10.4 MacLaurin and Taylor Series

Having found several power series (all variations of the geometric series) that converge to familiar functions such as $\ln(1+x)$ and $\arctan(x)$, we turn our attention to more general functions, asking:

- Does the function have a power series expansion?
- Where does this power series converge?
- Where does this power series converge to the original function?

Once we determine a power series for a new function, we can use it to approximate function values, compute integrals and evaluate limits.

MacLaurin Series

Our first result tells us that *if* a function has a power series expansion, the coefficients of that power series must follow a familiar pattern.

Theorem:

If a function $f(x)$ has a power series representation $f(x) = \sum_{k=0}^{\infty} a_k x^k$ valid for $|x| < R$

then the coefficients of the power series must be:

$$a_k = \frac{f^{(k)}(0)}{k!}$$

Proof. Suppose that:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

Putting $x = 0$ into this equation yields:

$$f(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + \cdots + a_n \cdot 0^n + \cdots = a_0$$

so we know that $a_0 = f(0) = \frac{f^{(0)}(0)}{0!}$, proving the coefficient formula for $k = 0$. Differentiating the equation in the hypothesis yields:

We use the conventions that $0! = 1$ and that $f^{(0)}(x) = f(x)$.

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n \cdot a_n x^{n-1} + \cdots$$
$$\Rightarrow f'(0) = a_1 + 2a_2 \cdot 0 + 3a_3 \cdot 0^2 + \cdots + n \cdot a_n \cdot 0^{n-1} + \cdots = a_1$$

so that $a_1 = f'(0) = \frac{f'(0)}{1!}$, proving the coefficient formula for $k = 1$. Differentiating again yields:

$$f''(x) = 2a_2 + 3 \cdot 2a_3 x + \cdots + n(n-1) \cdot a_n x^{n-2} + \cdots$$
$$\Rightarrow \quad f''(0) = 2a_2 + 3 \cdot 2a_3 \cdot 0 + \cdots + n(n-1) \cdot a_n \cdot 0^{n-2} + \cdots = 2a_2$$

so that $a_2 = \dfrac{f''(0)}{2} = \dfrac{f''(0)}{2!}$, proving the coefficient formula for $k = 2$.
Differentiating yet again:

$$f'''(x) = 3 \cdot 2 \cdot 1 a_3 + \cdots + n(n-1)(n-1) \cdot a_n x^{n-3} + \cdots$$
$$\Rightarrow\ f'''(0) = 3 \cdot 2 \cdot 1 a_3 + \cdots + n(n-1)(n-2) \cdot a_n \cdot 0^{n-3} + \cdots = 3 \cdot 2 \cdot 1 a_3$$

so that $a_3 = \dfrac{f'''(0)}{3 \cdot 2 \cdot 1} = \dfrac{f'''(0)}{3!}$, proving the coefficient formula for $k = 3$.
In general:

$$f^{(n)}(x) = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 a_n + [\text{terms containing powers of } x]$$
$$\Rightarrow\quad f^{(n)}(0) = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 a_n + [0] = n! \cdot a_n$$

so that $a_n = \dfrac{f^{(n)}(0)}{n!}$. □

You may recognize this coefficient pattern from Section 8.7, where we called the polynomial:

$$P(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}$$

the **MacLaurin polynomial** for $f(x)$. If we continue to add terms (forever) to this polynomial, we get the **MacLaurin series** for $f(x)$.

The **MacLaurin series** for $f(x)$ is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!} + \cdots.$$

This definition of a MacLaurin series and the preceding result about the form of its coefficients do *not* say that every function can be written as a power series. But *if* a function can be written as a power series, its coefficients must follow the above pattern. Fortunately, many important functions (such as $\sin(x)$ and e^x) can be written as power series.

The preceding proof also does *not* tell us where a MacLaurin series converges: we will need to apply techniques from Chapter 9 (typically the Ratio Test) to determine the interval of convergence for a MacLaurin series. Nor does the proof tell us that the series actually converges to the original function at any point (other than $x = 0$): to show that the series actually converges to the original function on its interval of convergence, we will need a result to be proved in Section 10.5.

Example 1. Find the MacLaurin series for $f(x) = \sin(x)$ and determine the radius of convergence of the series.

Solution. $f(x) = \sin(x) \Rightarrow f(0) = \sin(0) = 0 \Rightarrow a_0 = f(0) = 0$. Computing the derivatives of $\sin(x)$:

$$f'(x) = \cos(x) \Rightarrow f'(0) = \cos(0) = 1 \Rightarrow a_1 = \frac{f'(0)}{1!} = \frac{1}{1} = 1$$
$$f''(x) = -\sin(x) \Rightarrow f''(0) = -\sin(0) = 0 \Rightarrow a_2 = \frac{f''(0)}{2!} = \frac{0}{2} = 0$$
$$f'''(x) = -\cos(x) \Rightarrow f'''(0) = -\cos(0) = -1 \Rightarrow a_3 = \frac{f'''(0)}{3!} = \frac{-1}{6}$$
$$f^{(4)}(x) = \sin(x) \Rightarrow f^{(4)}(0) = \sin(0) = 0 \Rightarrow a_4 = \frac{f^{(4)}(0)}{4!} = \frac{0}{24} = 0$$

This derivative pattern repeats, cycling through the values 1, 0, −1 and 0 so that $a_5 = \frac{1}{5!}$, $a_6 = \frac{0}{6!} = 0$, $a_7 = \frac{-1}{7!}$, $a_8 = 0$, $a_9 = \frac{1}{9!}$, and so on:

Notice that the MacLaurin series for $\sin(x)$, an odd function, contains only odd powers of x. Also notice that it alternates between positive and negative coefficients.

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

To find the radius of convergence, apply the Ratio Test:

$$\left| \frac{\frac{(-1)^{k+1}}{(2(k+1)+1)!}x^{2(k+1)+1}}{\frac{(-1)^k}{(2k+1)!}x^{2k+1}} \right| = \left| \frac{x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{x^{2k+1}} \right| = \frac{(2k+1)!}{(2k+3)!}x^2$$
$$= \frac{x^2}{(2k+3)(2k+2)} \longrightarrow 0 < 1$$

for any value of x, so the interval of convergence is $(-\infty, \infty)$, hence the radius of convergence is $R = \infty$. ◀

You have two options here: proceed as in Example 1, or differentiate the result of Example 1.

Practice 1. Find the MacLaurin series for $f(x) = \cos(x)$ and determine the radius of convergence of the series.

Example 2. Find the MacLaurin series for $f(x) = e^x$ and determine the radius of convergence of the series.

Solution. With $f(x) = e^x$, $f'(x) = e^x \Rightarrow f''(x) = e^x \Rightarrow f'''(x) = e^x$ and in fact $f^{(k)}(x) = e^x$ for any integer $k \geq 0$, so $f^{(k)}(0) = e^0 = 1$ for all such k. Therefore the coefficients of the MacLaurin series for $f(x) = e^x$ all have the form $a_k = \frac{1}{k!}$, so we can write:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n + \cdots$$

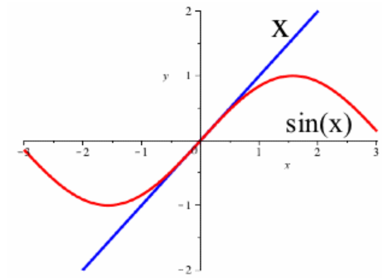
To find the radius of convergence for this series, apply the Ratio Test:

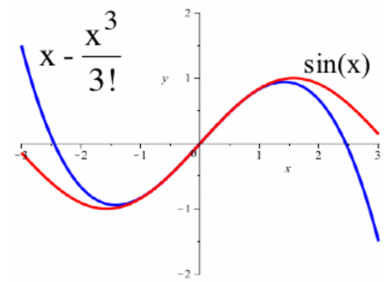
$$\left| \frac{\frac{1}{(k+1)!}x^{k+1}}{\frac{1}{k!}x^k} \right| = \frac{k!}{(k+1)!}|x| = \frac{|x|}{k+1} \longrightarrow 0$$

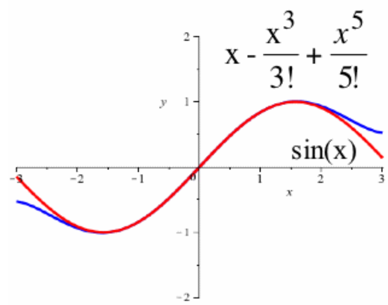
for any value of x, so the interval of convergence this MacLaurin series is $(-\infty, \infty)$ and its radius of convergence is $R = \infty$. ◀

Approximation Using MacLaurin Series

The MacLaurin series for $\sin(x)$ converges for every value of x (although we have not yet shown that it actually converges to $\sin(x)$ anywhere other than $x = 0$). The margin figure shows the graphs of $\sin(x)$ and the first few MacLaurin polynomials x, $x - \frac{1}{6}x^3$ and $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ for $-\pi \leq x \leq \pi$. While these low-degree MacLaurin polynomials appear to approximate $\sin(x)$ well near $x = 0$, the farther x gets from 0, the worse the approximation.







Example 3. Use a MacLaurin series to represent $\sin(0.5)$ as a numerical series. Approximate the value of $\sin(0.5)$ by computing the partial sum of the first three non-zero terms of this series and give a bound on the "error" between this approximation and the exact value of $\sin(0.5)$.

Solution. Putting $x = 0.5$ into the MacLaurin Series for $\sin(x)$:

$$\sin(0.5) = (0.5) - \frac{1}{3!}(0.5)^3 + \frac{1}{5!}(0.5)^5 - \frac{1}{7!}(0.5)^7 + \frac{1}{9!}(0.5)^9 - \cdots$$

so that:

$$\begin{aligned}\sin(0.5) &\approx (0.5) - \frac{1}{3!}(0.5)^3 + \frac{1}{5!}(0.5)^5 \\ &= \frac{1}{2} - \frac{1}{48} + \frac{1}{3840} \approx 0.479427083333\end{aligned}$$

Because the series in question is an alternating series, the difference between the approximation of $\sin(0.5)$ and the exact value of $\sin(0.5)$ is less than the absolute value of the next term in the alternating series:

$$\text{“error”} < \frac{1}{7!}(0.5)^7 = \frac{1}{645120} \approx 0.00000155$$

If you use the first four nonzero terms to approximate $\sin(0.5)$:

$$\sin(0.5) \approx (0.5) - \frac{1}{3!}(0.5)^3 + \frac{1}{5!}(0.5)^5 - \frac{1}{7!}(0.5)^7 \approx 0.47942553323$$

then the "error" is less than $\frac{1}{9!}(0.5)^9 = \frac{1}{185794560} \approx 5.4 \times 10{-}9$. ◀

We were able to obtain a bound for the error in the approximation of $\sin(0.5)$ because the series in question was an alternating series, a type of series for which we have an error bound.

Many power series, however, are not alternating series. In Section 10.5 we will develop a general error bound for MacLaurin series.

Practice 2. Use the sum of the first two nonzero terms of the MacLaurin series for $\cos(x)$ to approximate the value of $\cos(0.2)$. Give a bound on the "error" between this approximation and the exact value of $\cos(0.2)$.

Practice 3. Evaluate the partial sums of the first six terms of the numerical series for $e = e^1$ and $\frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$. Compare these partial sums with the values your calculator gives.

The numerical series for e is not an alternating series, so we do not have a bound for the approximation yet. We will in the next section.

Calculator Note: When you press the buttons on your calculator to evaluate $\sin(0.5)$ or $\cos(0.2)$, the calculator does not look up the answer in a table. Instead, it has been programmed with series representations for sine, cosine and other functions, and it calculates a partial sum of an appropriate series to obtain a numerical answer. It adds enough terms so that the eight or nine digits shown on the display are (usually) correct. In Section 10.5 we examine these methods in more detail and consider how to determine the number of terms needed in the partial sum to achieve the desired number of accurate digits in the answer.

Substitution in MacLaurin Series

Now that we know MacLaurin series for $\sin(x)$, $\cos(x)$ and e^x, we can use techniques from Section 10.3 to quickly determine MacLaurin series representations of more complicated fucntions.

Example 4. Represent $\sin\left(x^3\right)$ and $\int \sin\left(x^3\right)\,dx$ as power series. Use the first three non-zero terms of the second series to approximate $\int_0^1 \sin\left(x^3\right)\,dx$ and obtain a bound for the "error."

Solution. Starting with the MacLaurin series for $\sin(u)$:

$$\sin(u) = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7 + \cdots$$

put $u = x^3$ to get:

$$\begin{aligned}\sin\left(x^3\right) &= x^3 - \frac{1}{3!}\left(x^3\right)^3 + \frac{1}{5!}\left(x^3\right)^5 - \frac{1}{7!}\left(x^3\right)^7 + \cdots \\ &= x^3 - \frac{1}{3!}x^9 + \frac{1}{5!}x^{15} - \frac{1}{7!}x^{21} + \cdots\end{aligned}$$

Integrating this result term by term yields:

$$\begin{aligned}\int \sin\left(x^3\right)\,dx &= \int \left[x^3 - \frac{1}{6}x^9 + \frac{1}{120}x^{15} - \frac{1}{5040}x^{21} + \cdots\right]dx \\ &= C + \frac{1}{4}x^4 - \frac{1}{60}x^{10} + \frac{1}{1920}x^{16} - \frac{1}{110880}x^{22} + \cdots\end{aligned}$$

Approximating the definite integral:

$$\int_0^1 \sin\left(x^3\right)\,dx \approx \frac{1}{4} - \frac{1}{60} + \frac{1}{1920} \approx 0.2338542$$

A bound for the "error" between this approximation and the exact value of the definite integral is $\frac{1}{22 \cdot 7!} = \frac{1}{110880} \approx 0.0000090$. Using just one more term:

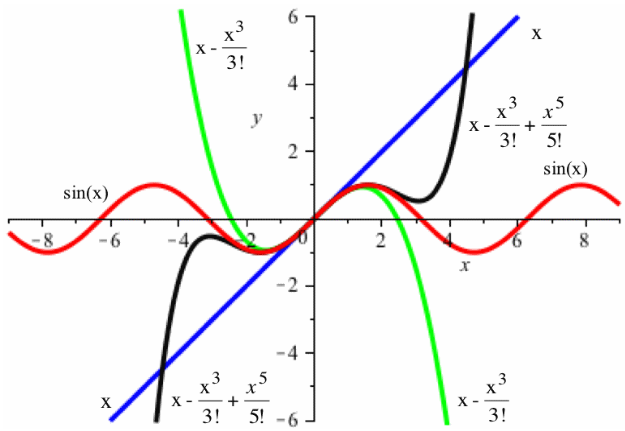
$$\int_0^1 \sin\left(x^3\right)\,dx \approx \frac{1}{4} - \frac{1}{60} + \frac{1}{1920} - \frac{1}{110880} \approx 0.233845515$$

gives an estimate within $\frac{1}{28 \cdot 9!} \approx 0.000000098$ of the exact value. ◀

Practice 4. Represent $x \cdot \cos\left(x^3\right)$ and $\int x \cdot \cos\left(x^3\right)\,dx$ as MacLaurin series. Use the first two nonzero terms of the second series to approximate $\int_0^{\frac{1}{2}} x \cdot \cos\left(x^3\right)\,dx$ and obtain a bound for the "error."

Taylor Series

The coefficients for a MacLaurin series (or polynomial) for a function $f(x)$ depend only on the values $f(0)$ and $f^{(k)}(0)$. As a consequence, the MacLaurin polynomials for $f(x)$ typically do a very good job of approximating the values of the original function near $x = 0$, as you can observe in this graph of $\sin(x)$ and its first few MacLaurin polynomials:



The figure above also demonstrates, however, that for values of x not close to 0, the values of the MacLaurin polynomials for $f(x)$ can be quite far from the values of the original function $f(x)$. Even though we know that the MacLaurin series for $\sin(x)$ converges for any value of x, for values of x far away from 0 we might need to add up hundreds of terms in order to achieve a good approximation of $\sin(x)$. For example, the first two nonzero terms of the MacLaurin series for $\sin(x)$ approximate $\sin(0.1)$ correctly to six decimal places, but you need 11 terms to approximate $\sin(5)$ with the same accuracy.

If you need to approximate a function with polynomials near a value away from $x = 0$, you can either use significantly more terms in a MacLaurin polynomial for that function, or you can "shift" the power series to center it at another value $x = c$. We call these "shifted" power series **Taylor series** and their partial sums **Taylor polynomials**. Typically the Taylor polynomials of a function centered at $x = c$ provide good approximations to $f(x)$ when x is close to c.

Theorem:

If a function $f(x)$ has a power series representation $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ valid for $|x - c| < R$

then the coefficients of the power series must be:

$$a_k = \frac{f^{(k)}(c)}{k!}$$

This theorem generalizes the corresponding result about MacLaurin series from the beginning of this section. To prove the new theorem, merely replace 0 with c in the original proof.

The **Taylor series** for $f(x)$ centered at $x = c$ is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

Taylor series and MacLaurin series were developed by the Scottish mathematician and astronomer James Gregory (1638–1675), but the results were not published until after his death. The English mathematician Brook Taylor (1685–1731) independently rediscovered these results and included them in a 1715 book. The Scottish mathematician and engineer Colin MacLaurin (1698–1746) quoted Taylor's work in his widely read 1742 *Treatise on Fluxions*, with the result that Taylor series centered at $c = 0$ became known as MacLaurin series.

A Maclaurin series is merely a Taylor series centered at $c = 0$, hence a MacLaurin series is a special case of Taylor series.

You should notice that the first term of the Taylor series for $f(x)$ is simply the value of the function f at the point $x = c$: it provides the best constant-function approximation of $f(x)$ near $x = c$. The sum of the first two terms of the Taylor series for a function $f(x)$:

$$f(c) + f'(c) \cdot (x - c)$$

resembles our usual formula in an equation of the line tangent to the graph of $f(x)$ at $x = c$ and gives the linear approximation of $f(x)$ near $x = c$ that we first examined in Chapter 2. The Taylor series formula extends these approximations to higher-degree polynomials, and the partial sums of the Taylor series provide higher-degree polynomial approximations of $f(x)$ near $x = c$.

Multiplying Power Series

We can add and subtract power series term by term, and you have already multiplied a power series by monomials such as x and x^2 to create new power series. Occasionally, you may find it useful to multiply a power series by another power series. The method for multiplying series is the same method used to multiply a polynomial by another polynomial, but it becomes very tedious to obtain more than the first few terms of the resulting product.

Example 5. Find the first five nonzero terms of the MacLaurin series for $\frac{1}{1-x} \cdot \sin(x)$.

Solution. Starting with the MacLaurin series for $\frac{1}{1-x}$ and $\sin(x)$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$
$$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots$$

multiply each term in the first series by each term in the second series:

$$\begin{aligned}
\frac{1}{1-x}\cdot\sin(x) &= \left[1+x+x^2+x^3+\cdots\right]\cdot\left[x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots\right] \\
&= 1\cdot\left[x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots\right]+x\left[x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots\right]+x^2\left[x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots\right] \\
&\qquad +x^3\left[x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots\right]+x^4\left[x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots\right]+\cdots \\
&= \left[x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots\right]+\left[x^2-\frac{1}{6}x^4+\cdots\right]+\left[x^3-\frac{1}{6}x^5+\cdots\right]+\left[x^4\cdots\right]+\left[x^5\cdots\right] \\
&= x+x^2+\frac{5}{6}x^3+\frac{5}{6}x^4+\frac{101}{120}x^5+\cdots
\end{aligned}$$

We know that *if* the function $\frac{1}{1-x}\cdot\sin(x)$ has a MacLaurin series, these must be the first five non-zero terms of that series. In order to show that this power series actually converges to $\frac{1}{1-x}\cdot\sin(x)$ on its interval of convergence, we need a theorem due to Abel (proved in more advanced courses) that says the product of two convergent power series also converges on their common interval of convergence. ◀

Practice 5. Find the first three nonzero terms of the MacLaurin series for $e^x\cdot\sin(x)$.

It is also possible to divide one power series by another power series using a procedure similar to "long division" of a polynomial by a polynomial, but we will not discuss that (quite tedious) process here.

Wrap-Up

The table below collects information about several important MacLaurin series developed in this section and the previous one.

$$e^x = \sum_{k=0}^{\infty}\frac{1}{k!}x^k = 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots \qquad \text{valid on } (-\infty,\infty)$$

$$\sin(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} = x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\frac{1}{9!}x^9-\cdots \qquad \text{valid on } (-\infty,\infty)$$

$$\cos(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\frac{1}{8!}x^8-\cdots \qquad \text{valid on } (-\infty,\infty)$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty}x^k = 1+x+x^2+x^3+x^4+\cdots \qquad \text{valid on } (-1,1)$$

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\cdot\frac{x^k}{k} = x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots \qquad \text{valid on } (-1,1]$$

$$\arctan(x) = \sum_{k=0}^{\infty}(-1)^k\cdot\frac{x^{2k+1}}{2k+1} = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots \qquad \text{valid on } [-1,1]$$

10.4 Problems

In Problems 1–14, use the MacLaurin series coefficient formula to find the first several terms of the MacLaurin series for the given function, then compare the result with the series representation found in Section 10.3.

1. $\ln(1+x)$
2. $\ln(1+x)$
3. $\arctan(x)$
4. $\dfrac{1}{1-x}$

In Problems 5–8, find the first several terms of the MacLaurin series for the given function.

5. $\cos(x)$ to the x^6 term
6. $\tan(x)$ to the x^5 term
7. $\sec(x)$ to the x^4 term
8. e^{3x} to the x^4 term

In Problems 9–13, find the first several terms of the Taylor series for the given function centered at the given point c.

9. $\ln(x)$ for $c=1$
10. $\sin(x)$ for $c=\pi$
11. $\sin(x)$ for $c=\dfrac{\pi}{2}$
12. $\sqrt{x}$ for $c=1$
13. $\sqrt{x}$ for $c=9$

In Problems 14–17, use the first three nonzero terms of a MacLaurin series to approximate the given numerical values. Then compare the approximation with the value your calculator provides.

14. $\sin(0.1)$, $\sin(0.2)$, $\sin(0.5)$, $\sin(1)$ and $\sin(2)$
15. $\cos(0.1)$, $\cos(0.2)$, $\cos(0.5)$, $\cos(1)$ and $\cos(2)$
16. $\ln(1.1)$, $\ln(1.2)$, $\ln(1.3)$, $\ln(2)$ and $\ln(3)$
17. $\arctan(0.1)$, $\arctan(0.2)$, $\arctan(0.5)$, $\arctan(1)$, $\arctan(2)$

In Problems 18–23, find the first three nonzero terms of a power series for the integral.

18. $\int \cos\left(x^2\right)\,dx$
19. $\int \sin\left(x^2\right)\,dx$
20. $\int \cos\left(x^3\right)\,dx$
21. $\int \sin\left(x^3\right)\,dx$
22. $\int e^{x^2}\,dx$
23. $\int e^{-x^2}\,dx$
24. $\int e^{x^3}\,dx$
25. $\int e^{-x^3}\,dx$
26. $\int \ln(x)\,dx$
27. $\int x\sin(x)\,dx$
28. $\int x\ln(x)\,dx$
29. $\int x^2\sin(x)\,dx$

In Problems 30–37, use a series representation to help compute the limit.

30. $\displaystyle\lim_{x\to 0}\frac{1-\cos(x)}{x}$
31. $\displaystyle\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$
32. $\displaystyle\lim_{x\to 0}\frac{\ln(x)}{x-1}$
33. $\displaystyle\lim_{x\to 0}\frac{1-e^x}{x}$
34. $\displaystyle\lim_{x\to 0}\frac{1+x-e^x}{x^2}$
35. $\displaystyle\lim_{x\to 0}\frac{\sin(x)}{x}$
36. $\displaystyle\lim_{x\to 0}\frac{x-\sin(x)}{x^3}$
37. $\displaystyle\lim_{x\to 0}\frac{x-\frac{1}{6}x^3-\sin(x)}{x^5}$
38. Use MacLaurin series for e^x and e^{-x} to find a series representation for $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$.
39. Use MacLaurin series for e^x and e^{-x} to find a series representation for $\sinh(x)=\dfrac{e^x-e^{-x}}{2}$.
40. Use results from the previous two problems to show that $\mathbf{D}\left(\cosh(x)\right)=\sinh(x)$.
41. Use results from previous problems to show that $\mathbf{D}\left(\sinh(x)\right)=\cosh(x)$.

Euler's Formula: So far we have only discussed series involving real numbers, but sometimes it is useful to replace the variable in a power series with a complex number. Problems 42–44 ask you to make such a substitution and then to obtain and use one of the most famous formulas in mathematics: Euler's formula. Recall that $i=\sqrt{-1}$ is called the **complex unit** and that its powers follow the pattern $i^2=-1$, $i^3=\left(i^2\right)(i)=-i$, $i^4=(i^2)^2=(-1)^2=1$, $i^5=(i^4)(i)=i$, and so on.

42. (a) Substitute $x=i\theta$ into the MacLaurin series for e^x to obtain a series for $e^{i\theta}$.
    (b) Simplify each power of i to rewrite the series for $e^{i\theta}$.

(c) Sort the terms in the simplified series into those terms that do not contain i and those terms that do contain i. Then rewrite the series for $e^{i\theta}$ in the form:

$$e^{i\theta} = [\text{terms that do not contain } i] + i \cdot [\text{terms that do contain } i]$$

(d) You should recognize the sum in each bracket as the MacLaurin series for an elementary function. Rewrite the series for $e^{i\theta}$ as:

$$e^{i\theta} = [\text{function of } \theta] + i \cdot [\text{other function of } \theta]$$

43. In Problem 42 you should have obtained the result:

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$$

Use Euler's formula to compute the values of $e^{i\left(\frac{\pi}{2}\right)}$ and $e^{\pi i}$.

44. Use Euler's formula to show that $e^{\pi i} + 1 = 0$. This is one of the most remarkable and beautiful formulas in mathematics because it connects five of the most fundamental constants: the additive identity 0, the multiplicative identity 1, the complex unit i and the two most commonly used transcendental numbers (π and e) in a simple yet non-obvious way.

The Binomial Theorem

You have probably seen the pattern for expanding $(1+x)^n$ where n is a non-negative integer:

$$\begin{aligned}
(1+x)^0 &= 1 \\
(1+x)^1 &= 1 + x \\
(1+x)^2 &= 1 + 2x + x^2 \\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 \\
(1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5
\end{aligned}$$

using either Pascal's triangle (see margin) or binomial coefficients:

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k! \cdot (n-k)!}$$

for any positive integers n and k with $k \le n$, defining $\binom{n}{0} = 1$. Binomial coefficients allow us to write the expansion of $(1+x)^n$ for non-negative integer powers n in a very compact way:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Notice that each entry in the interior of **Pascal's triangle** is the sum of the two numbers immediately above it.

When n is a positive integer, $(1+x)^n$ expands into a polynomial of degree n, but what happens when n is a negative integer? Or a non-integer? Newton himself investigated this question, leading him to a general pattern that allowed him to quickly write a MacLaurin series expansion for $(1+x)^m$ when m is any real number:

Binomial Series Theorem:

If m is any real number and $|x| < 1$

then $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$ where:

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$$

As before, we define: $\binom{m}{0} = 1$

The remaining problems guide you through an investigation and (the idea behind a) proof of this theorem.

45. Calculate $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$ and $\binom{3}{3}$, then verify that:
    (a) they agree with the entries in the third row of Pascal's triangle.
    (b) they agree with the coefficients in the expansion of $(1+x)^3$.
46. Calculate $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$ and $\binom{4}{4}$, then verify that:
    (a) they agree with the entries in the fourth row of Pascal's triangle.
    (b) they agree with the coefficients in the expansion of $(1+x)^4$.
47. Determine the first five terms of the MacLaurin series for $(1+x)^{\frac{5}{2}}$.
48. Determine the first five terms of the MacLaurin series for $(1+x)^{-\frac{3}{2}}$.
49. Determine the first five terms of the MacLaurin series for $(1+x)^{-\frac{1}{2}}$ and use this result to find the first five non-zero terms in the MacLaurin series for $\arcsin(x)$.
50. Use the first result from the preceding problem to approximate $\sqrt{2}$.
51. Determine the first four terms of the MacLaurin series for $(1+x)^m$. (This is the beginning of a proof of the Binomial Series Theorem.)

10.4 Practice Answers

1. Differentiating the MacLaurin series for $\sin(x)$ yields:

$$\begin{aligned}\cos(x) &= \mathbf{D}\left(\sin(x)\right)\\ &= \mathbf{D}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \cdots\right)\\ &= 1 - \frac{3}{3!}x^2 + \frac{5}{5!}x^4 - \frac{7}{7!}x^6 + \frac{9}{9!}x^8 - \frac{11}{11!}x^{10} + \cdots\\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \cdots\\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}\end{aligned}$$

2. Putting $x = 0.2$ into the MacLaurin series obtained in Practice 1:

$$\cos(0.2) \approx 1 - \frac{1}{2!}(0.2)^2 = 1 - \frac{0.04}{2} = 0.98$$

Because the full MacLaurin series:

$$\cos(0.2) = 1 - \frac{1}{2!}(0.2)^2 + \frac{1}{4!}(0.2)^4 - \frac{1}{6!}(0.2)^6 + \cdots$$

is a convergent alternating series, the "error" when approximating $\cos(0.2)$ by 0.98 is no bigger than the absolute value of the next term in the series, which is:

$$\frac{1}{4!}(0.2)^4 = \frac{0.0016}{24} \approx 0.000067$$

so that $|\cos(0.2) - 0.98| < 0.000067$.

In fact, $\cos(0.2) \approx 0.9800665778$.

3. Starting with the MacLaurin series:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

and using the first six terms with $x = 1$:

$$e = e^1 \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.71666666666$$

Your calculator should report the approximation: $e^1 \approx 2.718281828$

To approximate $\frac{1}{\sqrt{e}}$, substitute $x = -\frac{1}{2}$:

$$\begin{aligned}e^{-\frac{1}{2}} &\approx 1 - \frac{1}{2} + \frac{1}{2!}\left(-\frac{1}{2}\right)^2 + \frac{1}{3!}\left(-\frac{1}{2}\right)^3 + \frac{1}{4!}\left(-\frac{1}{2}\right)^4 + \frac{1}{5!}\left(-\frac{1}{2}\right)^5\\ &\approx 0.6065104167\end{aligned}$$

Your calculator should report the approximation: $e^{-\frac{1}{2}} \approx 0.6065306597$

4. Substitute $u = x^3$ into the MacLaurin series for $\cos(u)$ and multiply the result by x:

$\cos(u) = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 - \frac{1}{6!}u^6 + \frac{1}{8!}u^8 - \cdots$

$$\begin{aligned} x \cdot \cos\left(x^3\right) &= x \cdot \left[1 - \frac{1}{2!}(x^3)^2 + \frac{1}{4!}(x^3)^4 - \frac{1}{6!}(x^3)^6 + \frac{1}{8!}(x^3)^8 - \cdots\right] \\ &= x - \frac{1}{2!}x^7 + \frac{1}{4!}x^{13} - \frac{1}{6!}x^{19} + \frac{1}{8!}x^{25} - \cdots \end{aligned}$$

then integrate term by term:

$$\int x \cdot \cos\left(x^3\right)\, dx = C + \frac{1}{2}x^2 - \frac{1}{8 \cdot 2!}x^8 + \frac{1}{14 \cdot 4!}x^{14} - \frac{1}{20 \cdot 6!}x^{20} + \cdots$$

and use the first two nonzero terms of this antiderivative to estimate the value of the definite integral:

$$\int_0^{\frac{1}{2}} x \cdot \cos\left(x^3\right)\, dx \approx \left[\frac{1}{2}x^2 - \frac{1}{8 \cdot 2!}x^8\right]_0^{\frac{1}{2}} = \frac{1}{8} - \frac{1}{8 \cdot 2 \cdot 2^8} = \frac{511}{4096}$$

or about 0.124755859375. Because the series for the exact value of the integral is an alternating series, the "error" is no bigger than:

$$\frac{1}{14 \cdot 4!}\left(\frac{1}{2}\right)^{14} \approx 0.000000182$$

5. Multiply the two MacLaurin series:

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\ \sin(x) &= x - \frac{1}{6}x^3 + \frac{120^5}{x} + \cdots \end{aligned}$$

to get a MacLaurin series for $e^x \cdot \sin(x)$:

$$\begin{aligned} &\left[1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right] \cdot \left[x - \frac{1}{6}x^3 + \frac{120^5}{x} + \cdots\right] \\ &= 1 \cdot \left[x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right] + x \cdot \left[x - \frac{1}{6}x^3 + \frac{120^5}{x} + \cdots\right] \\ &\quad + \frac{1}{2}x^2 \cdot \left[x - \frac{1}{6}x^3 + \frac{120^5}{x} + \cdots\right] \\ &\quad + \frac{1}{6}x^3 \cdot \left[x - \frac{1}{6}x^3 + \frac{120^5}{x} + \cdots\right] \cdots \\ &= \left[x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right] + \left[x^2 - \frac{1}{6}x^4 + \cdots\right] \\ &\quad + \left[\frac{1}{2}x^3 - \frac{1}{12}x^5 + \cdots\right] + \left[\frac{1}{6}x^4 + \cdots\right] + \cdots \\ &= x + x^2 + \frac{1}{3}x^3 + 0x^4 - \frac{3}{40}x^5 + \cdots \end{aligned}$$

so the sum of the first three nonzero terms is $x + x^2 + \frac{1}{3}x^3$.