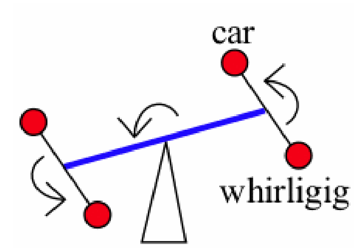
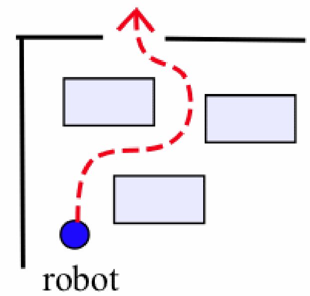
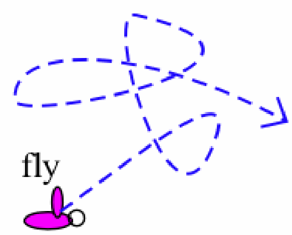
11.3 *Parametric Equations*

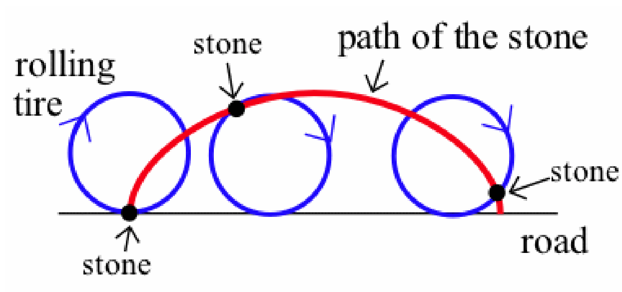
Some motions and paths are inconvenient, difficult or impossible for us to describe using a graph of the form $y = f(x)$.







- A rider on a "whirligig" (see top margin figure) at a carnival travels in circles at the end of a rotating bar.
- A robot delivering supplies in a factory (second margin figure) must avoid obstacles.
- A fly buzzing around the room (third margin figure) or a molecule in a solution follow erratic paths.
- A stone caught in the tread of a rolling wheel has a smooth path with some sharp corners (see figure below).

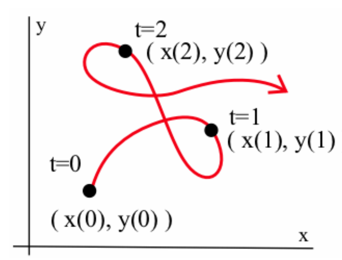


Parametric equations provide a way to describe all of these motions and paths. And parametric equations generalize easily to describe paths and motions in three (or more) dimensions.

We used parametric equations briefly in Sections 2.5 and 5.2. We consider them more carefully now, looking at functions given parametrically by data, graphs and formulas, and examining how to build formulas to describe certain motions parametrically (including the cycloid, one of the most famous curves in mathematics).

The next section uses calculus with parametric equations to find slopes of tangent lines, arclengths and areas.

In two dimensions, parametric equations describe the location of a point (x, y) on a graph or path as a function of a single independent variable t, a "parameter" often representing time. The coordinates x and y are functions of the variable t: $x = f(t)$ and $y = g(t)$ (see margin). (In three dimensions, we add a z-coordinate that is also a function of t: $z = h(t)$.) Among other applications, we can use parametric equations to analyze the forces acting on an object separately in each coordinate direction and then combine the results to determine the overall behavior of the object.



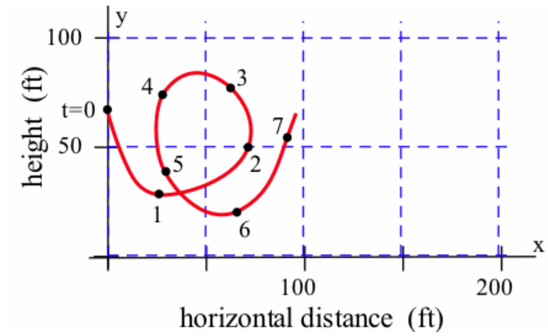
Graphing Parametric Curves

The data we need to create a graph can be given as a table of values, as graphs of $(t, f(t))$ and $(t, y(t))$, or as formulas for $f(t)$ and $g(t)$.

t	x	y
0	0	70
1	30	20
2	70	50
3	60	75
4	30	70
5	32	35
6	60	15
7	90	55
8	105	85
9	125	100
10	130	80
11	150	65
12	180	75
13	200	30

Example 1. The margin table records the location of a roller coaster car relative to its starting location. Use the data to sketch a graph of the car's path during the first seven seconds of motion.

Solution. The figure below plots the (x, y) locations of the car at one-second intervals from $t = 0$ to $t = 7$ seconds. We can connect these points using a smooth curve that shows one possible path of the car. ◄

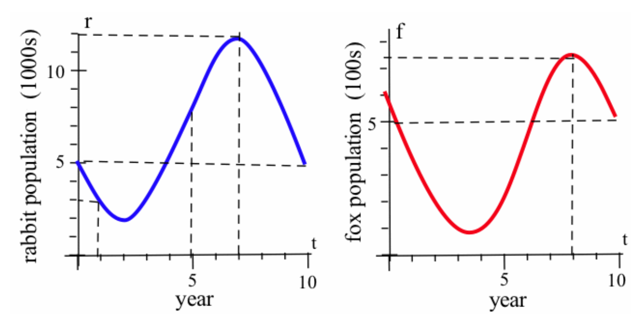


Practice 1. Use the remaining data in the margin table to sketch a possible path of the roller coaster car from $t = 7$ to $t = 13$ seconds.

Clearly the path of the roller coaster in the preceding Example is not the graph of a function $y = f(x)$. But every graph of the form $y = f(x)$ has an easy parametric representation: set $x(t) = t$ and $y(t) = f(t)$.

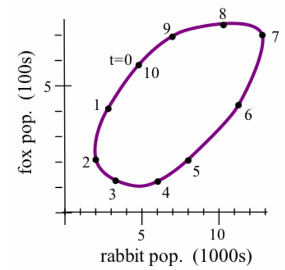
Sometimes a parametric graph can show patterns that are not clearly visible in individual graphs.

Example 2. The figures below are graphs of the populations of rabbits and foxes on an island. Use these graphs to sketch a parametric graph of rabbits (x-axis) versus foxes (y-axis) for $0 \le t \le 10$ years.

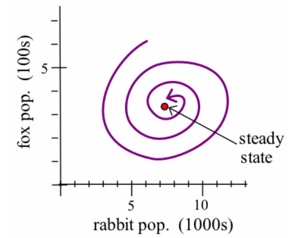


Solution. The separate rabbit and fox population graphs give us information about each population separately, but the parametric graph helps us see the effects of the interaction between the rabbits and the foxes more clearly. For each time t, you can read the rabbit and fox populations from the separate graphs (for example, when $t = 1$, there are roughly 3,000 rabbits and 400 foxes so $x \approx 3000$ and $y \approx 400$) and then combine this information to plot a single point.

If you repeat this process for a large number of values of t, you get a graph (see margin) of the "motion" of the rabbit and fox populations over a period of time. We can then ask questions about why the populations might exhibit this behavior. ◀



The type of graph created in the preceding Example is very common for "predator-prey" interactions. Some two-species populations approach a "steady state" or "fixed point" (see second margin figure), while others repeat a cyclical pattern over time (as in Example 2).



Practice 2. What would happen if the rabbit-fox graph touched the horizontal axis?

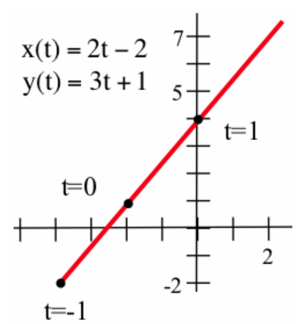
Example 3. Graph the parametric equations $x(t) = 2t - 2$ and $y(t) = 3t + 1$ in the xy-plane.

Solution. The margin table shows the values of x and y for several values of t and the graph shows these points plotted in the xy-plane. The graph appears to be a line. Often it is difficult or impossible to write y as a simple function of x, but in this situation we can do so:

$$x = 2t - 2 \;\Rightarrow\; t = \frac{1}{2}x + 1 \;\Rightarrow\; y = 3\left(\frac{1}{2}x + 1\right) + 1 = \frac{3}{2}x + 4$$

t	x	y
-1	-4	-2
0	-2	1
1	0	4
2	2	7

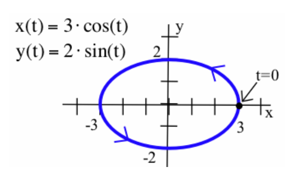
This agrees with what we see in the graph: a line with slope $\frac{3}{2}$ and y-intercept at $(0, 4)$. ◀



Practice 3. Graph $x(t) = 3 - t$ and $y(t) = t^2 + 1$ in the xy-plane. Then write y as a function of x alone and identify the shape of the graph.

Example 4. Graph $x(t) = 3\cos(t)$ and $y(t) = 2\sin(t)$ in the xy-plane for $0 \le t \le 2\pi$, then show that these parametric equations satisfy the relation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ for all values of t.

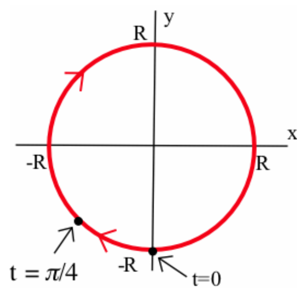
Solution. The graph, an ellipse, appears in the margin. Substituting $3\cos(t)$ for x and $2\sin(t)$ for y into the left-hand side of the given relation yields $\cos^2(t) + \sin^2(t) = 1$, as required. ◀



Practice 4. Graph the equations $x(t) = \sin(t)$ and $y(t) = 5\cos(t)$ in the xy-plane for $0 \le t \le 2\pi$, then show that these equations satisfy the relation $x^2 + \dfrac{y^2}{25} = 1$ for all values of t.

Example 5. Describe the motion of an object whose location at time t is given by $x(t) = -R \cdot \sin(t)$ and $y(t) = -R \cdot \cos(t)$.

Solution. At $t = 0$, the object starts at $x(0) = -R\sin(0) = 0$ and $y(0) = -R\cos(0) = -R$. By plotting $x(t)$ and $y(t)$ for several other



values of t (see margin figure), we can see that the object is rotating clockwise around the origin. Because:

$$x^2 + y^2 = [-R\sin(t)]^2 + [-R\cos(t)]^2 = R^2\left[\sin^2(t) + \cos^2(t)\right] = R^2$$

the object must traverse a circle of radius R centered at the origin. ◀

Practice 5. Each set of parametric equations below give the position of an object travelling around a circle of radius 1 centered at the origin.

(a) $x(t) = \cos(2t)$, $y(t) = \sin(2t)$

(b) $x(t) = -\cos(3t)$, $y(t) = \sin(3t)$

(c) $x(t) = \sin(4t)$, $y(t) = -\cos(4t)$

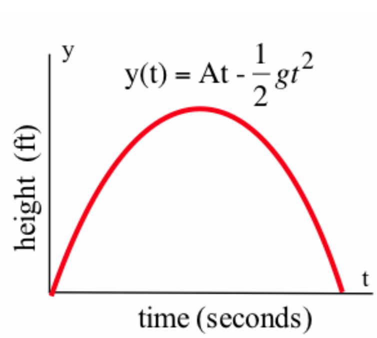
For each object, determine:

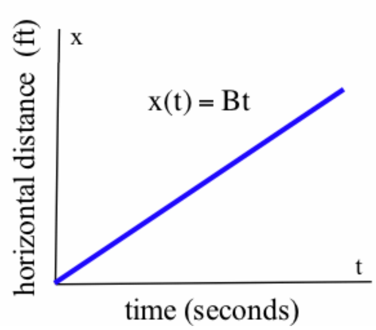
- the location of the object at time $t = 0$.
- whether the object is traveling clockwise or counterclockwise.
- the time it takes for the object to make one revolution.

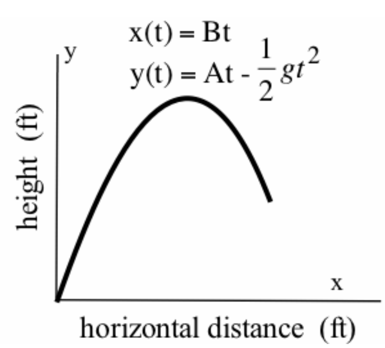
Putting Motions Together

If we know how an object moves horizontally and how it moves vertically, we can combine these motions to see how the object moves through the xy-plane.

If you throw an object straight upward with an initial velocity of A feet per second, then its height after t seconds is $y(t) = A \cdot t - \frac{1}{2}g \cdot t^2$ feet where $g = 32$ feet/sec^2 is the (downward) acceleration of gravity (see below left). If you throw an object horizontally with an initial velocity of B feet per second, then its horizontal distance from the starting place after t seconds is $x(t) = B \cdot t$ feet (see below center).

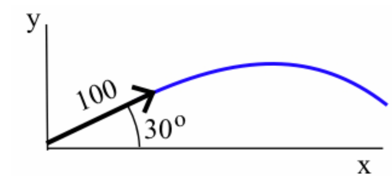






Example 6. Write parametric equations for the location at time t (above right) of an object thrown at an angle of 30° with the ground (horizontal) with an initial velocity 100 feet per second.

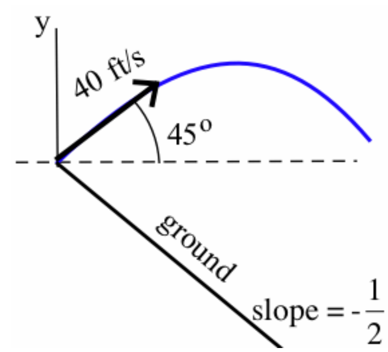
Solution. If the object travels 100 feet along a line at an angle of 30° to the horizontal ground (see margin), then it travels $100 \cdot \sin(30°) = 50$ feet upward and $100 \cdot \cos(30°) \approx 86.6$ feet sideways, so $A = 50$ and $B = 86.6$ (using the notation from the discussion above). The location of the object at time t is therefore given by $x(t) = 86.6t$ and $y(t) = 50t - \frac{1}{2}gt^2 = 50t - 16t^2$. ◀



Practice 6. You throw a ball upward at an angle of 45° with an initial velocity of 40 ft/sec.

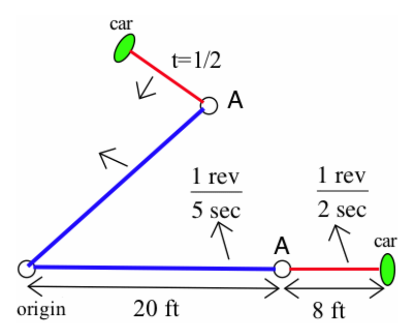
(a) Write the parametric equations for the position of the ball as a function of time.

(b) Use the parametric equations to find when and then where the ball will hit the sloped ground (as shown in the margin figure).



Sometimes we record the location or motion of an object using an instrument that is itself in motion (for example, tracking a pod of migrating whales from a moving ship) and we want to determine the path of the object independent of the location of the instrument. In that case, the "absolute" location of the object with respect to the origin is the sum of the relative location of the object (the pod of whales) with respect to the instrument (the ship) and the location of the instrument (the ship) with respect to the origin. The same approach allows us to describe the motion of linked objects, such as connected gears.

Example 7. A car on a carnival ride (see margin) makes one counterclockwise revolution (with radius $r = 8$ feet) about the pivot point A every two seconds. The pivot A is at the end of a longer arm (with radius $R = 20$ feet) that makes one counterclockwise revolution about its pivot point (the origin) every five seconds. If the ride begins with the two arms outstretched along the positive x-axis, sketch the path you think the car will follow. Then find a pair of parametric equations that describe the location of the car at time t.



Solution. The location of the car relative to its pivot point A is given by $x_c(t) = 8\cos\left(\frac{2\pi}{2}t\right) = 8\cos(\pi t)$ and $y_c(t) = 8\sin\left(\frac{2\pi}{2}t\right) = 8\sin(\pi t)$.

The position of the pivot point A relative to the origin is given by $x_p(t) = 20\cos\left(\frac{2\pi}{5}t\right)$ and $y_p(t) = 20\sin\left(\frac{2\pi}{5}t\right)$, so the location of the car, relative to the origin, is given by:

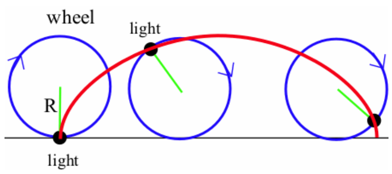
$$x(t) = x_p(t) + x_c(t) = 8\cos(\pi t) + 20\cos\left(\frac{2\pi}{5}t\right)$$

$$y(t) = y_p(t) + y_c(t) = 8\sin(\pi t) + 20\sin\left(\frac{2\pi}{5}t\right)$$

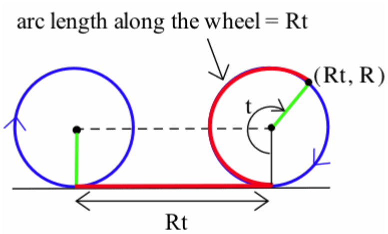
Use technology to graph the path of the car for over the first five seconds and compare that path to your initial guess. ◀

The Cycloid

Our final example devlops parametric equations for a curve called a **cycloid**, one of the most famous and interesting curves in mathematics.

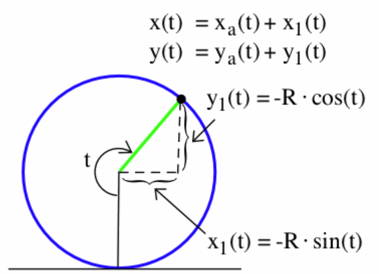


Example 8. A light is attached to the edge of a wheel of radius R, which rolls along a level road (see margin). Find parametric equations to describe the location of the light.



Solution. We can describe the location of the axle of the wheel, then the location of the light relative to the axle, and finally put the results together to get the location of the light.

The axle of the wheel is always R inches off the ground, so the y-coordinate of the axle is given by $y_a(t) = R$. When the wheel has rotated t radians about its axle, the wheel has rolled a distance of $R \cdot t$ along the road, so the x-coordinate of the axle is given by $x_a(t) = R \cdot t$.

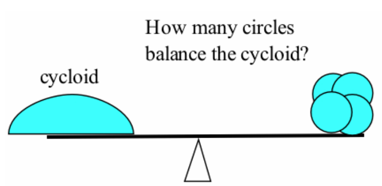


The position of the light relative to the axle is given by $x_l(t) = -R\sin(t)$ and $y_l(t) = -R\cos(t)$ so the position of the light relative to the origin is given by:

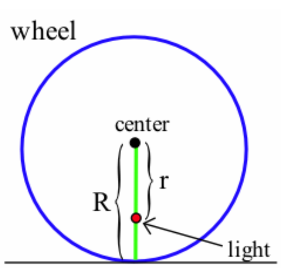
$$x(t) = x_a(t) + x_l(t) = Rt - R\sin(t) = R\,[t - \sin(t)]$$
$$y(t) = y_a(t) + y_l(t) = R - R\cos(t) = R\,[1 - \cos(t)]$$

Use technology (choose a value for R) to graph these equations. ◀

Many great mathematicians and physicists (Mersenne, Galileo, Newton, Bernoulli, Huygens and others) examined the cycloid, determined its properties and used it in physical applications. Marin Mersenne (1588–1648) thought the path might be part of an ellipse (it isn't). In 1634, Gilles Personne de Roberval (1602–1675) determined the parametric form of the cycloid and found the area under the cycloid, as did Descartes and Fermat, before Newton (1642–1727) was even born: they used various specialized geometric approaches to solve the area problem. Around the same time, Galileo determined the area experimentally by cutting a cycloidal region from a sheet of lead and balancing it against a number of disks (with the same radius as the circle that generated the cycloid) cut from the same material. How many disks do you think balance the cycloidal region's area?



The cycloid's most amazing properties, however, involve motion along a cycloid-shaped path. Those discoveries had to wait for Newton and calculus.



Practice 7. A light is attached r inches from the axle to a wheel of radius R inches ($r < R$) that rolls along a level road (see margin). Use the approach of Example 8 to find parametric equations to describe the location of the light. The resulting curve is called a **curate cycloid**.

11.3 Problems

For Problems 1–4, use the data in the table to create three graphs: (a) $(t, x(t))$ (b) $(t, y(t))$ and (c) the parametric graph $(x(t), y(t))$. (Connect the points with line segments to create the graph.)

1.

t	x	y
0	2	1
1	2	0
2	−1	0
3	1	−1

2.

t	x	y
0	0	1
1	1	1
2	1	−1
3	2	0

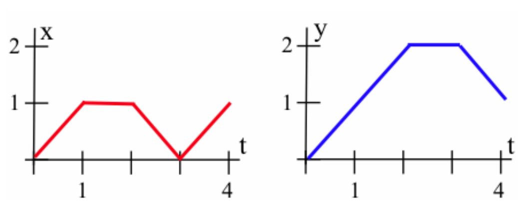
3.

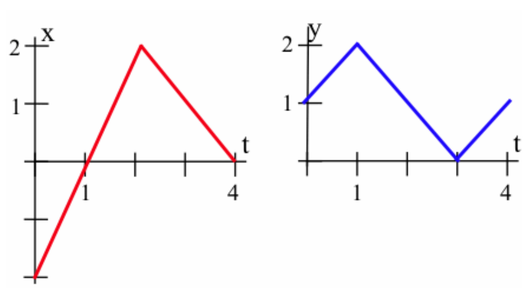
t	x	y
0	1	2
1	−1	−1
2	1	2
3	0	2

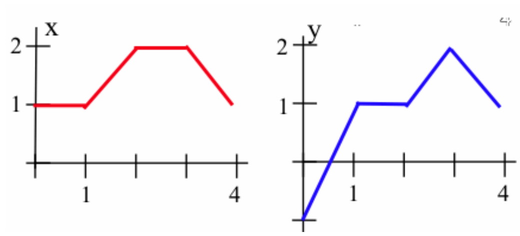
4.

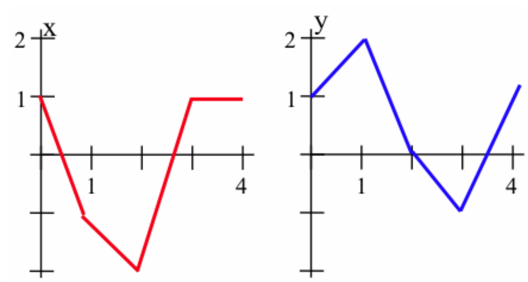
t	x	y
0	0	1
1	−1	0
2	0	−2
3	3	1

For 5–8, use the given graphs of $(t, x(t))$ and $(t, y(t))$ to sketch the parametric graph $(x(t), y(t))$.

5. 

6. 

7. 

8. 

9. Graph the parametric equations $x(t) = 3t - 2$, $y(t) = 1 - 2t$. What shape is this graph?

10. Graph the parametric equations $x(t) = 2 - 3t$, $y(t) = 3 + 2t$. What shape is this graph?

11. Calculate the slope of the line through the points $P = (x(0), y(0))$ and $Q = (x(1), y(1))$ for the equations $x(t) = at + b$ and $y(t) = ct + d$.

12. Graph $x(t) = 3 + 2\cos(t)$, $y(t) = -1 + 3\sin(t)$ for $0 \le t \le 2\pi$. Describe the shape of the graph.

13. Graph $x(t) = -2 + 3\cos(t)$, $y(t) = 1 - 4\sin(t)$ for $0 \le t \le 2\pi$. Describe the shape of the graph.

14. Graph each set of parametric equations, then describe the similarities and the differences among these graphs.

    (a) $x(t) = t^2$, $y(t) = t$

    (b) $x(t) = \sin^2(t)$, $y(t) = \sin(t)$

    (c) $x(t) = t$, $y(t) = \sqrt{t}$.

15. Graph each set of parametric equations, then describe the similarities and the differences among these graphs.

    (a) $x(t) = t$, $y(t) = t$

    (b) $x(t) = \sin(t)$, $y(t) = \sin(t)$

    (c) $x(t) = t^2$, $y(t) = t^2$.

16. Graph the parametric equations:

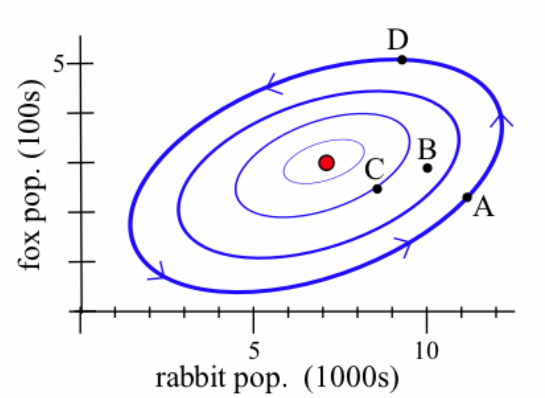
$$x(t) = \left(4 - \frac{1}{t}\right)\cos(t), \quad y(t) = \left(4 - \frac{1}{t}\right)\sin(t)$$

for $t \ge 1$, then describe the behavior of the graph.

17. Graph $x(t) = \dfrac{\cos(t)}{t}$, $y(t) = \dfrac{\sin(t)}{t}$ for $t \geq \dfrac{\pi}{4}$, then describe the behavior of the graph.

18. Graph $x(t) = t + \sin(t)$, $y(t) = t^2 + \cos(t)$ for $0 \leq t \leq 2\pi$, then describe the shape of the graph.

Problems 19–22 refer to the rabbit–fox population graph below, which shows several different population cycles depending on the various numbers of rabbits and foxes. Wildlife biologists sometimes try to control animal populations by "harvesting" some of the animals, but this needs to be done with care. The thick dot on the graph is the fixed point for this two-species population.



19. If there are currently 11,000 rabbits and 200 foxes (point A), and wildlife officials "harvest" 1,000 rabbits (removing them from the population), does the harvest shift the populations onto a cycle closer to or farther from the fixed point?

20. If there are currently 10,000 rabbits and 300 foxes (point B), and officials "harvest" 100 foxes, does the harvest shift the populations onto a cycle closer to or farther from the fixed point?

21. If there are currently 8,000 rabbits and 250 foxes (point C), and 1,000 rabbits die during a hard winter, does the wildlife biologist need to take action to main the population balance? Justify your response.

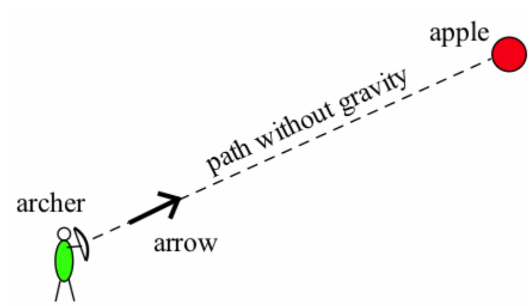
22. If there are currently 9,000 rabbits and 500 foxes (point D), and 2,000 rabbits die during a hard winter, does the wildlife biologist need to take action to main the population balance? Justify your response.

23. If $x(t) = at + b$ and $y(t) = ct + d$ with $a \neq 0$ and $c \neq 0$, write y as a function of x alone and show that the parametric graph $(x(t), y(t))$ is a line. What is the slope of that line?

24. Each set of parametric equations given below satisfy $x^2 + y^2 = 1$ and, for $0 \leq t \leq 2\pi$, describe the position of an object moving around a circle with radius 1 with center at the origin. Explain how the motions of the objects differ.
    (a) $x(t) = \cos(t)$, $y(t) = \sin(t)$
    (b) $x(t) = \cos(-t)$, $y(t) = \sin(-t)$
    (c) $x(t) = \cos(2t)$, $y(t) = \sin(2t)$
    (d) $x(t) = \sin(t)$, $y(t) = \cos(t)$
    (e) $x(t) = \cos\left(t + \frac{\pi}{2}\right)$, $y(t) = \sin\left(t + \frac{\pi}{2}\right)$

25. From a tall building, you observe a person walking along a straight path while twirling a light (parallel to the ground) at the end of a string.
    (a) If the person is walking slowly, sketch the path of the light.
    (b) How would the path of the light change if the person were running?
    (c) Sketch the path of the light for a person walking along a parabolic path.
    (d) Sketch the path of the light for a person running along a parabolic path.

26. William Tell aims his arrow directly at an apple and releases the arrow at exactly the same instant that the apple stem breaks. In a world without gravity (or air resistance), the apple remains in place after the stem breaks and the arrow flies straight to hit the apple (see below).



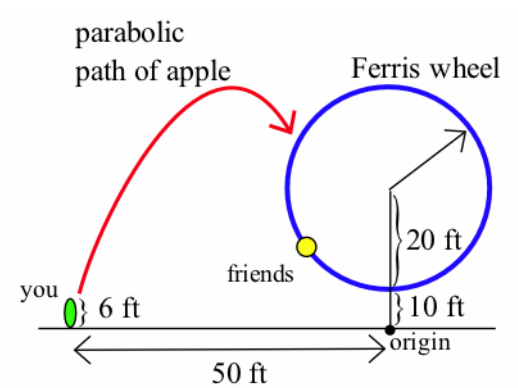
    (a) Sketch the path of the apple and the arrow in a world with gravity (but still no air resistance).
    (b) Does the arrow still hit the apple? Explain.

27. Find the radius R of a circle that generates a cycloid that starts at the point $(0,0)$ and:
    (a) passes through the point $(10\pi, 0)$ on its first complete revolution $(0 \le t \le 2\pi)$.
    (b) passes through the point $(5,2)$ on its first complete revolution. (Technology is helpful here.)
    (c) passes through the point $(2,3)$ on its first complete revolution. (Technology is helpful here.)
    (d) passes through the point $(4\pi, 8)$ on its first complete revolution.

28. Your friends are riding on the Ferris wheel illustrated below, and a t seconds after the ride begins, their location is given parametrically as:

$$\left(-20\sin\left(\frac{2\pi}{15}t\right), 30-20\cos\left(\frac{2\pi}{15}t\right)\right)$$

    (a) Is the Ferris wheel turning clockwise of counterclockwise?
    (b) How many seconds does it take the Ferris wheel to make one complete revolution?



29. You are standing 50 feet to the left of the Ferris wheel from Problem 28. You toss an apple from a height of six feet above the ground at an angle of 45°. Write parametric equations for the location of the apple (relative to the origin indicated in the figure above) at time t if:
    (a) you give the apple an initial velocity of 30 feet per second.
    (b) you give the apple an initial velocity of v feet per second.

30. Help—the Ferris wheel won't stop! To keep your friends on the Ferris wheel in Problems 28–29 from getting hungry, you toss an apple to them (at time $t = 0$). Find a formula for the distance between the apple and your friends at time t. Somehow (technology may be useful), find a value for the initial velocity v of the apple that will ensure that it comes close enough for your friends to catch it (within two feet should do the trick).

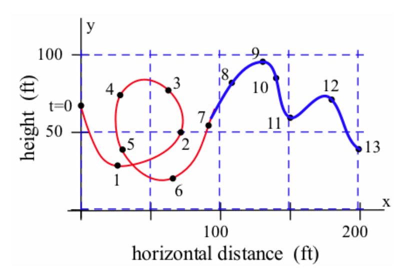
31. A wheel of radius R sits on a ledge, with a rod of length $1.5R$ attached to the center of the wheel and hanging down over the ledge. Find parametric equations for the path (called a **prolate cycloid**) of a light at the end of the rod.

32. A wheel of radius R rolls along the inside of a circle of radius $3R$. Find parametric equations for the path (called a **hypocycloid**) of a light on the edge of the wheel.

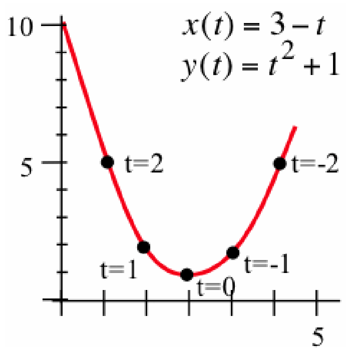
33. A wheel of radius R rolls along the outside of a circle of radius $3R$. Find parametric equations for the path (called an **epicycloid**) traced out by a light on the edge of the wheel.

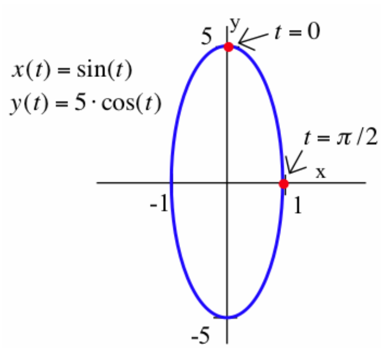
11.3 Practice Answers

1. A possible path for the car appears in the margin.



2. At the time the (rabbit, fox) parametric graph touches the horizontal axis there will be 0 foxes, so the fox population becomes extinct.





3. If $x = 3 - t$ and $y = t^2 + 1$ then $t = 3 - x$ and $y = (3 - x)^2 + 1 = x2 - 6x + 10$. The graph (see margin) is parabola, opening upward, with vertex at $(3, 1)$.

4. For all t:

$$\frac{x^2}{1} + \frac{y^2}{25} = \frac{\sin^2(t)}{1} + \frac{25\cos^2(t)}{25} = \sin^2(t) + \cos^2(t) = 1$$

A parametric graph of $x(t) = \sin(t)$ and $y(t) = 5\cos(t)$ appears in the second margin figure.

5. A starts at $(1, 0)$, travels counterclockwise, and takes $\frac{2\pi}{2} = \pi$ seconds to make one revolution. B starts at $(-1, 0)$, travels clockwise, and takes $\frac{2\pi}{3}$ seconds to make one revolution. C starts at $(0, -1)$, travels counterclockwise, and takes $\frac{2\pi}{4} = \frac{\pi}{2}$ seconds to make one revolution.

6. (a) $x(t) = 40\cos(45°)\,t = 20\sqrt{2}\,t$ and $y(t) = 40\sin(45°)\,t - 16t^2 = 20\sqrt{2}\,t - 16t^2$

(b) Along the ground $y = -\frac{1}{2}x$, so the ball hits the ground when:

$$y(t) = -\frac{1}{2}x(t) \;\Rightarrow\; 20\sqrt{2}\,t - 16t^2 = -10\sqrt{2}\,t \;\Rightarrow\; t = \frac{15\sqrt{2}}{8}$$

(assuming $t \neq 0$). The location of the ball is therefore given by:

$$x\left(\frac{15\sqrt{2}}{8}\right) = 20\sqrt{2}\left(\frac{15\sqrt{2}}{8}\right) = 75$$
$$y\left(\frac{15\sqrt{2}}{8}\right) = 20\sqrt{2}\left(\frac{15\sqrt{2}}{8}\right) - 16\left(\frac{15\sqrt{2}}{8}\right)^2 = -37.5$$

so the ball hits the ground at location $(75, -37.5)$ after (approximately) 2.652 seconds.

7. The axle is located at $x_a(t) = Rt$ and $y_a(t) = R$ while the location of the light relative to the axle is given by $x_l(t) = -r\sin(t)$ and $y_l(t) = -r\cos(t)$, hence the position of the light relative to the origin is given by:

$$x(t) = x_a(t) + x_l(t) = Rt - r\sin(t)$$
$$y(t) = y_a(t) + y_l(t) = R - r\cos(t)$$