10.5 Approximation Using Taylor Polynomials

If a function has a power series representation, we now have formula to determine the coefficients of that power series. Using techniques from Chapter 9, we can then find the interval of convergence for the power series. And *if* evaluating a power series at a point results in an alternating numerical series, we can even use the Estimation Bound for Alternating Series to get a bound on the "error" between a partial sum approximation and the exact value of the series:

$$|(\text{exact value}) - (\text{approximation})| < |\text{next term in the series}|$$

If evaluating the power series at a point does not result in an alternating numerical series, we do not yet have a bound on the size of the error of the approximation.

In this section we obtain a bound on the error when approximating a Taylor series for any $f(x)$ with a corresponding Taylor polynomial:

$$\text{"error"} = |(\text{exact value of } f(x)) - (\text{approximation of } f(x))|$$

The bound we get is valid even if the Taylor series is not an alternating series, and the pattern for the error bound looks very much like the next term in the series (the first unused term in the partial sum of the Taylor series). In computer and calculator applications, this error bound can help us work efficiently by allowing us to use only the number of terms we really need.

As a very important bonus, this error bound will allow us to show that a function is equal to its Taylor series on its interval of convergence (for most "well-behaved" functions).

Taylor Polynomials

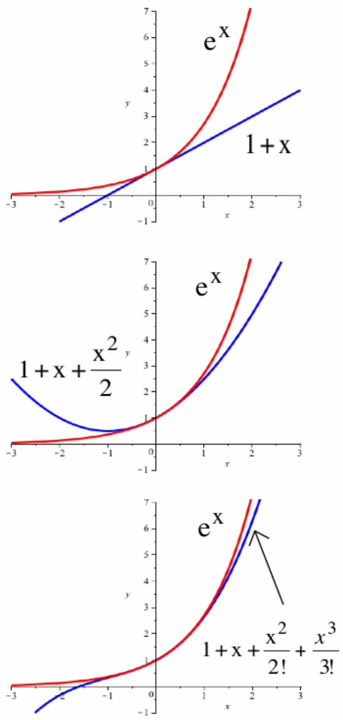
For a function $f(x)$, the n-th degree **Taylor Polynomial** (centered at $x = c$) is:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)\cdot(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \end{aligned}$$

Example 1. Write the first four Taylor Polynomials, $P_0(x)$ through $P_3(x)$, centered at $x = 0$ for e^x, then graph them for $-1 < x < 1$.

Solution. With $c = 0$, a Taylor polynomial is a MacLaurin polynomial and the MacLaurin series for e^x is:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$



so $P_0(x) = 1$, $P_1(x) = 1 + x$, $P_2(x) = 1 + x + \frac{1}{2!}x^2$ and:

$$P_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

Graphs of e^x, $P_1(x)$, $P_2(x)$ and $P_3(x)$ appear in the margin. ◀

Practice 1. Write the Taylor polynomials $P_0(x)$, $P_2(x)$ and $P_4(x)$ centered at $x = 0$ for $\cos(x)$, then graph them for $-\pi < x < \pi$. (What are the Taylor polynomials $P_1(x)$ and $P_3(x)$?)

When we center a Taylor polynomial at $x = c \neq 0$, the Taylor polynomial approximates the function well for values of x near c.

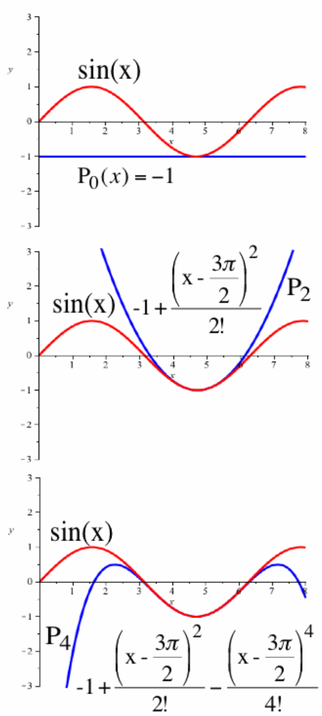
Example 2. Write the Taylor polynomials $P_0(x)$, $P_2(x)$ and $P_4(x)$ centered at $x = \frac{3\pi}{2}$ for $\sin(x)$, then graph them for $0 < x < 8$.

Solution. The Taylor series centered at $x = \frac{3\pi}{2}$ for $\sin(x)$ is:

$$\sin(x) = -1 + \frac{1}{2!}\left(x - \frac{3\pi}{2}\right)^2 - \frac{1}{4!}\left(x - \frac{3\pi}{2}\right)^4 + \frac{1}{6!}\left(x - \frac{3\pi}{2}\right)^6 - \cdots$$

so $P_0(x) = -1$, $P_2(x) = -1 + \frac{1}{2}\left(x - \frac{3\pi}{2}\right)^2$ and:

$$P_4(x) = -1 + \frac{1}{2}\left(x - \frac{3\pi}{2}\right)^2 - \frac{1}{24}\left(x - \frac{3\pi}{2}\right)^4$$



Graphs of $\sin(x)$, $P_0(x)$, $P_2(x)$ and $P_4(x)$ appear in the margin. ◀

Practice 2. Write the Taylor polynomials $P_0(x)$, $P_1(x)$ and $P_3(x)$ centered at $x = \frac{\pi}{2}$ for $\cos(x)$, then graph them for $-1 < x < 4$.

Looking at the graphs from the preceding Examples, you should notice that how well a function can be approximated by its Taylor polynomial appears to depend on:

- The number of terms, n, in the Taylor polynomial.
- How close the point of approximation, x, is to the the center of the Taylor series, c.

The Remainder

Approximation of functions using Taylor polynomials is a useful tool, but in many applied situations you want to know how good the approximation is, or how many terms of a series you require to obtain a needed level of accuracy. If two terms of a series provide the desired accuracy for your application, it is a waste of time, resources and money to use 100 terms. On the other hand, sometimes even 100 terms may not provide the accuracy you need. Fortunately, it is possible to obtain a guarantee on how close a particular Taylor polynomial approximates an exact value. Then you can work efficiently and use the smallest possible number of terms in a Taylor polynomial. The next theorem gives a pattern for the size of the "error" in a Taylor polynomial approximation.

Taylor's Formula with Remainder

If f has $n+1$ derivatives in an interval I containing c, and x is in I,

then there is a number z, strictly between c and x, so that $f(x) = P_n(x) + R_n(x)$ where:

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} \cdot (x-c)^{n+1}$$

This result says that $f(x)$ is equal to its n-th degree Taylor polynomial plus a **remainder**, and the remainder $R_n(x)$ has the form given in the theorem. Notice that the formula for $R_n(x)$ looks very much like the pattern for the $(n+1)$-st term of the Taylor series for $f(x)$, except that it involves $f^{(n+1)}(z)$ instead of $f^{(n+1)}(c)$.

This particular formula for $R_n(x)$ is called the **Lagrange form** of the remainder, named for the French-Italian mathematician and astronomer Joseph-Louis Lagrange (1736–1813).

The main idea of the proof of the Taylor's Formula with Remainder is straightforward, but the details are somewhat technical, so we will set aside the proof for the moment.

Notice that the formula for the remainder, $\frac{f^{(n+1)}(z)}{(n+1)!} \cdot (x-c)^{n+1}$, involves three pieces: $(n+1)!$, $(x-c)^{n+1}$ and $f^{(n+1)}(z)$ for some z (strictly) between x and c. Notice also that:

- When you make the number of terms, n, in the Taylor polynomial bigger, the $(n+1)!$ in the denominator of the formula becomes bigger, making the remainder smaller.

Usually. Both $f^{(n+1)}(z)$ and $(x-c)^{n+1}$ also depend on n and affect the size of the remainder.

- When you make the point of approximation, x, closer to the the center of the Taylor series, c, the factor $(x-c)^{n+1}$ becomes smaller, making the remainder smaller.
- With $n = 0$, Taylor's Formula becomes the Mean Value Theorem:

$$f(x) = f(c) + f'(z) \cdot (x-c) \Rightarrow f'(z) = \frac{f(x) - f(c)}{x-c}$$

This last fact allows us to think of Taylor's Formula with Remainder as a generalization of the Mean Value Theorem.

We used Rolle's Theorem to prove the Mean Value Theorem, so we will rely on repeated applications of Rolle's Theorem to prove Taylor's Formula.

Applying the Taylor Remainder Formula

In practice, you will typically use Taylor's Formula with Remainder in one of two ways:

- You know the Taylor polynomial $P_n(x)$ for $f(x)$, so you know x, c and n, allowing you to evaluate $(n+1)!$ and $(x-c)^{n+1}$ exactly. That leaves the piece $f^{(n+1)}(z)$ for some z between x and c. If you can find a bound for the value of $\left|f^{(n+1)}(z)\right|$ for all z between x and c, then you can use this bound with the known values of $(n+1)!$ and $(x-c)^{n+1}$ to obtain a *bound* for the remainder term $R_n(x)$.
- Someone tells you the amount of acceptable "error," so you know the values of x, c and $R_n(x)$. You then need to find a value of n that guarantees the required accuracy.

Corollary: A Bound for the Remainder $R_n(x)$

If f has $n+1$ derivatives in an interval I containing c,
x is in I, and $\left|f^{(n+1)}(z)\right| \le M$ for all z between x and c,

then "error" $= |f(x) - P_n(x)| = |R_n(x)| \le \dfrac{M}{(n+1)!} \cdot |x-c|^{n+1}$

Example 3. You need to approximate the values of e^x using the MacLaurin polynomial $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$. Find a bound for the "error" of the approximation, $R_3(x)$, if x is in the interval:

(a) $[-1,1]$ (b) $[-3,2]$ (c) $[-0.2,0.3]$

Solution. You know that $f(x) = e^x$, $c = 0$ (corresponding to a Maclaurin series), $n = 3 \Rightarrow (n+1)! = 4! = 24$ and $f^{(n+1)}(x) = f^{(4)}(x) = e^x$.

(a) For x in the interval $[-1,1]$:

$$\left|(x-c)^{n+1}\right| = \left|x^4\right| \le |1|^4 = 1 \quad \text{and} \quad \left|f^{(n+1)}(x)\right| = |e^x| \le e^1$$

because e^x is increasing on $[-1,1]$. A "crude" but "easy to use" bound for e^1 is $e^1 < 3^1 = 3 = M$. Then:

$$|R_3(x)| < \frac{M}{(n+1)!} \cdot |x-c|^{n+1} < \frac{3}{24} \cdot 1 = 0.125$$

When $-1 < x < 1$, $P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ is within 0.125 of e^x.

For a more precise approximation of e, you can use:

$$e^1 < 2.72^1 = 2.72$$

resulting in the bound:

$$|R_3(x)| < \frac{2.72}{24} = \frac{17}{150} \approx 0.1133$$

(b) For x in the interval $[-3,2]$:

$$\left|(x-c)^{n+1}\right| = \left|x^4\right| \le \left|(-3)^4\right| = 81 \quad \text{and} \quad \left|f^{(n+1)}(x)\right| = |e^x| \le e^2$$

because e^x is increasing on $[-3, 2]$. A "crude" but "easy to use" bound for e^2 is $e^2 < 3^2 = 9 = M$. Then:

$$|R_3(x)| < \frac{M}{(n+1)!} \cdot |x-c|^{n+1} < \frac{9}{24} \cdot 81 = 30.375$$

Obviously you cannot have much confidence when using $P_3(x)$ to approximate e^x on the interval $[-3, 2]$.

For a more precise bound, you can use:

$$e^2 < 2.72^2 < 7.4$$

resulting in:

$$|R_3(x)| < \frac{(2.72)^2}{24} \cdot 81 < 24.9696$$

(c) For x in the interval $[-0.2, 0.3]$:

$$\left|(x-c)^{n+1}\right| = \left|x^4\right| \le \left|(0.3)^4\right| = 0.0081 \text{ and } \left|f^{(n+1)}(x)\right| = |e^x| \le e^{0.3}$$

because e^x is increasing on $[-0.2, 0.3]$. A good bound for $e^{0.3}$ is $e^{0.3} < 2.72^{0.3} < 1.4 = M$. Then:

$$|R_3(x)| < \frac{M}{(n+1)!} \cdot |x-c|^{n+1} < \frac{1.4}{24} \cdot 0.0081 = 0.0004725$$

When $-0.2 < x < 0.3$, $P_3(x)$ is within 0.0004725 of e^x.

When the interval is small, you can be confident that $P_3(x)$ provides a good approximation of e^x, but as the interval grows, so does your bound on the remainder. ◀

To guarantee a good approximation on a larger interval, you typically need $(n+1)!$ to be larger, so you need to use a higher-degree Taylor Polynomial $P_n(x)$.

Practice 3. Find a value of n to guarantee that $P_n(x)$ is within 0.001 of e^x for x in the interval $[-3, 2]$.

Example 4. You need to approximate the values of $f(x) = \sin(x)$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with an error less that 10^{-10}. How many terms of the MacLaurin series for $\sin(x)$ do you need?

Solution. For every value of n, $\left|f^{(n+1)}(x)\right|$ is either equal to $|\sin(x)|$ or $|\cos(x)|$, so $M = 1$ works as a bound for $\left|f^{(n+1)}(z)\right|$. Hence:

$$\text{"error"} = |R_n(x)| < \frac{1}{(n+1)!} \cdot |x-0|^{n+1} \le \frac{\left(\frac{\pi}{2}\right)^{n+1}}{(n+1)!}$$

so we need to find a value of n such that:

$$\frac{\left(\frac{\pi}{2}\right)^{n+1}}{(n+1)!} < 10^{-10}$$

holds. A bit of numerical experimentation on a calculator (see margin) shows that $n+1 = 16$ works, so we can take $n = 15$ and use:

$$\frac{\left(\frac{\pi}{2}\right)^{14}}{14!} \approx 6.39 \times 10^{-9}$$

$$\frac{\left(\frac{\pi}{2}\right)^{15}}{15!} \approx 6.69 \times 10^{-10}$$

$$\frac{\left(\frac{\pi}{2}\right)^{16}}{16!} \approx 6.57 \times 10^{-11}$$

$$P_{15}(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \frac{1}{13!}x^{13} - \frac{1}{15!}x^{15}$$

to approximate $\sin(x)$. If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $|P_{15}(x) - \sin(x)| < 10^{-10}$ will hold. ◀

Practice 4. How many terms of the MacLaurin series for e^x do you need in order to approximate e^x to within 10^{-10} for $0 \leq x \leq 1$?

Proving Taylor's Formula

As observed previously, Taylor's Formula generalizes the Mean Value Theorem, so we will apply Rolle's Theorem repeatedly to prove it. Given values of x, n and c, we need to find a z strictly between x and c so that:

$$f(x) - P_n(x) = R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} \cdot (x-c)^{n+1}$$

Proof. Given fixed values of x, n and c, define α so that:

$$\alpha = \frac{R_n(x)}{(x-c)^{n+1}} \quad \Rightarrow \quad R_n(x) = \alpha \cdot (x-c)^{n+1}$$

We now need to show that $\alpha = \dfrac{f^{(n+1)}(z)}{(n+1)!}$ for an appropriate value of z. Next, define a new function $g(t)$ as:

$$g(t) = f(t) - P_n(t) - \alpha \cdot (t-c)^{n+1}$$

Because $f(c) = P_n(c)$, we know that:

$$g(c) = f(c) - P_n(c) - (c-c)^{n+1} = 0$$

Recall that we have constructed $P_n(x)$ so the values of it (and its first n derivatives) agree with the values of $f(x)$ (and its first n derivatives) at $x = c$.

and because $f^{(k)}(c) = P_n^{(k)}(c)$ for $1 \leq k \leq n$, we also know that:

$$g^{(k)}(c) = f^{(k)}(c) - P_n^{(k)}(c) - (n+1)(n)\cdots(n-k+2)(0)^{n-k+1} = 0$$

for $1 \leq k \leq n$. Furthermore, we know that:

$$g(x) = f(x) - P_n(x) - \alpha \cdot (x-c)^{n+1} = f(x) - P_n(x) - R_n(x) = 0$$

Refer to Section 3.2 to review the statement of Rolle's Theorem.

Now apply Rolle's Theorem to $g(t)$: we know that $g(c) = 0$ and $g(x) = 0$, so there is a number z_1 between c and x such that $g'(z_1) = 0$.

Next apply Rolle's Theorem to $g'(t)$: we know that $g'(c) = 0$ and $g'(z_1) = 0$, so there is a z_2 between c and z_1 with $g''(z_2) = 0$.

Keeping track of these z_k's, we know that: $c < z_{n+1} < z_n < \cdots < z_3 < z_2 < z_1 < x$

Continue applying Rolle's Theorem to $g''(t)$, $g'''(t)$, $\ldots$, $g^{(n)}(t)$ to get similar numbers z_3, z_4, $\ldots$, z_{n+1}. Let $z = z_{n+1}$. We know that z is strictly between c and x, and that $g^{(n+1)}(z) = 0$. But:

$$g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - (n+1)! \cdot \alpha$$

so using the fact that $g^{(n+1)}(z) = 0$ we have:

$$f^{(n+1)}(z) = (n+1)! \cdot \alpha \;\Rightarrow\; \alpha = \frac{f^{(n+1)}(z)}{(n+1)!}$$

as required. □

Showing a Taylor Series Converges to a Function

We know that if $f(x) = e^x$ has a MacLaurin series, the only possibility for that MacLaurin series is $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$, and we know that this series converges for all values of x. But does this power series converge to e^x? And if so, where? The Remainder Bound Corollary can help us answer these questions.

Example 5. Show that $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ converges to e^x for all values of x.

Solution. For any fixed value of x, we need to show that:

$$\sum_{k=0}^{\infty} \frac{1}{k!}x^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}x^k = e^x$$

With $f(x) = e^x$ and $P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}x^k$, this is equivalent to showing:

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \left[f(x) - P_n(x) \right] = 0$$

For $f(x) = e^x$, we know that $f^{(n+1)}(x) = e^x$ for any n. If $x > 0$, then $f^{(n+1)}(z) = e^z < e^x$ for any z between x and 0; and if $x < 0$, then $f^{(n+1)}(z) = e^z < e^0 = 1$ for any z between x and 0. Define M to be the larger of e^x and 1, so that $\left| f^{(n+1)}(z) \right| \leq M$ for any such z. Then the Remainder Bound Corollary guarantees that:

$$|R_n(x)| \leq \frac{M}{(n+1)!} \cdot |x-0|^{n+1} = M \cdot \frac{|x|^{n+1}}{(n+1)!}$$

Because we already know the series $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ converges, we know (by the Corollary to the Test for Divergence) that $\lim_{k\to\infty} \frac{x^k}{k!} = 0$. Hence:

$$\lim_{n\to\infty} |R_n(x)| \leq M \cdot \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

We now know that $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ converges to e^x for any value of x. ◀

Practice 5. Show that $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$ converges to $\sin(x)$ for all x.

Using the Remainder Bound Corollary, you can show that most functions for which you can compute infinitely many derivatives at $x = c$ are equal to their Taylor series centered at $x = c$ everywhere that the series converges. Problems 27–28 provide an example of a function $f(x)$ with a MacLaurin series that converges everywhere, but which converges to $f(x)$ only at $x = 0$.

10.5 Problems

In Problems 1–10, calculate the Taylor polynomials P_0, P_1, P_2, P_3 and P_4 for the given function centered at the given value of c. Then graph the function and the Taylor polynomials on the given interval.

1. $f(x) = \sin(x)$, $c = 0$, $[-2, 4]$
2. $f(x) = \cos(x)$, $c = 0$, $[-3, 3]$
3. $f(x) = \ln(x)$, $c = 1$, $[0.1, 3]$
4. $f(x) = \arctan(x)$, $c = 0$, $[-3, 3]$
5. $f(x) = x$, $c = 1$, $[0, 3]$
6. $f(x) = x$, $c = 9$, $[0, 20]$
7. $f(x) = (1 + x)^{-\frac{1}{2}}$, $c = 0$, $[-2, 3]$
8. $f(x) = e^{2x}$, $c = 0$, $[-2, 4]$
9. $f(x) = \sin(x)$, $c = \dfrac{\pi}{2}$, $[-1, 5]$
10. $f(x) = \sin(x)$, $c = \pi$, $[-1, 5]$

In Problems 11–18, use the given function $f(x)$ and the given value of n to determine a formula for $R_n(x)$ and find a bound for $|R_n(x)|$ on the given interval. This bound for $|R_n(x)|$ is our "guaranteed accuracy" for $P_n(x)$ to approximate $f(x)$ on the given interval. Use $c = 0$ (so each $P_n(x)$ will be a MacLaurin polynomial).

11. $f(x) = \sin(x)$, $n = 5$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
12. $f(x) = \sin(x)$, $n = 9$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
13. $f(x) = \sin(x)$, $n = 5$, $[-\pi, \pi]$
14. $f(x) = \sin(x)$, $n = 9$, $[-\pi, \pi]$
15. $f(x) = \cos(x)$, $n = 10$, $[-1, 2]$
16. $f(x) = \cos(x)$, $n = 10$, $[-1, 5]$
17. $f(x) = e^x$, $n = 6$, $[-1, 2]$
18. $f(x) = e^x$, $n = 10$, $[-1, 3]$

In Problems 19–24, determine the number of terms of the Taylor series for $f(x)$ you need to use in order to approximate $f(x)$ to within the specified error on the given interval. (For each function, use $c = 0$.)

19. $f(x) = \sin(x)$ within 0.001 on $[-1, 1]$
20. $f(x) = \sin(x)$ within 0.001 on $[-3, 3]$
21. $f(x) = \sin(x)$ within 0.00001 on $[-1.6, 1.6]$
22. $f(x) = \cos(x)$ within 0.001 on $[-2, 2]$
23. $f(x) = e^x$ within 0.001 on $[0, 2]$
24. $f(x) = e^x$ within 0.001 on $[-1, 4]$
25. Show that the MacLaurin series for $\cos(x)$ converges to $\cos(x)$ for all values of x.
26. Show that the Taylor series for e^x centered at $x = 3$ converges to e^x for all values of x.
27. Define the function $f(x)$ as:
$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
    (a) Show that: $f'(0) = \lim_{h \to 0} \dfrac{e^{-h^{-2}}}{h}$
    (b) Use the change of variable $y = \dfrac{1}{h}$ along with L'Hôpital's Rule to show that $f'(0) = 0$.
28. Define $f(x)$ as in Problem 27.
    (a) Show that $f^{(k)}(0) = 0$ for all $k \geq 0$.
    (b) If $P_n(x)$ is a MacLaurin polynomial for $f(x)$, show that $P_n(x) = 0$ for all n and all x.
    (c) On what interval does the MacLaurin series for $f(x)$ converge?
    (d) On what interval is the MacLaurin series for $f(x)$ equal to $f(x)$?

Series Approximations of π

The following problems illustrate some of the ways series have been used to obtain very precise approximations of π. Several of these methods use the MacLaurin series for $\arctan(x)$:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

which converges rapidly if $|x|$ is close to 0.

Method I: $\tan\left(\frac{\pi}{4}\right) = 1$, so:

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$
$$\Rightarrow \quad \pi = 4\arctan(1) = \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right]$$

29. (a) Approximate π as $4\arctan(1) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9}$ and compare this result with the value your calculator gives for π.

(b) The series for $\arctan(1)$ is an alternating series, so we have an "easy" error bound. Use the error bound for an alternating series to find a bound for the error if you were to use 50 terms of the series for $\arctan(1)$ (instead of five).

(c) Using the error bound for an alternating series, how many terms of the $4\arctan(1)$ series do you need in order to guarantee that the series approximation of π is within 0.0001 of the exact value of π? (The $4\arctan(1)$ series converges so slowly that it is not used to approximate π.)

Method II: $\tan(\alpha + \beta) = \dfrac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$, so:

$$\tan\left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = 1$$
$$\Rightarrow \frac{\pi}{4} = \arctan(1) = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

Because the series for $\arctan\left(\frac{1}{2}\right)$ and $\arctan\left(\frac{1}{3}\right)$ converge much more rapidly than the series for $\arctan(1)$, this approximation method leads to a more efficient estimate of π.

30. (a) Approximate π as using the first four terms of the series for $\arctan\left(\frac{1}{2}\right)$ and $\arctan\left(\frac{1}{3}\right)$ and compare this result with the value your calculator gives for π.

(b) The series for $\arctan\left(\frac{1}{2}\right)$ and $\arctan\left(\frac{1}{3}\right)$ are each alternating series. Use the error bound for an alternating series to find a bound for the error if you use 10 terms of each series.

(c) How many terms of each series do you need in order to guarantee that the series approximation of π is within 0.0001 of the exact value of π?

Method III: Putting $\beta = \alpha$ in the angle addition formula for $\tan(x)$ used in Method II and letting $\tan(\alpha) = \frac{1}{5}$ yields:

$$\begin{aligned} \tan(2\alpha) &= \frac{2\tan(\alpha)}{1-\tan^2(\alpha)} = \frac{\frac{2}{5}}{\frac{24}{25}} = \frac{5}{12} \\ \Rightarrow \quad \tan(4\alpha) &= \frac{2\tan(2\alpha)}{1-\tan^2(2\alpha)} = \frac{\frac{5}{6}}{\frac{119}{144}} = \frac{120}{119} \\ \Rightarrow \quad \tan\left(4\alpha - \frac{\pi}{4}\right) &= \frac{\tan(4\alpha)-1}{1+\tan(4\alpha)\cdot 1} = \frac{\frac{1}{119}}{1+\frac{120}{119}} = \frac{1}{239} \\ \Rightarrow \quad 4\alpha - \frac{\pi}{4} &= \arctan\left(\frac{1}{239}\right) \\ \Rightarrow \frac{\pi}{4} &= 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) \end{aligned}$$

Mathematician and astronomer John Machin (1686–1751) first obtained this result around 1706. He used it to approximate π to 100 decimal places.

31. (a) Approximate π using the first three terms of the series for $\arctan\left(\frac{1}{5}\right)$ and $\arctan\left(\frac{1}{239}\right)$ and compare this result with the value your calculator gives for π.
    (b) Explain why Method III yields a series that converges more rapidly (requiring fewer terms for a "good" approximation of π) than Methods I and II.

Method IV: Carl Friedrich Gauss (1777–1855) worked out many such formulas involving arctan, including this one with three terms:

By 1958, the advent of computers allowed mathematicians working a century after Gauss' death to approximate π accurate to more than 10,000 decimal places.

$$\frac{\pi}{4} = 12\arctan\left(\frac{1}{18}\right) + 8\arctan\left(\frac{1}{57}\right) - 5\arctan\left(\frac{1}{239}\right)$$

32. (a) Approximate π using the first three terms of of the series for $\arctan\left(\frac{1}{18}\right)$, $\arctan\left(\frac{1}{57}\right)$ and $\arctan\left(\frac{1}{239}\right)$ and compare this result with the value your calculator gives for π.
    (b) Explain why Method IV yields a series that converges more rapidly (requiring fewer terms for a "good" approximation of π) than Methods I, II and III.

Calculator Notes

Imagine that you are in charge of designing or selecting an algorithm for a calculator to employ when its user pushes the **sin** button. You know that if the value of θ is relatively close to 0, then using a "few" terms of the MacLaurin series for $\sin(x)$ will approximate the value of $\sin(\theta)$ accurate to 10 digits (the size of the display of the calculator). If $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then:

$$\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!}$$

will give the value of $\sin(\theta)$ with an "error" less than:

$$\frac{1}{13!} \cdot \left(\frac{\pi}{2}\right)^1 3 < \frac{(0.76)^{13}}{13!} < 5 \times 10^{-12}$$

You could rewrite the polynomial above as:

$$\theta\left(1 - \frac{\theta^2}{2\cdot 3}\left(1 - \frac{\theta^2}{4\cdot 5}\left(1 - \frac{\theta^2}{6\cdot 7}\left(1 - \frac{\theta^2}{8\cdot 9}\left(1 - \frac{\theta^2}{10\cdot 11}\right)\right)\right)\right)\right)$$

This new pattern may look more complicated, but it uses fewer multiplications and avoids very large values such as 11! and θ^{11}.

If $|\theta| > 1$, θ^{11} will be very large; if $|\theta| < 1$, it will be very small.

This algorithm should work well for values of θ near 0, but you also want your algorithm to provide the same accuracy when θ is larger, say 10 or 101.7. Rather than computing many more terms of the Maclaurin series for $\sin(x)$, some algorithms simply shift the problem closer to 0. First, you can use the fact that $\sin(x) = \sin(x - 2\pi)$ to keep shifting the problem until the argument resides in the interval $[0, 2\pi]$:

$$\begin{aligned} \sin(10) &= \sin(10 - 2\pi) \approx \sin(3.71681469) \\ \sin(101.7) &= \sin(101.7 - 2\pi) = \sin(101.7 - 4\pi) = \cdots \\ &= \sin(101.7 - 32\pi) \approx \sin(1.169035085) \end{aligned}$$

Once the argument is between 0 and 2π, you can use additional trigonometric facts. If the $\theta > \pi$, use $\sin(x) = -\sin(x - \pi)$ to replace θ with $\theta - \pi$ (and keep track of the change in sign of the answer). Finally, you can shift the problem into the interval $\left[0, \frac{\pi}{2}\right]$: if the new value of θ is larger than $\frac{\pi}{2}$, use $\sin(x) = \sin(\pi - x)$ to replace θ with $\pi - \theta$.

Calculators encounter other major problems, however, when evaluating the sine or exponential function of a very large number. Because calculators only store the leading finite number of digits of a number (usually 10 or 12 digits), the calculator cannot distinguish between large numbers that differ only past that leading number of stored digits: one particular calculator correctly says that $\left(10^{12} + 1\right) - 10^{12} = 1$, but it incorrectly reports that $\left(10^{13} + 1\right) - 10^{13} = 0$. Because it calculates that "$10^{13} + 1 = 10^{13}$," it also would falsely report the same values for $\sin\left(10^{13} + 1\right)$ and $\sin\left(10^{13}\right)$.

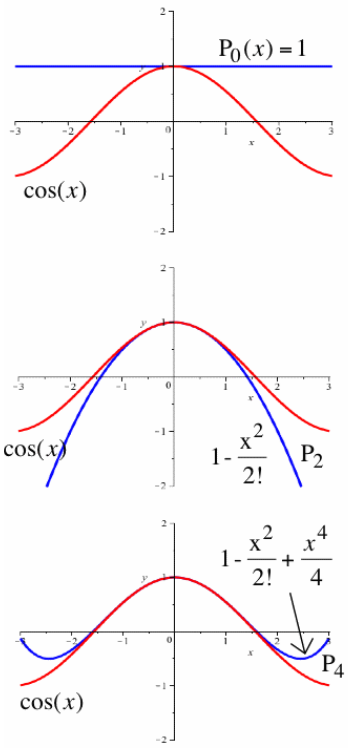
In fact, the people who programmed this particular type of calculator recognized that problem, so the calculator produces an error message if it is asked to calculate $\sin\left(10^{11}\right)$. This calculator reports that $e^{230} \approx 7.7 \times 10^{99}$ but yields an error message when asked to compute e^{231} because the largest number it can display is 9.9×10^{99} and e^{231} exceeds that value. What happens on your calculator?

10.5 Practice Answers

1. The MacLaurin series for $\cos(x)$ is:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots = \sum_{k-0}^{\infty} \frac{(-1)^K}{(2k)!} x^{2k}$$

so $P_0(x) = 1$, $P_2(x) = 1 - \dfrac{x^2}{2}$ and $P_4(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$.



See margin for graphs. Because all odd-indexed coefficients are 0:

$$P_1(x) = P_0(x) = 1$$

$$P_3(x) = P_2(x) = 1 - \frac{x^2}{2}$$

$$P_5(x) = P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

2. With $f(x) = \cos(x)$ we know that $f\left(\frac{3\pi}{2}\right) = 0$ and:

$$f'(x) = -\sin(x) \Rightarrow f'\left(\frac{3\pi}{2}\right) = 1$$

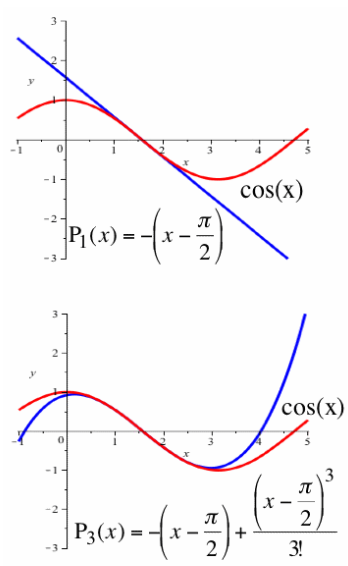
$$f''(x) = -\cos(x) \Rightarrow f''\left(\frac{3\pi}{2}\right) = 0$$

$$f'''(x) = \sin(x) \Rightarrow f'''\left(\frac{3\pi}{2}\right) = -1$$

so that $P_0(x) = 0$, $P_1(x) = \left(x - \frac{3\pi}{2}\right)$ and:

$$P_3(x) = \left(x - \frac{3\pi}{2}\right) - \frac{1}{6}\left(x - \frac{3\pi}{2}\right)^3$$

See lower margin figure for graphs.



3. Using the result of Example 3(b), we know that:

$$|R_n(x)| < \frac{9}{(n+1)!} \cdot 3^{n+1} = \frac{3^{n+3}}{(n+1)!}$$

so we need:

$$\frac{3^{n+3}}{(n+1)!} < 0.001 \Rightarrow \frac{(n+1)!}{3^{n+3}} > 1000$$

Experimenting with a calculator reveals that $n = 13$ works.

4. Because $0 \le x \le 1$ and $e^z < e^1 = e < 2.72$ for $0 < z < 1$:

$$|R_n(x)| = \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!} |x - 0|^{n+1} < \frac{2.72}{(n+1)!}$$

hence we need:

$$\frac{2.72}{(n+1)!} < 10^{-10} \Rightarrow (n+1)! > 2.72 \times 10^{10} \Rightarrow n \ge 13$$

5. Any derivative of $f(x) = \sin(x)$ equals $\pm\sin(x)$ or $\pm\cos(x)$, so $\left|f^{(n+1)}(z)\right| \le 1$ for any z. Hence:

$$|R_n(x)| = \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!} |x - 0|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!}$$

As noted in the solution to Example 5, this expression approaches 0 as $n \to \infty$ (no matter the value of x).