $-3 \le x < 3$, so $R = 3$. ◀

Practice 3. Find the radius of convergence of the series in Practice 1.

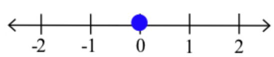
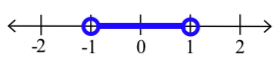
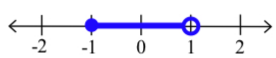
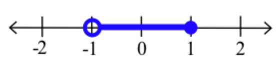
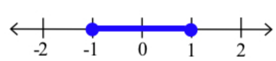
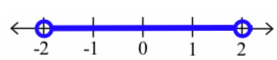
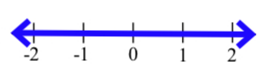
The convergence or divergence of a power series at an endpoint of the interval of convergence does not affect the value of the radius of convergence R, and the value of R does not tell you anything about the convergence of the power series at the endpoints of the interval of convergence (at $x = R$ and $x = -R$).

Summary

From the preceding discussion (and from the Idea for a Proof of the Corollary to the Interval of Convergence Theorem), we know that exactly one of three situations can occur for the radius of convergence R of a power series:

- it converges only for $x = 0$ (so that $R = 0$)
- it converges for when $|x| < R$ and diverges when $|x| > R$
- it converges for all values of x (so that $R = \infty$)

The following table displays information about the intervals and radii of convergence for several power series. Four of the series in the table have the same radius of convergence ($R = 1$) but slightly different intervals of convergence.

series	radius	interval	
$\sum_{k=1}^{\infty} k! \cdot x^k$	$R = 0$	$\{0\}$	
$\sum_{k=1}^{\infty} x^k$	$R = 1$	$(-1, 1)$	
$\sum_{k=1}^{\infty} \frac{x^k}{k}$	$R = 1$	$[-1, 1)$	
$\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$	$R = 1$	$(-1, 1]$	
$\sum_{k=1}^{\infty} \frac{x^k}{k^2}$	$R = 1$	$[-1, 1]$	
$\sum_{k=1}^{\infty} \frac{x^k}{2^k}$	$R = 2$	$(-2, 2)$	
$\sum_{k=1}^{\infty} \frac{x^k}{k!}$	$R = \infty$	$(-\infty, \infty)$	

A power series looks like a very long polynomial. The domain of a regular polynomial with a finite number of terms, however, is $(-\infty, \infty)$, but a power series may converge only on a finite interval, in which case it will have a much smaller domain than any related polynomial. As we continue to work with power series we need to be alert to where the power series converges (and behaves like a finite polynomial) and where the power series diverges. We need to know the interval of convergence of the power series and, typically, we use the Ratio Test to find that interval.

10.1 Problems

In Problems 1–15, determine all values of x for which each given power series converges, then graph the interval of convergence for the series on a number line.

1. $\sum_{k=1}^{\infty} x^k$
2. $\sum_{k=1000}^{\infty} x^k$
3. $\sum_{k=1}^{\infty} 3^k \cdot x^k$
4. $\sum_{k=1}^{\infty} (5x)^k$
5. $\sum_{k=1}^{\infty} \frac{x^k}{k}$
6. $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$
7. $\sum_{k=1}^{\infty} k \cdot x^k$
8. $\sum_{k=1}^{\infty} k^2 \cdot x^k$
9. $\sum_{k=1}^{\infty} k \cdot x^{2k+1}$
10. $\sum_{k=0}^{\infty} k! \cdot x^k$
11. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$
12. $\sum_{k=0}^{\infty} \frac{2^k \cdot x^k}{k!}$
13. $\sum_{k=1}^{\infty} k \left(\frac{x}{4}\right)^{2k}$
14. $\sum_{k=0}^{\infty} \frac{k! \cdot x^k}{2^k}$
15. $\sum_{k=0}^{\infty} \frac{x^k}{2^k}$

16. Your friend claims that the interval of convergence for a power series of the form $\sum_{k=0}^{\infty} a_k x^k$ is the interval $(-2, 3)$. Without checking your friend's work, how can you be certain that your friend is wrong?

In Problems 17–24, find the radius of convergence for the series from the given Problem.

17. Problem 1
18. Problem 3
19. Problem 5
20. Problem 7
21. Problem 9
22. Problem 11
23. Problem 13
24. Problem 15

In Problems 25–28, use the patterns you noticed in earlier problems and Examples to build a power series with the given interval of convergence. (There are many possible correct answers—find one.)

25. $(-5, 5)$
26. $[-3, 3)$
27. $[-2, 2]$
28. $(-4, 4]$

In Problems 29–32, given the interval of convergence for a power series, find its radius of convergence.

29. $(-5, 5)$
30. $[-3, 3)$
31. $[-2, 2]$
32. $(-4, 4]$

In Problems 33–41, find the interval of convergence for each series. Then, for x in the interval of convergence, find the sum of the series as a function of x. (Hint: You already know how to find the sum of a geometric series.)

33. $\sum_{k=0}^{\infty} x^k$
34. $\sum_{k=0}^{\infty} (-x)^k$
35. $\sum_{k=0}^{\infty} (2x)^k$
36. $\sum_{k=0}^{\infty} (3x)^k$
37. $\sum_{k=1}^{\infty} x^k$
38. $\sum_{k=0}^{\infty} x^{2k}$
39. $\sum_{k=0}^{\infty} x^{3k}$
40. $\sum_{k=0}^{\infty} \left(\frac{x}{4}\right)^k$
41. $\sum_{k=0}^{\infty} (4x)^k$

10.1 Practice Answers

1. Applying the Ratio Test to $\sum_{k=1}^{\infty} \frac{5^k \cdot x^k}{k}$:

$$\left| \frac{\frac{5^{k+1} \cdot x^{k+1}}{k+1}}{\frac{5^k \cdot x^k}{k}} \right| = \left| \frac{5^{k+1} \cdot x^{k+1}}{k+1} \cdot \frac{5^k \cdot x^k}{k} \right| = 5|x| \cdot \left(\frac{k}{k+1} \right) \to 5|x|$$

For the series to converge, we need:

$$5|x| < 1 \quad \Rightarrow \quad |x| < \frac{1}{5} \quad \Rightarrow \quad -\frac{1}{5} < x < \frac{1}{5}$$

The Ratio Test also tells us that the series diverges when:

$$5|x| > 1 \quad \Rightarrow \quad |x| > \frac{1}{5} \quad \Rightarrow \quad x < -\frac{1}{5} \text{ or } x > \frac{1}{5}$$

Now we need to check the points where $|x| = \frac{1}{5}$. When $x = \frac{1}{5}$:

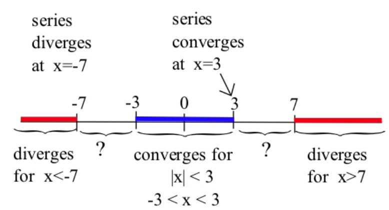
$$\sum_{k=1}^{\infty} \frac{5^k \cdot x^k}{k} = \sum_{k=1}^{\infty} \frac{5^k \cdot \left(\frac{1}{5}\right)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

which diverges (it's the harmonic series), and when $x = -\frac{1}{5}$:

$$\sum_{k=1}^{\infty} \frac{5^k \cdot x^k}{k} = \sum_{k=1}^{\infty} \frac{5^k \cdot \left(-\frac{1}{5}\right)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

which converges conditionally (by the Alternating Series Test). So the series converges on the interval $\left[-\frac{1}{5}, \frac{1}{5}\right)$.

2. The series converges at $x = -1$ and $x = 2$, diverges at $x = -8$ and $x = 17$, and may converge or diverge at the other points. See the margin figure for a graph of the regions of known convergence and known divergence.



3. The radius of convergence is $R = \frac{1}{5}$ (half the length of the interval of convergence).