10.3 Representing Functions as Power Series

We know from our work with geometric series that the function:

$$G(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \cdots$$

has domain $-1 < x < 1$ (that is, the geometric series converges for $|x| < 1$) and for values of x in that domain we know that:

$$G(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} = (1-x)^{-1}$$

The function $\frac{1}{1-x}$ has a much larger domain, $(-\infty, 1) \cup (1, \infty)$, than the corresponding power series function $G(x)$. But on their common domain, $(-1, 1)$, these two functions agree.

Can we find power series representations for other functions? If so, on what interval does the power series converge to the same value as the function? We will investigate the answers to these questions throughout the next few sections.

In this section, we obtain power series representations for several functions related to $\frac{1}{1-x}$ using our knowledge of the geometric series. We will also examine some applications of these power series representations of functions.

Substitution in Power Series

One simple but powerful method for obtaining a power series for a function is to make a substitution into a known power series representation. If we begin with the geometric series:

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k = 1 + u + u^2 + u^3 + u^4 + \cdots$$

and make the substitution $u = -x$ we get:

$$\frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k = 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \cdots$$

which we can rewrite as:

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k \cdot x^k = 1 - x + x^2 - x^3 + x^4 + \cdots$$

This new power series is also a geometric series, and it converges when (and only when) $|-x| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$.

Similar substitutions (along with some straightforward algebra) lead to a variety of other power series representations.

Example 1. Find power series for $\frac{1}{1-x^2}$ and $\frac{x}{1-x}$.

Solution. For the first function, use the substitution $u = x^2$ in the geometric series formula:

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k = 1 + u + u^2 + u^3 + u^4 + \cdots$$
$$\Rightarrow \frac{1}{1-x^2} = \sum_{k=0}^{\infty} (x^2)^k = 1 + (x^2) + (x^2)^2 + (x^2)^3 + (x^2)^4 + \cdots$$
$$\Rightarrow \frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots$$

which converges if $x^2 < 1 \Rightarrow \sqrt{x^2} < \sqrt{1} \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$. For the second series, rewrite the function as a product:

$$\frac{x}{1-x} = x \cdot \frac{1}{1-x} = x \cdot \sum_{k=0}^{\infty} x^k = x \cdot \left[1 + x + x^2 + x^3 + x^4 + \cdots\right]$$
$$= \sum_{k=0}^{\infty} x^{k+1} = x + x^2 + x^3 + x^4 + x^5 + \cdots$$

This is also a geometric series with ratio x, so it converges when $|x| < 1 \Rightarrow -1 < x < 1$. ◀

Practice 1. Find power series for $\frac{1}{1-x^3}$, $\frac{1}{1+x^2}$ and $\frac{5x}{1+x}$.

Differentiation and Integration of Power Series

One feature of polynomials that makes them very easy to differentiate and integrate is that we can differentiate and integrate them term-by-term. The same result holds true for power series.

Term-by-Term Differentiation of Power Series

If $f(x)$ is defined by a power series:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

that converges for $-R < x < R$, then:

$$f'(x) = \sum_{k=1}^{\infty} k \cdot a_k x^{k-1} = a_1 + 2 \cdot a_2 x + 3 \cdot a_3 x^2 + 4 \cdot a_4 x^3 + \cdots$$

and this new power series also converges for $-R < x < R$.

The power series for f and the power series for f' may differ in whether they converge or diverge at the endpoints of the interval of convergence, but they both converge for $-R < x < R$.

The proof of this statement is rather long and highly technical, so we will omit it, but this result allows us to find power series of even more functions based on the geometric series.

Example 2. Find a power series for $\frac{1}{(1-x)^2}$.

Solution. Because $(1-x)^{-2}$ is the derivative of $(1-x)^{-1}$, we can write:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{k=0}^{\infty} x^k\right] = \sum_{k=0}^{\infty} \frac{d}{dx}\left(x^k\right)$$
$$= \sum_{k=0}^{\infty} k \cdot x^{k-1} = \sum_{k=1}^{\infty} k \cdot x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

See Problem 41.

You can use the Ratio Test on the new power series to verify that its interval of convergence is $(-1, 1)$, the same as the original series. ◀

Term-by-Term Integration of Power Series

If $f(x)$ is defined by a power series:

$$f(x) = \sum_{k=0}^{\infty} a_k\, x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

that converges for $-R < x < R$, then:

$$\int f(x)\,dx = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1}\, x^{k+1} = C + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots$$

and this new power series also converges for $-R < x < R$.

The power series for f and its antiderivative may differ in whether they converge or diverge at the endpoints of the interval of convergence, but they both converge for $-R < x < R$.

Proof. Let $F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1}\, x^{k+1}$ so that, using term-by-term differentiation:

$$F'(x) = \sum_{k=0}^{\infty} (k+1) \cdot \frac{a_k}{k+1}\, x^k = \sum_{k=0}^{\infty} a_k\, x^k = f(x)$$

Here we use the result of the previous theorem, which we did not prove.

Therefore $\int f(x)\,dx = F(x) + C$. To find the interval of convergence for $F(x)$, note that:

$$k \geq 0 \;\Rightarrow\; k+1 \geq 1 \;\Rightarrow\; \frac{1}{k+1} \leq 1 \;\Rightarrow\; \frac{|a_k|}{k+1}|x|^k \leq |a_k| \cdot |x^k|$$

Because $\sum_{k=0}^{\infty} a_k\, x^k$ converges for $|x| < R$, it converges absolutely on this open interval, hence so does $\sum_{k=0}^{\infty} \frac{a_k}{k+1}\, x^k$ (by the Basic Comparison Test), and so does:

$$x \cdot \sum_{k=0}^{\infty} \frac{a_k}{k+1}\, x^k = \sum_{k=0}^{\infty} \frac{a_k}{k+1}\, x^{k+1}$$

(Convergence at the endpoints $x = -R$ and $x = R$ must be determined on a case-by-case basis.) □

Example 3. Find power series for $\ln(1-x)$ and $\arctan(x)$

Solution. We need to recognize that these functions are integrals of functions whose power series we already know. For the first function:

$$\ln(1-x) = \int \frac{-1}{1-x}\,dx = -\int \left[\sum_{k=0}^{\infty} x^k\right] dx = C - \sum_{k=0}^{\infty} \frac{1}{k+1} \cdot x^{k+1}$$
$$= C - x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$$

Substituting $x = 0$ into both sides of this equation yields:

$$0 = \ln(1) = \ln(1-0) = C - 0 - \frac{1}{2}\cdot 0^2 - \frac{1}{3}\cdot 0^3 - \cdots = C \;\Rightarrow\; C = 0$$

so that:

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots = -\sum_{k=0}^{\infty} \frac{1}{k+1}\cdot x^{k+1}$$

You should be able to check that the interval of convergence for this new power series is $-1 \le x < 1$, which agrees with the interval of convergence of the original series (except at the left endpoint).

See Problem 37.

For the second series, we can use the second result from Practice 1:

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k \cdot x^{2k} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

and then apply term-by-term integration to this power series to get:

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \int \left[\sum_{k=0}^{\infty} (-1)^k \cdot x^{2k}\right] dx$$
$$= C + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

To determine the value of C, substitute $x = 0$ to get:

$$0 = \arctan(0) = C + 0 - \frac{1}{3}\cdot 0^3 + \frac{1}{5}\cdot 0^5 - \frac{1}{7}\cdot 0^7 + \cdots = C \;\Rightarrow\; C = 0$$

so that:

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

You should be able to check that the interval of convergence for this new power series is $-1 \le x \le 1$, which agrees with the interval of convergence of the original series (except at the endpoints). ◀

See Problem 39.

Practice 2. Find a power series for $\ln(1+x)$.

Applications of Power Series

An important application of power series is the use of these "infinite polynomials" in place of a complicated integrand to help evaluate difficult integrals.

Example 4. Express the definite integral $\int_0^{\frac{1}{2}} \arctan\left(x^2\right)\, dx$ as a numerical series. Then approximate the value of the integral by calculating the sum of the first four terms of that numerical series.

Solution. From Example 3, we know that:

$$\arctan(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} u^{2k+1} = u - \frac{1}{3}u^3 + \frac{1}{5}u^5 - \frac{1}{7}u^7 + \cdots$$

so substituting $u = x^2$ into this power series give us:

$$\arctan(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{4k+2} = x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} - \frac{1}{7}x^{14} + \cdots$$

Term-by-term integration of this power series yields:

$$\int_0^{\frac{1}{2}} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{4k+2}\right] dx = \int_0^{\frac{1}{2}} \left[x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} - \frac{1}{7}x^{14} + \cdots\right] dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left[\frac{x^{4k+3}}{4k+3}\right]_0^{\frac{1}{2}} = \left[\frac{1}{3}x^3 - \frac{1}{21}x^7 + \frac{1}{55}x^{11} - \frac{1}{105}x^{15} + \cdots\right]_0^{\frac{1}{2}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(4k+3)\cdot 2^{4k+3}} = \frac{1}{3\cdot 2^3} - \frac{1}{21\cdot 2^7} + \frac{1}{55\cdot 2^{11}} - \frac{1}{105\cdot 2^{15}} + \cdots$$

Adding up the first four terms of this series gives the approximation:

$$\int_0^{\frac{1}{2}} \arctan(x^2)\, dx \approx 0.04130323$$

Because this numerical series is an alternating series, the Alternating Series Estimation Bound tells us that the difference between this approximation and the actual value of the integral is smaller than the absolute value of the next term in the series:

$$\left| \int_0^{\frac{1}{2}} \arctan(x^2)\, dx - 0.04130323 \right| \le \frac{1}{9\cdot 19\cdot 2^{19}} \approx 0.0000000112$$

We can therefore say that the value of the definite integral is between 0.0413032188 and 0.0413032415. ◀

Practice 3. Express the definite integral $\int_0^{0.2} x^2 \ln(1+x)\, dx$ as a numerical series. Then approximate the value of the integral by calculating the sum of the first four terms of the numerical series.

Wrap-Up

We obtained all of the power series used in this section from the geometric series via substitution, differentiation and integration. Many important functions, however, are not related to a geometric series, so future sections will discuss methods for representing more general functions using power series. The following table collects some of the power series representations we have obtained in this section.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k \cdot x^k = 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k \cdot x^{2k} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{x^k}{k} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+1}}{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k \cdot x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

10.3 Problems

In Problems 1–14, use substitution and a known power series to find a power series for the function.

1. $\dfrac{1}{1-x^4}$
2. $\dfrac{1}{1-x^5}$
3. $\dfrac{1}{1+x^4}$
4. $\dfrac{1}{1+x^5}$
5. $\dfrac{1}{5+x}$
6. $\dfrac{1}{3-x}$
7. $\dfrac{x^2}{1+x^3}$
8. $\dfrac{x}{1+x^4}$
9. $\ln\left(1+x^2\right)$
10. $\ln\left(1+x^3\right)$
11. $x\arctan(x^2)$
12. $\arctan(x^3)$
13. $\dfrac{1}{(1-x^2)^2}$
14. $\dfrac{1}{(1+x^2)^2}$
15. $\dfrac{1}{(1-x)^3}$
16. $\dfrac{1}{(1-x^2)^3}$
17. $\dfrac{1}{(1+x^2)^3}$
18. $\dfrac{1}{(1-x)^4}$

In 19–26, represent each integral as a series, then calculate the sum of the first three terms.

19. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1-x^3}\,dx$
20. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1+x^3}\,dx$
21. $\displaystyle\int_0^{\frac{3}{5}} \ln(1+x)\,dx$
22. $\displaystyle\int_0^{\frac{1}{2}} \ln(1+x^2)\,dx$
23. $\displaystyle\int_0^{\frac{1}{2}} x^2\arctan(x)\,dx$
24. $\displaystyle\int_0^{\frac{1}{2}} \arctan(x^3)\,dx$
25. $\displaystyle\int_0^{0.3} \frac{1}{(1-x)^2}\,dx$
26. $\displaystyle\int_0^{0.7} \frac{x^3}{(1-x)^2}\,dx$

In 27–32, represent each numerator as a power series, then use the power series to help find the limit.

27. $\lim_{x\to 0} \frac{\arctan(x)}{x}$

28. $\lim_{x\to 0} \frac{\ln(1-x)}{2x}$

29. $\lim_{x\to 0} \frac{\ln(1+x)}{2x}$

30. $\lim_{x\to 0} \frac{\arctan\left(x^2\right)}{x}$

31. $\lim_{x\to 0} \frac{\arctan\left(x^2\right)}{x^2}$

32. $\lim_{x\to 0} \frac{\arctan(x) - x}{x^3}$

33. $\lim_{x\to 0} \frac{\ln\left(1-x^2\right)}{3x}$

34. $\lim_{x\to 0} \frac{\ln\left(1+x^2\right)}{3x}$

In Problems 35–42, determine a power series for each function and then determine the interval of convergence of each power series.

35. $\frac{1}{1+x}$

36. $\frac{1}{1-x^2}$

37. $\ln(1-x)$

38. $\ln(1+x)$

39. $\arctan(x)$

40. $\arctan(x^2)$

41. $\frac{1}{(1-x)^2}$

42. $\frac{1}{(1+x)^2}$

10.3 Practice Answers

1. For the first function, use the substitution $u = x^3$ in the geometric series formula:

$$\begin{aligned} \frac{1}{1-u} &= \sum_{k=0}^{\infty} u^k = 1 + u + u^2 + u^3 + u^4 + \cdots \\ \Rightarrow \frac{1}{1-x^3} &= \sum_{k=0}^{\infty} (x^3)^k = 1 + (x^3) + (x^3)^2 + (x^3)^3 + (x^3)^4 + \cdots \\ \Rightarrow \frac{1}{1-x^3} &= \sum_{k=0}^{\infty} x^{3k} = 1 + x^3 + x^6 + x^9 + x^{12} + \cdots \end{aligned}$$

For the second function, use the substitution $u = -x^2$:

$$\begin{aligned} \frac{1}{1-u} &= \sum_{k=0}^{\infty} u^k = 1 + u + u^2 + u^3 + u^4 + \cdots \\ \Rightarrow \frac{1}{1-(-x^2)} &= \sum_{k=0}^{\infty} (-x^2)^k = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots \\ \Rightarrow \frac{1}{1+x^2} &= \sum_{k=0}^{\infty} (-1)^k \cdot x^{2k} = 1 - x^2 + x^4 - x^6 - \cdots \end{aligned}$$

For the third function, use the substitution $u = -x$:

$$\begin{aligned} \frac{1}{1-u} &= \sum_{k=0}^{\infty} u^k = 1 + u + u^2 + u^3 + u^4 + \cdots \\ \Rightarrow \frac{1}{1-(-x)} &= \sum_{k=0}^{\infty} (-x)^k = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots \\ \Rightarrow \frac{1}{1+x} &= \sum_{k=0}^{\infty} (-1)^k \cdot x^k = 1 - x + x^2 - x^3 - \cdots \end{aligned}$$

and then multiply both sides of this last equation by $5x$:

$$\frac{5x}{1+x} = 5x \cdot \sum_{k=0}^{\infty} (-1)^k \cdot x^k = 5x \left[1 - x + x^2 - x^3 - \cdots\right]$$
$$= \sum_{k=0}^{\infty} 5(-1)^k \cdot x^{k+1} = 5x - 5x^2 + 5x^3 - 5x^4 + \cdots$$

2. Use the first result from Example 3:

$$\ln(1-u) = -u - \frac{1}{2}u^2 - \frac{1}{3}u^3 - \frac{1}{4}u^4 - \cdots = -\sum_{k=0}^{\infty} \frac{1}{k+1} \cdot u^{k+1}$$

and substitute $u = -x$:

$$\ln(1-(-x)) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \cdot x^{k+1}$$

3. Multiply the result of Practice 2 by x^2:

$$x^2 \cdot \ln(1+x) = x^3 - \frac{1}{2}x^4 + \frac{1}{3}x^5 - \frac{1}{4}x^6 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \cdot x^{k+3}$$

and then apply term-by-term integration to get:

$$\int_0^{0.2} x^2 \cdot \ln(1+x)\, dx = \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)(k+4)} \cdot x^{k+4}\right]_0^{0.2}$$
$$= \left[\frac{1}{4}x^4 - \frac{1}{10}x^5 + \frac{1}{18}x^6 - \frac{1}{28}x^7 + \cdots\right]_0^{0.2}$$
$$= \frac{1}{4}(0.2)^4 - \frac{1}{10}(0.2)^5 + \frac{1}{18}(0.2)^6 - \frac{1}{28}(0.2)^7 + \cdots$$
$$\approx 0.000371$$