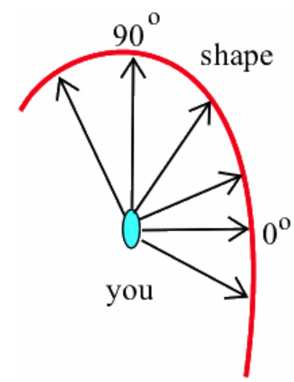
11

Polar and Parametric Curves

The rectangular coordinate system, while immensely useful, is not the only way to assign an address to a point in the plane — and sometimes it is not the most useful way to describe the location of a point or the shape of curve. This chapter examines two additional ways to plot points and describe curves in a plane: polar coordinates and parametric coordinates. We then extend calculus techniques you have already learned to compute arclengths, areas and rates of change for curves, regions and functions described using these new coordinate systems.



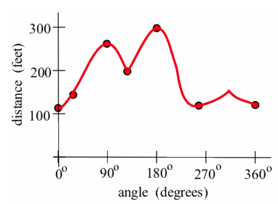
Taking Measurements

11.1 Polar Coordinates

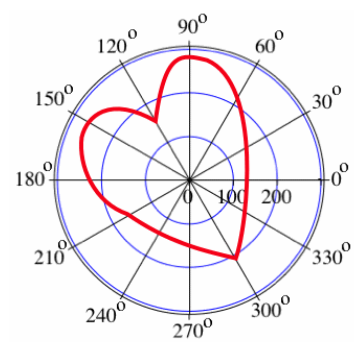
In many experimental situations, your location is fixed and you — or your instruments, such as radar — take readings in different directions (see margin). You can record this information in a table (below left) and graph it using rectangular coordinates with the angle on the horizontal axis and the measurement on the vertical axis (below right):

angle	distance
0°	120 ft
30°	140
60°	230
90°	270
120°	200

Numerical Data



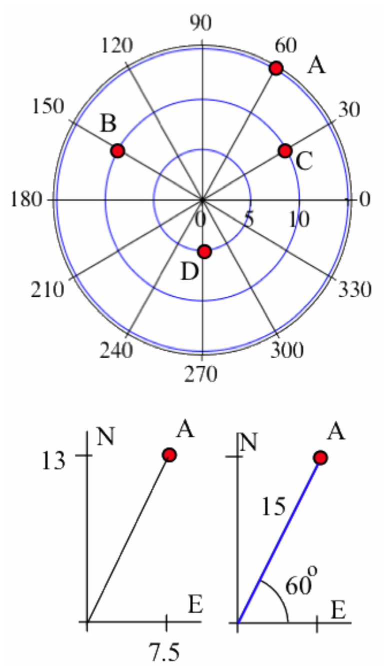
Rectangular Coordinate Graph of Data



Polar coordinate graph of data

Sometimes, however, you will find it more useful to plot the information in a manner similar to the way in which it was collected: as magnitudes along radial lines (see margin) using the **polar coordinate system**.

In this section we introduce polar coordinates and examine some of their uses. We graph points and functions in polar coordinates, consider how to change back and forth between the rectangular and polar coordinate systems, investigate slopes of lines tangent to polar graphs, and tackle some of the many applications in which polar coordinates arise: they provide a "natural" and easy way to represent certain types of information.



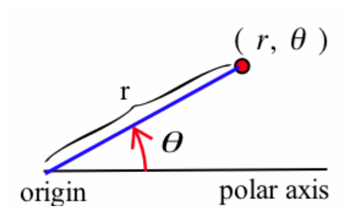
Example 1. SOS! You've just received a distress signal from a ship located at position A on your radar screen (see margin). Describe its location to your captain so your vessel can speed to the rescue.

Solution. You could convert the relative location of the other ship to rectangular coordinates and then tell your captain to sail due east for 7.5 miles and north for 13 miles, but that certainly is not the quickest way to reach the other ship. It would be better to tell the captain to sail for 15 miles in the direction of $60°$. If the distressed ship was at position B on the radar screen, your vessel should sail for 10 miles in the direction $150°$. ◀

Actual radar screens have $0°$ at the top of the screen, but the convention in mathematics is to put $0°$ in the direction of the positive x-axis and to measure positive angles counterclockwise from there. (And a real sailor uses the terms "bearing" and "range" instead of "direction" and "magnitude.")

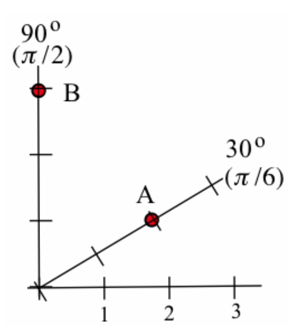
Practice 1. Describe the locations of the ships at positions C and D in the top margin figure by determining a distance and a direction to those ships from your current position at the center of the radar screen.

Points in Polar Coordinates



To construct a polar coordinate system we need a starting point (called the **origin** or **pole**) for the magnitude measurements and a starting direction (called the **polar axis**) for the angle measurements (see margin). A **polar coordinate pair** for a point P in the plane is an ordered pair (r, θ) where r is the directed distance along a radial line from O to P and θ is the angle formed by the polar axis and the segment OP (see margin). The angle θ is positive when the angle of the radial line OP is measured counterclockwise from the polar axis; θ is negative when measured clockwise from the polar axis.

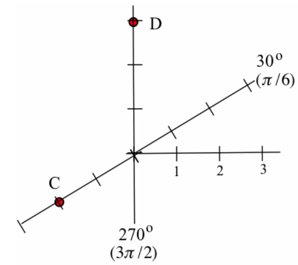
You can use either degree or radian measure for the angle in the polar coordinate system, but when we differentiate and integrate trigonometric functions of θ we will need angles to be given in radians. You should assume that all angles are in radian measure unless the you see the "°" symbol indicating "degrees."



Example 2. Plot the points with the given polar coordinates: $A(2, 30°)$, $B\left(3, \frac{\pi}{2}\right)$, $C\left(-2, \frac{\pi}{6}\right)$ and $D(-3, 270°)$.

Solution. To find the location of A, we look along the ray that makes an angle of $30°$ with the polar axis, then take two steps in that direction (assuming one step corresponds to one unit on the graph). The locations of A and B appear in the margin.

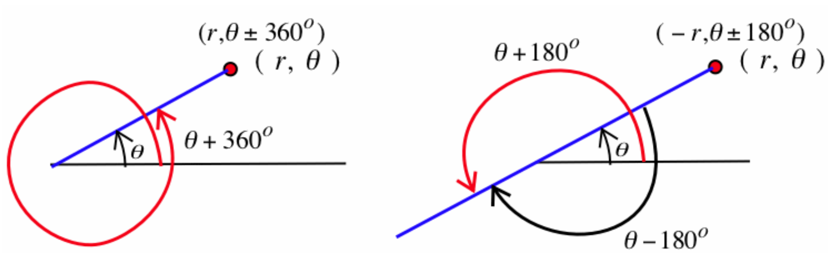
To find the location of C, look along the ray that makes an angle of $\frac{\pi}{6}$ with the polar axis, then we take two steps *backwards* (because $r = -2$ is negative). The locations of C and D appear in the margin. ◀



Notice that B and D have different addresses, $B\left(3, \frac{\pi}{2}\right)$ and $D(-3, 270°)$, but the same location.

Practice 2. Plot the points with polar coordinates $A\left(2, \frac{\pi}{2}\right)$, $B(2, -120°)$, $C\left(-2, \frac{\pi}{3}\right)$, $D(-2, -135°)$ and $E(2, 135°)$. Which two points coincide?

Each polar coordinate pair (r, θ) gives the location of one point, but each location has many different addresses in the polar coordinate system: the polar coordinates of a point are not unique. This non-uniqueness of addresses comes about in two ways. First, the angles θ, $\theta \pm 360°$, $\theta \pm 2 \cdot 360°, \ldots$ all describe the same radial line (see below left), so the polar coordinates (r, θ), $(r, \theta \pm 360°)$, $(r, \theta \pm 2 \cdot 360°), \ldots$ all locate the same point.

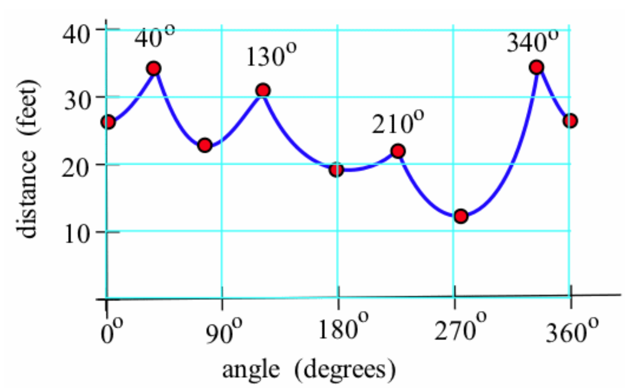


In the rectangular coordinate system we use (x, y) and $y = f(x)$, listing the independent variable first and the dependent variable second. In the polar coordinate system we use (r, θ) and $r = f(\theta)$, listing the dependent variable first and the independent variable second, a reversal from rectangular coordinate usage.

Secondly, the angle $\theta \pm 180°$ describes the radial line pointing in exactly the opposite direction from the radial line described by the angle θ (see above right), so the polar coordinates (r, θ) and $(-r, \theta \pm 180°)$ locate the same point. A polar coordinate pair gives the location of exactly one point, but the location of one point can be described by (infinitely) many different polar coordinate pairs.

Practice 3. The margin table contains measurements to the edge of a plateau taken by a remote sensor that crashed on the plateau. The figure below shows the data plotted in rectangular coordinates. Plot the data in polar coordinates and determine the shape of the plateau.

angle	distance
0°	28 feet
20°	30
40°	36
60°	27
80°	24
100°	24
130°	30
150°	22
230°	13
210°	21
180°	18
270°	10
340°	30
330°	18

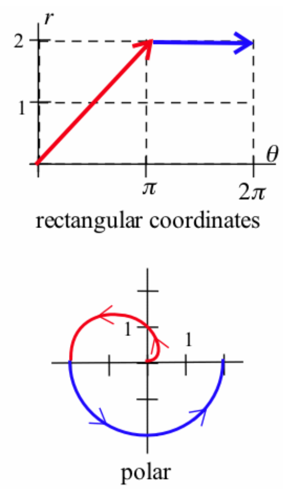


Rectangular Coordinate Graph of Data

Graphing Functions in the Polar Coordinate System

In the rectangular coordinate system, you have worked with functions given by tables of data, by graphs and by formulas. You can represent functions in the same ways using polar coordinates.

- If a table of data gives you values of a function, you can graph the function in polar coordinates by plotting individual points in a polar coordinate system and connecting the plotted points to see the shape of the graph. By hand, this is a tedious process; by calculator or computer, it is quick and easy.

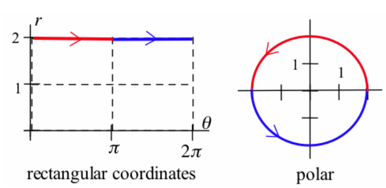


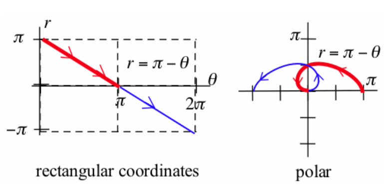
- If you have a rectangular coordinate graph of magnitude as a function of angle, you can read coordinates of points on the rectangular graph and replot them in polar coodinates. In essence, as you go from the rectangular coordinate graph to the polar coordinate graph you "wrap" the rectangular graph around the "pole" at the origin of the polar coordinate system (see margin).

- If you have a formula for a function, you (or your calculator) can graph the function to help obtain information about its behavior. Typically, you (or a calculator) creates a graph by evaluating the function at many points and then plotting the points in the polar coordinate system. Some of the following examples illustrate that functions given by simple formulas may have rather exotic graphs in the polar coordinate system.

If you already have a polar coordinate graph of a function, you can use the graph to answer questions about the behavior of the function. It is usually easy to locate the maximum value(s) of r on a polar coordinate graph and, by moving counterclockwise around the graph, you can observe where r is increasing, constant or decreasing.

Example 3. Graph $r = 2$ and $r = \pi - \theta$ in the polar coordinate system for $0 \le \theta \le 2\pi$.





Solution. First consider $r = 2$: In every direction θ, we simply move 2 units along the radial line and plot a point. The resulting polar graph (see margin) is a circle centered at the origin with a radius of 2. In the rectangular coordinate system, the graph of a constant $y = k$ is a horizontal line; in the polar coordinate system, the graph of a constant $r = k$ is a circle with radius $|k|$.

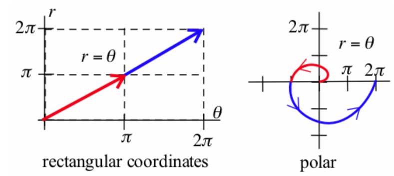
Next consider $r = \pi - \theta$: The rectangular=coordinate graph appears in the margin. Reading the values of r and θ from the rectangular coordinate graph and plotting them in polar coordinates results in the shape in the lower margin figure. The different line thicknesses used in the figures help you see which values from the rectangular graph become which parts of the loop in the polar graph. ◀

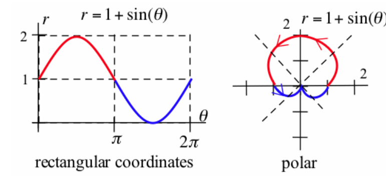
Practice 4. Graph $r = -2$ and $r = \cos(\theta)$ in polar coordinates.

Example 4. Graph $r = \theta$ and $r = 1 + \sin(\theta)$ in polar coordinates.

Solution. The rectangular coordinate graph of $r = \theta$ is a straight line (see top margin figure). Reading the values of r and θ from the rectangular coordinate graph and plotting them in polar coordinates results in a spiral, called an **Archimedean spiral**.

In the rectangular coordinate graph of $r = 1 + \sin(\theta)$ (see margin) the graph of the sine curve is shifted up 1 unit; in polar coordinates, the result of adding 1 to the sine function is much less obvious. ◀

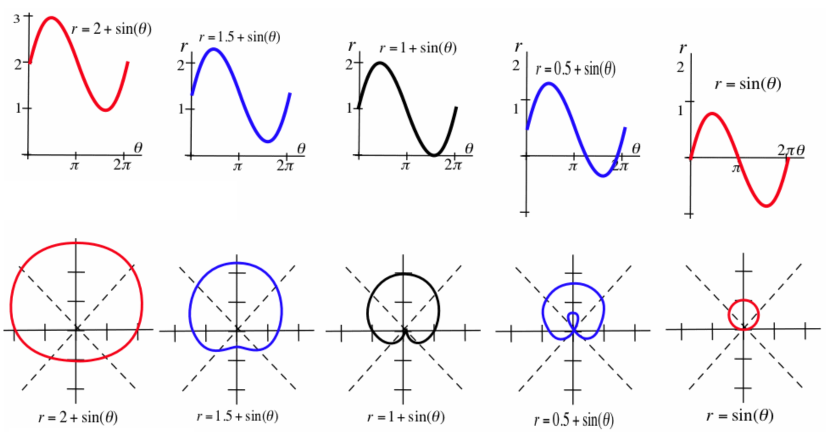




Practice 5. Plot the points in the margin table in polar coordinates and connect them with a smooth curve. Describe the shape in words.

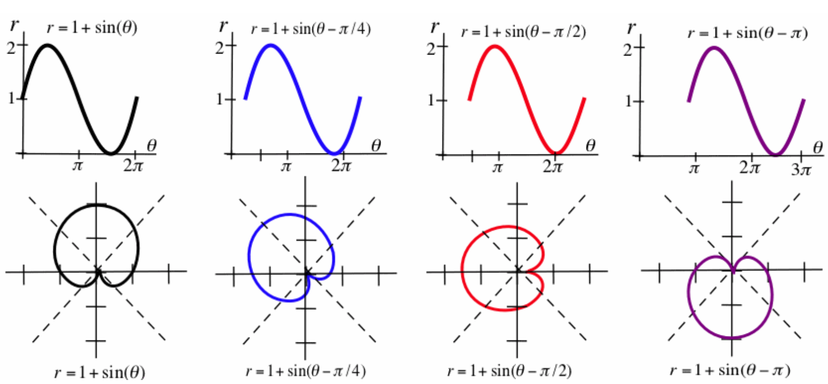
angle	distance
0	3.0 m
$\frac{\pi}{6}$	1.6
$\frac{\pi}{4}$	1.7
$\frac{\pi}{3}$	1.9
$\frac{\pi}{2}$	2.0

The graphs below show the effects of adding various constants to the rectangular and polar graphs of $r = \sin(\theta)$:



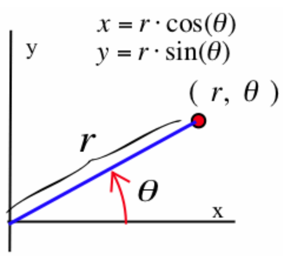
In rectangular coordinates, the result is a graph shifted up or down by k units; in polar coordinates, the result may be a graph with an entirely different shape.

The next set of graphs show the effects of adding a constant to the independent variable in rectangular and polar coordinates:

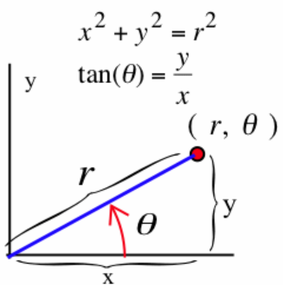


The result in rectangular coordinates is a horizontal shift of the original graph; the result in polar coordinates is a **rotation** of the original. Finding formulas for rotated figures in rectangular coordinates can be quite difficult, but rotations are easy in polar coordinates.

The formulas and names of several functions with exotic shapes in polar coordinates arise in the Problems. Many of them are difficult to graph "by hand," but by using a graphing calculator or computer you can appreciate the shapes and easily examine the effects of changing some of the constants in their formulas.



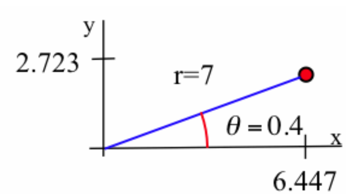
Converting Between Coordinate Systems



Sometimes you need both rectangular and polar coordinates in the same application, so it becomes necessary to change back and forth between the systems. If you place the two origins together and align the polar axis with the positive x-axis, the conversions involve straightforward applications of trigonometry and right triangles (see margin).

Polar to Rectangular: $x = r \cdot \cos(\theta)$, $y = r \cdot \sin(\theta)$
Rectangular to Polar: $r^2 = x^2 + y^2$, $\tan(\theta) = \dfrac{y}{x}$ (if $x \neq 0$)

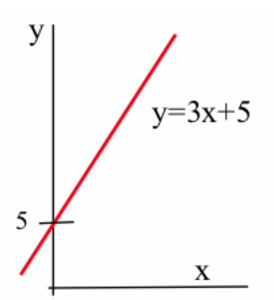
Example 5. Convert (a) the polar coordinate point $P(7, 0.4)$ to rectangular coordinates and (b) the rectangular coordinate point $R(12, 5)$ to polar coordinates.



Solution. (a) $r = 7$ and $\theta = 0.4$ so $x = 7 \cdot \cos(0.4) \approx 7(0.921) = 6.447$ and $y = 7 \cdot \sin(0.4) \approx 7(0.389) = 2.723$. (b) $x = 12$ and $y = 5$ so $r^2 = x^2 + y^2 = 144 + 25 = 169$, and $\tan(\theta) = \dfrac{y}{x} = \dfrac{5}{12}$; we can take $r = 13$ and $\theta = \arctan\left(\frac{5}{12}\right) \approx 0.395$. The polar coordinate addresses $(13, 0.395 \pm n \cdot 2\pi)$ and $(-13, 0.395 \pm (2n+1) \cdot \pi)$ give the location of the same point for any integer n. ◀

You can also use these conversion formulas to convert equations from one system to the other.

Example 6. Convert the linear equation $y = 3x + 5$ (see margin) from rectangular coordinates to polar coordinates.



Solution. Replacing x with $r \cdot \cos(\theta)$ and y with $r \cdot \sin(\theta)$:

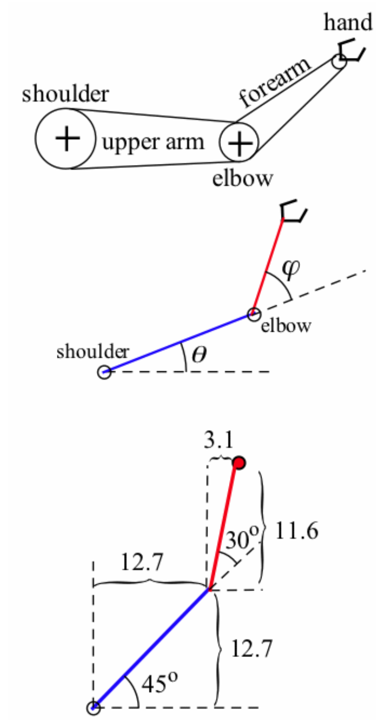
$$y = 3x + 5 \;\Rightarrow\; r \cdot \sin(\theta) = 3r \cdot \cos(\theta) + 5$$

$$\Rightarrow r \cdot [\sin(\theta) - 3\cos(\theta)] = 5 \;\Rightarrow\; r = \frac{5}{\sin(\theta) - 3\cos(\theta)}$$

This final representation is valid only when $\sin(\theta) - 3\cos(\theta) \neq 0$. ◀

Practice 6. Convert the polar coordinate equation $r^2 = 4r \cdot \sin(\theta)$ to a rectangular coordinate equation.

Example 7. A robotic arm has a hand at the end of a 12-inch forearm connected to an 18-inch upper arm (see margin). Determine the position of the hand, relative to the shoulder, if $\theta = 45° = \frac{\pi}{4}$ and $\varphi = 30° = \frac{\pi}{6}$.



Solution. The hand is $12 \cdot \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \approx 3.1$ inches to the right of the elbow and $12 \cdot \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \approx 11.6$ inches above the elbow. Similarly, the elbow is $18 \cdot \cos\left(\frac{\pi}{4}\right) \approx 12.7$ inches to the right of the shoulder and $18 \cdot \sin\left(\frac{\pi}{4}\right) \approx 12.7$ inches above the shoulder. Finally, the hand is approximately $3.1 + 12.7 = 15.8$ inches to the right of the shoulder and approximately $11.6 + 12.7 = 24.3$ inches above the shoulder. In polar coordinates, the hand is approximately 29 inches from the shoulder, at an angle of about 57° (about 0.994 radians) above the horizontal. ◄

Practice 7. Determine the position of the hand, relative to the shoulder, when $\theta = 30°$ and $\varphi = 45°$.

Which Coordinate System Should You Use?

There are no rigid rules. Use whichever coordinate system is easier or more "natural" for the problem or data you have. Sometimes it is unclear which system to use until you have graphed the data both ways. Some problems are easier if you switch back and forth between the systems. Generally, the polar coordinate system is easier if:

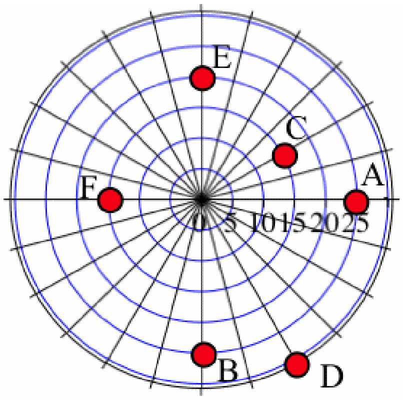
- the data consists of measurements in various directions (radar)
- your problem involves locations in relatively featureless locations (deserts, oceans, sky)
- rotations are involved

Typically, the rectangular coordinate system is easier if:

- the data consists of measurements given as functions of time or location (temperature, height)
- your problem involves locations in situations with an established grid (a city, a chess board)
- translations are involved

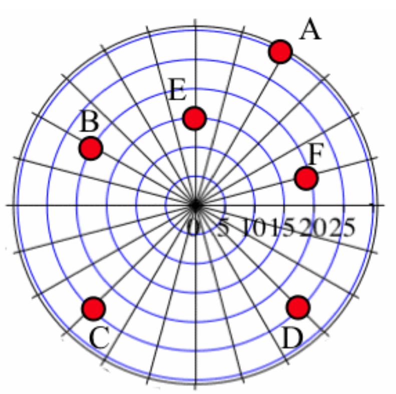
11.1 Problems

1. Give the locations in polar coordinates (using radians) of the points labeled A, B and C below.



2. Give the locations in polar coordinates of the points labeled D, E and F above.

3. Give the locations in polar coordinates of the points labeled A, B and C below.



4. Give the locations in polar coordinates of the points labeled D, E and F above.

In Problems 5–8, plot the points A–D in polar coordinates, connect the dots in order (A to B to C to D to A) using line segments, and name the approximate shape of the resulting figure.

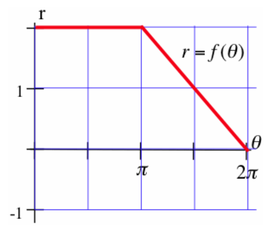
5. $A(3, 0°)$, $B(2, 120°)$, $C(2, 200°)$, $D(2.8, 315°)$

6. $A(3, 30°)$, $B(2, 130°)$, $C(3, 150°)$, $D(2, 280°)$

7. $A(2, 0.175)$, $B(3, 2.269)$, $C(2, 2.618)$, $D(3, 4.887)$

8. $A(3, 0.524)$, $B(2, 2.269)$, $C(3, 2.618)$, $D(2, 4.887)$

In Problems 9–14, use the given rectangular coordinate graph of the function $r = f(\theta)$ to sketch the polar coordinate graph of $r = f(\theta)$.

9.


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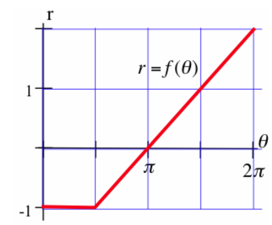

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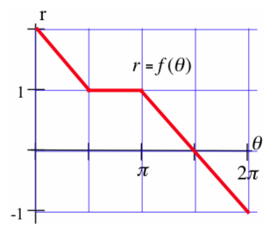

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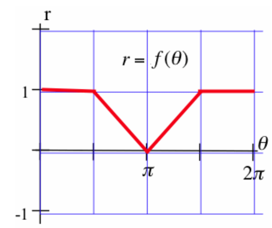

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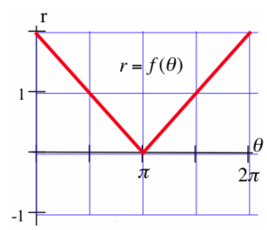

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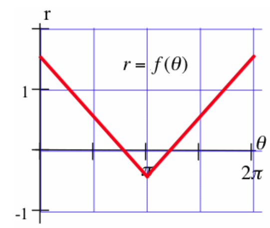
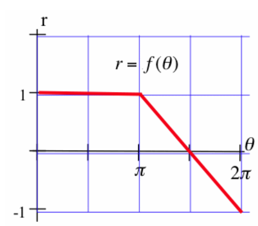
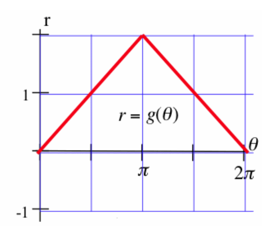


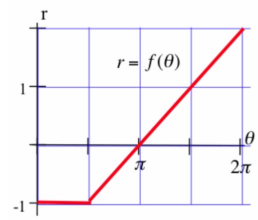
15. The rectangular coordinate graph of $r = f(\theta)$ appears below left.
    (a) Sketch the rectangular coordinate graphs of $r = 1 + f(\theta)$, $r = 2 + f(\theta)$ and $r = -1 + f(\theta)$.
    (b) Sketch the polar coordinate graphs of $r = 1 + f(\theta)$, $r = 2 + f(\theta)$ and $r = -1 + f(\theta)$.

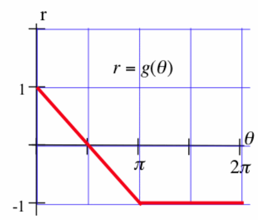




16. The rectangular coordinate graph of $r = g(\theta)$ appears above right.
    (a) Sketch the rectangular coordinate graphs of $r = 1 + g(\theta)$, $r = 2 + g(\theta)$ and $r = -1 + g(\theta)$.
    (b) Sketch the polar coordinate graphs of $r = 1 + g(\theta)$, $r = 2 + g(\theta)$ and $r = -1 + g(\theta)$.

17. The rectangular coordinate graph of $r = f(\theta)$ appears below left.
   (a) Sketch the rectangular coordinate graphs of $r = 1 + f(\theta)$, $r = 2 + f(\theta)$ and $r = -1 + f(\theta)$.
   (b) Sketch the polar coordinate graphs of $r = 1 + f(\theta)$, $r = 2 + f(\theta)$ and $r = -1 + f(\theta)$.





18. The rectangular coordinate graph of $r = g(\theta)$ appears above right.
   (a) Sketch the rectangular coordinate graphs of $r = 1 + g(\theta)$, $r = 2 + g(\theta)$ and $r = -1 + g(\theta)$.
   (b) Sketch the polar coordinate graphs of $r = 1 + g(\theta)$, $r = 2 + g(\theta)$ and $r = -1 + g(\theta)$.

19. If the rectangular coordinate graph of $r = f(\theta)$ has a horizontal asymptote of $r = 3$ as θ grows arbitrarily large, what does that tell you about the polar coordinate graph of $r = f(\theta)$ for large values of θ?

20. If $\lim_{\theta \to \frac{\pi}{6}} f(\theta) = \infty$ so that the rectangular coordinate graph of $r = f(\theta)$ has a vertical asymptote at $\theta = \frac{\pi}{6}$, what does that tell you about the polar coordinate graph of $r = f(\theta)$ for θ near $\frac{\pi}{6}$?

In Problems 21–40, graph the functions in polar coordinates for $0 \le \theta \le 2\pi$.

21. $r = -3$
22. $r = 5$
23. $\theta = \frac{\pi}{6}$
24. $\theta = \frac{5\pi}{3}$
25. $r = 4 \cdot \sin(\theta)$
26. $r = -2 \cdot \cos(\theta)$
27. $r = 2 + \sin(\theta)$
28. $r = -2 + \sin(\theta)$
29. $r = 2 + 3 \cdot \sin(\theta)$
30. $r = \sin(2\theta)$
31. $r = \tan(\theta)$
32. $r = 1 + \tan(\theta)$
33. $r = 3\sec(\theta)$
34. $r = 3\csc(\theta)$
35. $r = \dfrac{1}{\sin(\theta) + \cos(\theta)}$
36. $r = \dfrac{\theta}{2}$
37. $r = 2\theta$
38. $r = \theta^2$
39. $r = \dfrac{1}{\theta}$
40. $r = \sin(2\theta)\cos(3\theta)$

41. $r = \sin(m\theta) \cdot \cos(n\theta)$ produces lovely graphs for various small integer values of m and n. Use a calculator or computer to find values of m and n that result in shapes you find interesting.

42. Graph $r = \dfrac{1}{1 + 0.5 \cdot \cos(\theta + \alpha)}$ for $0 \le \theta \le 2\pi$ and for $\alpha = 0$, $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Describe how the graphs are related.

43. Graph $r = \dfrac{1}{1 + 0.5 \cdot \cos(\theta - \alpha)}$ for $0 \le \theta \le 2\pi$ and for $\alpha = 0$, $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Describe how the graphs are related.

44. Graph $r = \cos(n\theta)$ for $0 \le \theta \le 2\pi$ and for $n = 1$, 2, 3 and 4. Count the number of "petals" on each graph. Predict the number of "petals" for the graphs of $r = \sin(n\theta)$ for $n = 5$, 6 and 7, then test your prediction by creating those graphs.

45. Repeat the steps in Problem 44 using $r = \cos(n\theta)$.

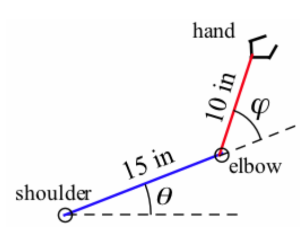
In Problems 46–49, convert the rectangular coordinate locations to polar coordinates.

46. $(0,3)$, $(5,0)$, $(1,2)$
47. $(-2,3)$, $(2,-3)$, $(0,-4)$
48. $(0,-2)$, $(4,4)$, $(3,-3)$
49. $(3,4)$, $(-1,-3)$, $(-7,12)$

In Problems 50–53, convert the polar coordinate locations to rectangular coordinates.

50. $(3,0)$, $(5,90^\circ)$ and $(1,\pi)$
51. $(-2,3)$, $(2,-3)$ and $(0,-4)$
52. $(0,3)$, $(5,0)$ and $(1,2)$
53. $(2,3)$, $(-2,-3)$ and $(0,4)$

For 54–60, refer to the robotic arm shown below.



54. Determine the position of the hand, relative to the shoulder, when $\theta = 60°$ and $\varphi = -45°$.
55. Determine the position of the hand, relative to the shoulder, when $\theta = -30°$ and $\varphi = 30°$.
56. Determine the position of the hand, relative to the shoulder, when $\theta = 0.6$ and $\varphi = 1.2$.
57. Determine the position of the hand, relative to the shoulder, when $\theta = -0.9$ and $\varphi = 0.4$.
58. If the robot's shoulder pivots so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, but the elbow is broken and φ is always 0, sketch the points the hand can reach.
59. If the robot's shoulder pivots so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, and the elbow pivots so $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$, sketch the points the hand can reach.
60. If the robot's shoulder pivots so $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, and the elbow pivots completely so $-\pi \le \varphi \le \pi$, sketch the points the hand can reach.
61. Graph $r = \dfrac{1}{1 + a \cdot \cos(\theta)}$ for $0 \le \theta \le 2\pi$ and $a = 0.5, 0.8, 1, 1.5$ and 2. What shapes do the various values of a produce?
62. Repeat Problem 61 with $r = \dfrac{1}{1 + a \cdot \sin(\theta)}$.
63. Show that the polar form of the linear equation $Ax + By + C = 0$ is:
$$r \cdot (A \cdot \cos(\theta) + B \cdot \sin(\theta)) + C = 0$$
64. Show that the equation of the line through the polar coordinate points (r_1, θ_1) and (r_2, θ_2) is:
$$r\,[r_1 \sin(\theta - \theta_1) + r_2 \sin(\theta_2 - \theta)] = r_1 r_2 \sin(\theta_2 - \theta_1)$$
65. Show that the graph of $r = a \cdot \sin(\theta) + b \cdot \cos(\theta)$ is a circle through the origin with center $\left(\frac{b}{2}, \frac{a}{2}\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

Some Exotic Curves (and Names)

An inexpensive resource for these shapes and names is *A Catalog Of Special Plane Curves* by J. Dennis Lawrence, Dover Publications, 1972; the page numbers given below refer to that book.

Many of the following curves were discovered and named even before polar coordinates came about. In most cases the curve describes the path of a point moving on or around some object. You may enjoy using your calculator or a computer to graph some of these curves, or you can invent your own exotic shapes.

Some classics:

- Cissoid ("like ivy") of Diocles (about 200 B.C.): $r = a \sin(\theta) \cdot \tan(\theta)$
- Right Strophoid ("twisting") of Barrow (1670): $r = a\,[\sec(\theta) - 2\cos(\theta)]$
- Trisectrix of MacLaurin (1742): $r = a \sec(\theta) - 4a \cos(\theta)$
- Lemniscate ("ribbon") of Bernoulli (1694): $r^2 = a^2 \cos(2\theta)$
- Conchoid ("shell") of Nicomedes (225 B.C.): $r = a + b \sec(\theta)$
- Hippopede ("horse fetter") of Proclus (about 75 B.C.):
$$r^2 = 4b\left[a - b\sin^2(\theta)\right] \quad \text{for} b = 3,\ a = 1,\ 2,\ 3,\ 4$$
- Devil's Curve of Cramer (1750):
$$r^2\left[\sin^2(\theta) - \cos^2(\theta)\right] = a^2 \sin^2(\theta) - b^2 \cos^2(\theta) \quad \text{for} \quad a = 2,\ b = 3$$
- Nephroid ("kidney") of Freeth: $r = a\left[1 + 2\sin(\theta^2)\right]$ for $a = 3$

Some of our own:

Based on their names, what shapes do you expect for the following curves?

- Piscatoid of Pat (1992): $r = \sec(\theta) - 3\cos(\theta)$ for $-1.1 \le \theta \le 1.1$, with window $-2 \le x \le 1$ and $-1 \le y \le 1$

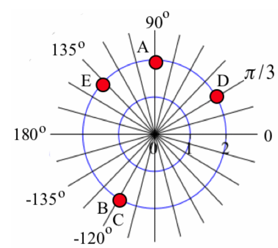
- Kermitoid of Kelcey (1992):

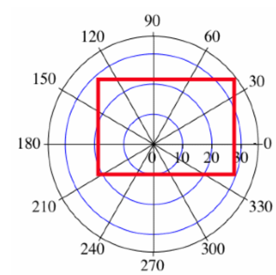
  $$r = 2.5\sin(2\theta)\,[\theta - 4.71] \cdot \text{INT}\left(\frac{\theta}{\pi}\right) + \left[5\sin^3(\theta) - 3\sin^9(\theta)\right] \cdot \left[1 - \text{INT}\left(\frac{\theta}{\pi}\right)\right]$$

  for $0 \le \theta \le 2\pi$ with window $-3 \le x \le 3$ and $-1 \le y \le 4$

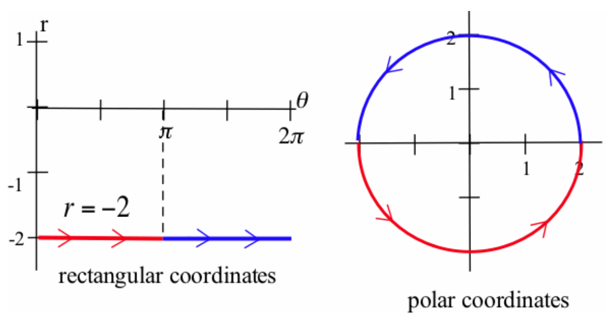
- Bovine Oculoid: $r = 1 + \text{INT}\left(\frac{\theta}{2\pi}\right)$ for $0 \le \theta \le 6\pi$ with window $-5 \le x \le 5$ and $-4 \le y \le 4$

11.1 Practice Answers

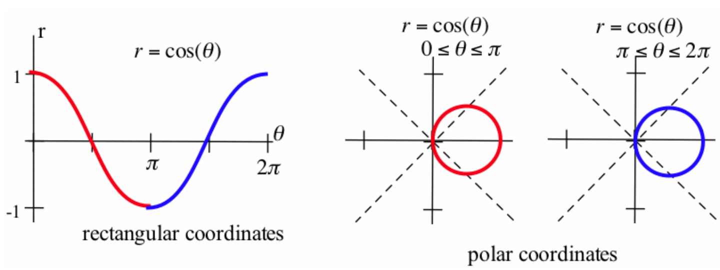


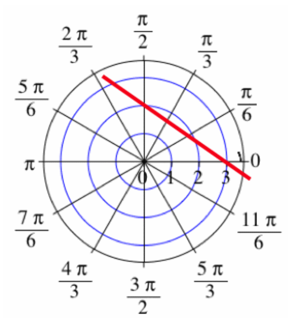


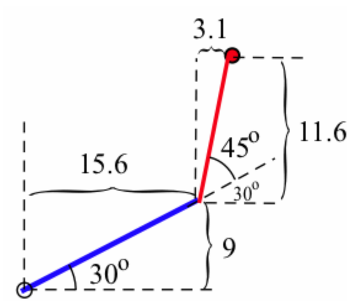
1. Point C is at a distance of 10 miles in the direction $30°$; D is 5 miles away at $270°$.
2. See first margin figure.
3. See second margin figure. The plateau is roughly rectangular.
4. The graphs for $r = -2$ appear below:



The graphs for $r = \cos(\theta)$ appear below. Note that in polar coordinates $r = \cos(\theta)$ traces out a circle *twice*: once as θ goes from 0 to π, and a second time as θ goes from π to 2π.







5. See margin figure. The points (almost) lie on a straight line.

6. $r^2 = x^2 + y^2$ and $r \cdot \sin(\theta) = y$, so:

$$r^2 = 4r \cdot \sin(\theta) \;\Rightarrow\; x^2 + y^2 = 4y$$

Putting this last equation into the standard form for a circle (by completing the square) yields $x^2 + (y-2)^2 = 4$, an equation for a circle with center at $(0, 2)$ and radius 2.

7. See margin figure. For point A, the "elbow," relative to O, the "shoulder": $x = 18\cos(30°) \approx 15.6$ inches and $y = 18\sin(30°) = 9$ inches. For point B, the "hand," relative to A: $x = 12\cos(75°) \approx 3.1$ inches and $y = 12\sin(75°) \approx 11.6$ inches. Then the rectangular coordinate location of B relative to O is $x \approx 15.6 + 3.1 = 18.7$ inches and $y \approx 9 + 11.6 = 20.6$ inches. The polar coordinate location of B relative to O is $r = \sqrt{x^2 + y^2} \approx 27.8$ inches and $\theta \approx 47.7°$ (or 0.83 radians).