11.4 Calculus with Parametric Equations

The previous section discussed parametric equations, their graphs and some of their uses. This section examines some of the ideas and techniques of calculus as they apply to parametric equations: slope of a tangent line, speed, arclength and area. Treatments of slope, speed, and arclength for parametric equations previously appeared in Sections 2.5 and 5.3, so the presentation here is brief. The material on area (new to this section) is a variation on the Riemann-sum development of the integral. This section ends with an investigation of some of the properties of the cycloid.

Slope

Also see Section 2.5.

If $x(t)$ and $y(t)$ are differentiable functions of t, then the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ measure the rates of change of x and y, respectively, with respect to t. The derivative $\frac{dy}{dx}$ measures the slope of the line tangent to the parametric graph $(x(t), y(t))$. To calculate $\frac{dy}{dx}$ we need to use the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dy}}{\frac{dx}{dt}}$$

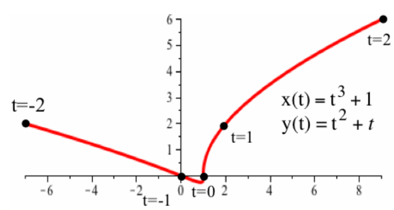
as long as $\frac{dx}{dt} \neq 0$.

Example 1. The location of an object in a plane, relative to the orgin, is given by the parametric equations $x(t) = t^3 + 1$ feet and $y(t) = t^2 + t$ feet at time t seconds.

(a) Evaluate $x(t)$ and $y(t)$ at $t = -2, -1, 0, 1$ and 2, then graph the path of the object for $-2 \leq t \leq 2$.

(b) Evaluate $\frac{dy}{dx}$ for $t = -2, -1, 0, 1$ and 2. Do your calculated values for $\frac{dy}{dx}$ agree with the shape of your graph from part (a)?

t	x	y	$\frac{dy}{dx}$
-2	-7	2	$-\frac{1}{4}$
-1	0	0	$-\frac{1}{3}$
0	1	0	UND
1	2	2	1
2	9	6	$\frac{5}{12}$



Solution. (a) When $t = -2$, $x(-2) = (-2)^3 + 1 = -7$ and $y(-2) = (-2)^2 + (-2) = 2$. The other values for $x(t)$ and $y(t)$ appear in the margin table; a graph of $(x(t), y(t))$ appears below.

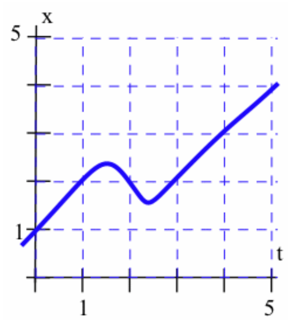
(b) $\frac{dy}{dt} = 2t + 1$ and $\frac{dx}{dt} = 3t^2$, so:

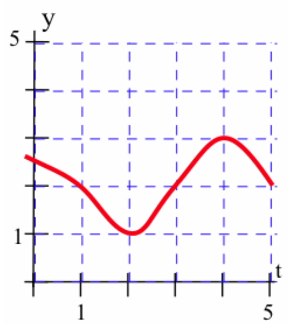
$$\frac{dy}{dx} = \frac{2t+1}{3t^2} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{t=2} = \frac{-3}{12} = -\frac{1}{4}$$

The other values for $\frac{dy}{dx}$ appear in the margin table (the value for $t = 0$ is undefined). ◀

Practice 1. Find an equation for the line tangent to the graph of the parametric equations from Example 1 at the point where $t = 3$.

An object can "visit" the same location more than once, and a parametric graph pass through the same point more than once.





Example 2. The first two margin figures show the x- and y-coordinates of an object at time t.

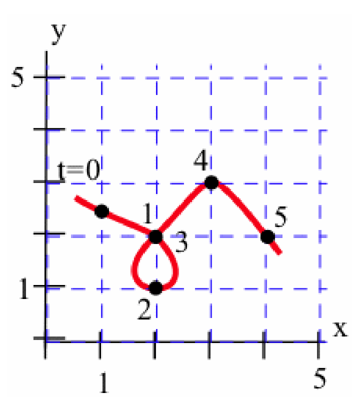
(a) Sketch the parametric graph of $(x(t), y(t))$, the position of the object at time t.

(b) Give the coordinates of the object when $t = 1$ and $t = 3$.

(c) Find the slopes of the tangent lines to the parametric graph when $t = 1$ and $t = 3$.

Solution. (a) By reading the x- and y-values on the graphs in margin figures, we can plot points on the parametric graph. The parametric graph appears in the bottom margin figure.

(b) When $t = 1$, $x = 2$ and $y = 2$ so the parametric graph goes through the point $(2, 2)$. When $t = 3$, the parametric graph goes through the same point $(2, 2)$, as observed in the parametric graph.



(c) When $t = 1$, $\frac{dy}{dt} \approx -1$ and $\frac{dx}{dt} \approx +1$, so:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \approx \frac{-1}{1} = -1$$

When $t = 3$, $\frac{dy}{dt} \approx +1$ and $\frac{dx}{dt} \approx +1$, so $\frac{dy}{dx} \approx 1$. These values agree with the appearance of the parametric graph. The object passes through the point $(2, 2)$ twice (when $t = 1$ and $t = 3$), but is traveling in a different direction each time. ◀

Practice 2. (a) Estimate the slopes of the lines tangent to the parametric graph from the previous Example when $t = 2$ and $t = 5$.

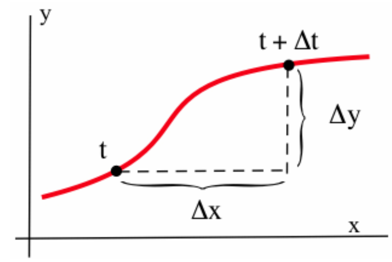
(b) At what time(s) does $\frac{dy}{dt} = 0$?

(c) When does the parametric graph have a maximum? A minimum?

(d) How are the maximum and minimum points on a parametric graph related to the derivatives of $x(t)$ and $y(t)$?

Speed

If you know how fast an object is moving in the x-direction $\left(\frac{dx}{dt}\right)$ and how fast it is moving in the y-direction $\left(\frac{dy}{dt}\right)$, it is straightforward to determine the **speed** of the object (how fast it is moving in the xy-plane).



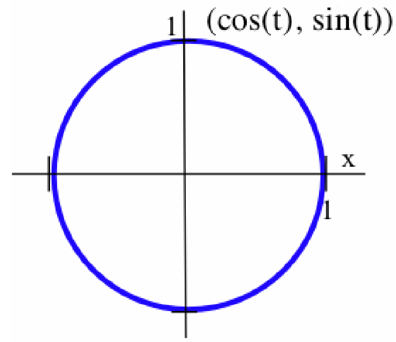
If, during a short interval of time Δt, the object's position changes by Δx in the x-direction and by Δy in the y-direction (see margin), then the object has moved a distance of $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ in time Δt, so the average speed during this brief time interval is:

$$\frac{\text{distance moved}}{\text{time change}} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} = \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2}$$

If $x(t)$ and $y(t)$ are differentiable functions of t, we can take the limit of the average speed (as Δt approaches 0) to get the instantaneous speed at time t:

$$\lim_{\Delta t \to 0} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example 3. At time t seconds the location (measured in feet) of an object in the xy-plane, relative to the origin is $(\cos(t), \sin(t))$. Sketch the path of the object and show that it is traveling at a constant speed.



Solution. The object is moving in a circular path (see margin). The speed of the object is:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

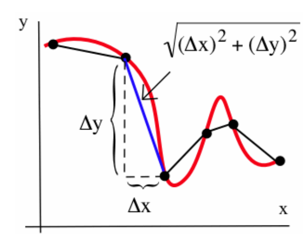
so its speed (1 foot per second) is indeed constant. ◀

Practice 3. Is the object in Example 2 traveling faster when $t = 1$ or when $t = 3$? When $t = 1$ or when $t = 2$?

Arclength

Also see Section 5.3.

In section 5.3 we approximated the total length L of a curve $\mathcal{C}$ by partitioning $\mathcal{C}$ into small pieces (see margin), approximating the length of each piece using the distance formula, and then adding the lengths of the pieces together to get:



$$L \approx \sum \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum \sqrt{\left(\frac{\Delta x}{\Delta x}\right)^2 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x \longrightarrow \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where $x = a$ and $x = b$ correspond to the endpoints of $\mathcal{C}$. We then used a similar approach for parametric equations:

$$L \approx \sum \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\, \Delta t$$

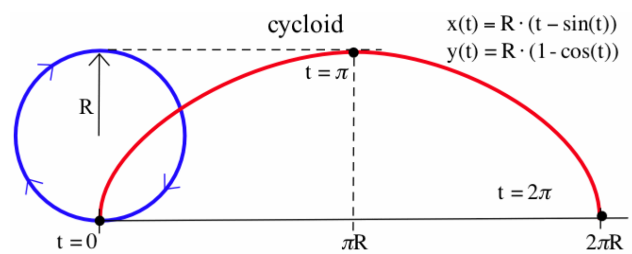
where $t = \alpha$ and $t = \beta$ correspond to the endpoints of $\mathcal{C}$.

Arclength Formula (Parametric Version)

If $\mathcal{C}$ is a curve given by $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$
and $x'(t)$ and $y'(t)$ exist and are continuous on $[\alpha, \beta]$
then the length L of $\mathcal{C}$ is given by:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 4. Find the length of the cycloid parametrized by $x(t) = R\,(t - \sin(t))$ and $y(t) = R\,(1 - \cos(t))$ for $0 \le t \le 2\pi$ (see below).



Solution. Computing $\dfrac{dx}{dt} = R\,(1 - \cos(t))$ and $\dfrac{dy}{dt} = R\sin(t)$ and using the parametric arclength formula yields:

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{R\,(1 - \cos(t))^2 + (R\sin(t))^2}\, dt$$

$$= R\int_0^{2\pi} \sqrt{1 - 2\cos(t) + \cos^2(t) + \sin^2(t)}\, dt = R\int_0^{2\pi} \sqrt{2 - 2\cos(t)}\, dt$$

The resulting integral:

$$\int_0^{2\pi} \sqrt{2 - 2\cos(t)}\, dt$$

appears challenging, but in this instance clever use of a trigonometric identity allows us to find an exact value. In most instances, however, the integrals resulting from arclength computation will require numerical approximation (as we observed in Section 5.3).

Replacing θ with $\dfrac{t}{2}$ in the formula $\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$ yields:

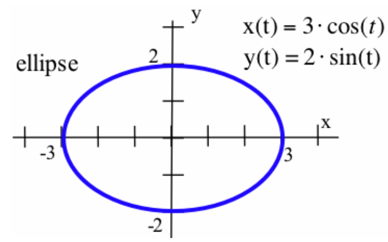
$$2 - 2\cos(t) = 4\sin^2\left(\frac{t}{2}\right)$$

so the integral becomes:

$$L = R\int_0^{2\pi} 2\sin\left(\frac{t}{2}\right)\, dt = \left[-4R\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} = 8R$$

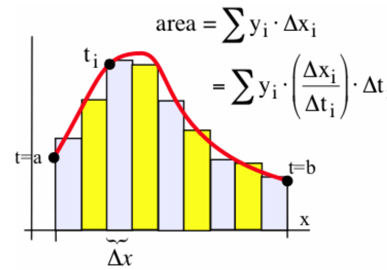
The length of a cycloid arch is 8 times the radius of the rolling circle that generated the cycloid. ◀

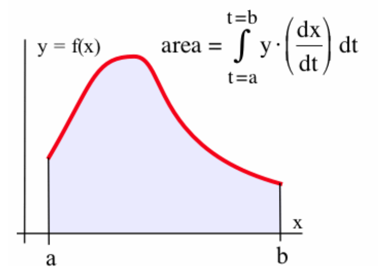
Practice 4. Represent the length of the ellipse parametrized by $x(t) = 3\cos(t)$ and $y = 2\sin(t)$ for $0 \le t \le 2\pi$ (see margin) as a definite integral, then use technology to approximate the value of the integral.



Area

When we first developed the definite integral, we approximated the area between the graph of a positive function $y = f(x)$ by partitioning the domain $a \le x \le b$ into n pieces of lengths Δx_k , finding the areas of the thin rectangles, and approximating the total area by adding the rectangle areas: $A \approx \sum y_k \Delta x_k$ (a Riemann sum). As $\Delta x \to 0$, the Riemann sum approached the definite integral $\int_{x=a}^{x=b} y\,dx$.





To compute the area of a similar region when the curve is defined by parametric equations, the process is similar, but the independent variable is now t and the domain is an interval $\alpha \le t \le \beta$. If $x(t)$ is an increasing function of t, any partition of the t-interval $[\alpha, \beta]$ into n pieces of lengths Δt_k induces a partition of the x-axis (see margin). We can use this induced partition of the x-axis to approximate the total area by:

$$A \approx \sum y_k \Delta x_k = \sum y_k \cdot \frac{\Delta x_k}{\Delta t_k} \Delta t_k \longrightarrow \int_{t=\alpha}^{t=\beta} y \cdot \frac{dx}{dt}\,dt$$

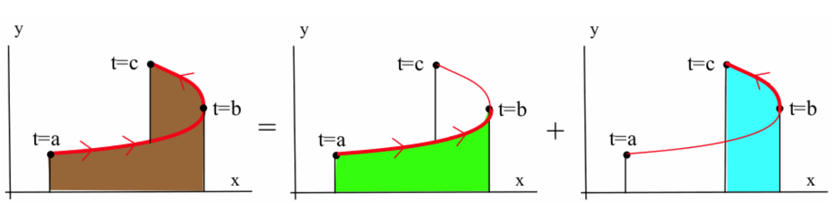
as $\Delta t \to 0$.

Area with Parametric Equations

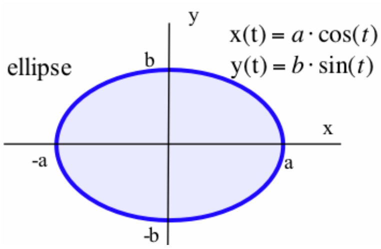
If	$\mathcal{C}$ is a curve given by $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$
and	$x'(t)$ and $y'(t)$ exist and are continuous on $[\alpha, \beta]$
and	$y(t)$ and $x'(t)$ do not change sign on $[\alpha, \beta]$
then	the area between $\mathcal{C}$ and the x-axis is given by:

$$A = \left| \int_{\alpha}^{\beta} y(t) \cdot \frac{dx}{dt}\,dt \right|$$

The requirement that y not change sign prevents the parametric graph from being above the x-axis for some values of t and below the x-axis for other t-values. The requirement that $\frac{dx}{dt}$ not change sign prevents the graph from "turning around" (see below).



If either requirement is not satisfied, some of the area will be added (above center) and some will be subtracted (above right).



Example 5. Find the area of the region $\mathcal{R}$ in the first quadrant enclosed by the ellipse described by the parametric equations $x(t) = a\cos(t)$ and $y = b\sin(t)$ (for $a > 0$ and $b > 0$) (see margin).

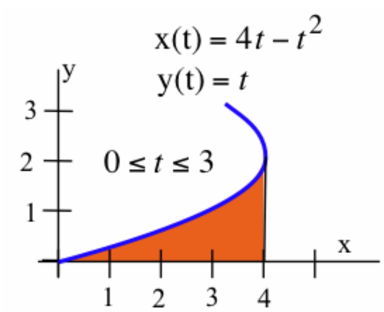
Solution. In the first quadrant (where $0 \le t \le \frac{\pi}{2}$), $y = b\sin(t) > 0$ and $\frac{dx}{dt} = -a\sin(t) < 0$, so the area of $\mathcal{R}$ is given by:

$$A = \left| \int_{t=0}^{t=\frac{\pi}{2}} y(t) \cdot \frac{dx}{dt}\, dt \right| = \left| \int_{t=0}^{t=\frac{\pi}{2}} b\sin(t) \cdot [-a\sin(t)]\, dt \right|$$

Evaluating this integral yields:

$$A = ab \left| \int_{t=0}^{t=\frac{\pi}{2}} \sin^2(t)\, dt \right| = ab \left| \int_{t=0}^{t=\frac{\pi}{2}} \left[\frac{1}{2} - \frac{1}{2}\cos(2t) \right] dt \right|$$
$$= ab \left| \left[\frac{1}{2}t - \frac{1}{4}\sin(2t) \right]_{t=0}^{t=\frac{\pi}{2}} \right| = \frac{\pi ab}{4}$$

The area enclosed by the entire ellipse is πab; if $a = b$, the ellipse is a circle with radius $r = a = b$ with area πr^2 (as expected). ◀



Practice 5. Let $x(t) = 4t - t^2$ and $y(t) = t$ (graphed in the margin).

(a) Represent the area of the shaded region as an integral.

(b) Evaluate the integral from part (a).

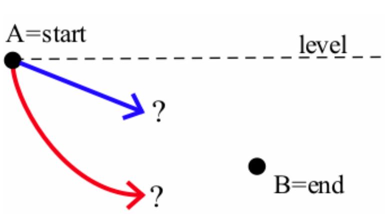
(c) Does $\int_0^3 t(4-2t)\, dt$ represent an area?

Properties of the Cycloid

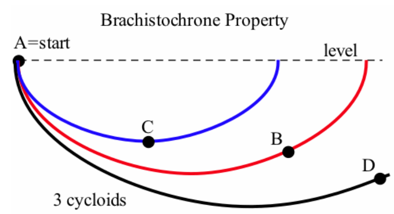
In Section 11.3 we developed parametric equations for the cycloid: $x(t) = R(t - \sin(t))$ and $y(t) = R(1 - \cos(t))$ For any $t \ge 0$, $y(t) \ge 0$ and $\frac{dx}{dt} = R(1 - \cos(t)) \ge 0$ so the area between one arch of the cycloid and the x-axis is:

$$A = \left| \int_{t=0}^{t=2\pi} y(t) \cdot \frac{dx}{dt}\, dt \right| = \int_0^{2\pi} [R(1-\cos(t))] \cdot [R(1-\cos(t))]\, dt$$
$$= R^2 \int_0^{2\pi} \left[1 - 2\cos(t) + \cos^2(t) dt \right] dt$$
$$= R^2 \left[\frac{1}{2}t - 2\sin(t) + \frac{1}{2}t + \frac{1}{4}\sin(2t) \right]_0^{2\pi} = R^2 [2\pi + \pi] = 3\pi R^2$$

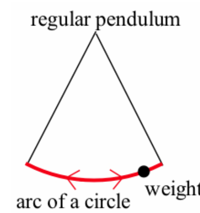
The area under one arch of a cycloid is 3 times the area of the circle that generates the cycloid. How does this compare with your guess from the end of Section 11.3?

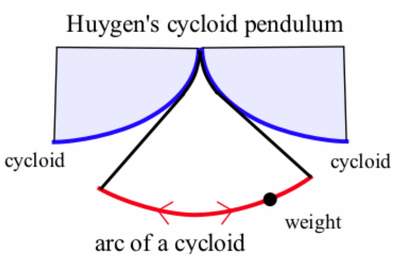


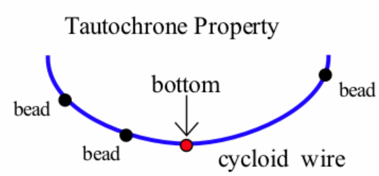
You and a friend decide to hold a contest to see who can build a slide that gets a person from point A to point B (see margin) in the shortest time. What shape should you make your slide: a straight line, part of a circle, or something else? Assuming that the slide is frictionless and that the only acceleration is due to gravity, Johann Bernoulli (1667–1748) showed that the shortest-time ("**brachistochrone**" for "brachi," meaning "short," and "chrone," meaning time) path is part of a cycloid that starts at A and also goes through the point B. The margin figure shows the cycloidal paths for A and B as well as the cycloidal paths for two other "finish" points, C and D.



Even before Bernoulli solved the brachistochrone problem, the astronomer, physicist and mathematician Christiaan Huygens (1629–1695) attempted to design an accurate pendulum clock. On a standard pendulum clock (see margin), the path of the bob is part of a circle, and the period of the swing depends on the displacement angle of the bob; as friction slows the bob, the displacement angle gets smaller and the clock slows down. Huygens designed a clock (below left) whose bob swung in a curve so that the period of the swing did not depend on the displacement angle:







The curve Huygens found to solve the same-time ("**tautochrone**" for "tauto," meaning "same," and "chrone," meaning "time") problem was part of the cycloid. Beads strung on a wire in the shape of a cycloid (above right) reach the bottom in the same amount of time, no matter where along the wire (except the bottom point) you release them.

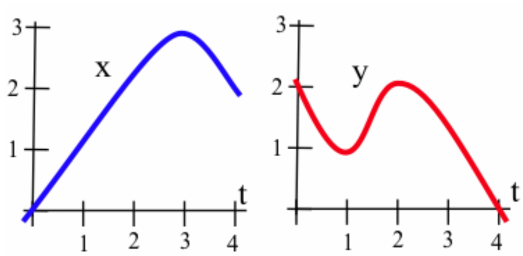
The brachistochane and tautochrone problems are examples from a field of mathematics called the **Calculus of Variations**. Typical optimization problems in calculus involve finding a point or number that maximizes or minimizes some quantity. Typical optimization problems in the Calculus of Variations involve finding a curve or function that maximizes or minimizes some quantity. For example, what curve or shape with a given length encloses the greatest area? (Answer: a circle.) Modern applications of Calculus of Variations include finding routes for airliners and ships to minimize travel time or fuel consumption depending on prevailing winds or currents.

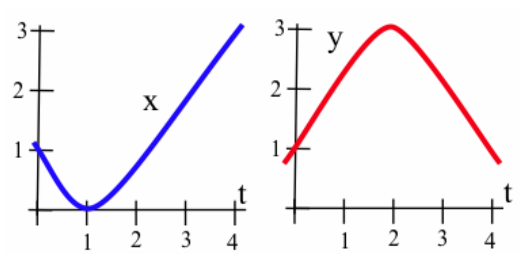
11.4 Problems

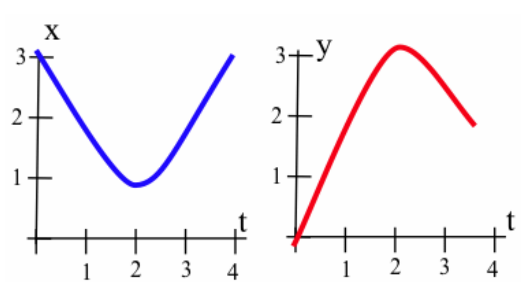
For Problems 1–8, (a) sketch the parametric graph $(x(t), y(t))$, (b) find the slope of the line tangent to the graph at the given values of t, and (c) find the points (x, y) at which $\frac{dy}{dx}$ is either 0 or undefined.

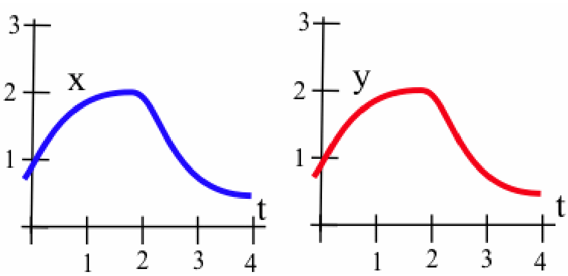
1. $x(t) = t - t^2$, $y(t) = 2t + 1$; $t = 0, 1, 2$
2. $x(t) = t^3 + t$, $y(t) = t^2$; $t = 0, 1, 2$
3. $x(t) = 1 + \cos(t)$, $y(t) = 2 + \sin(t)$; $t = 0, \frac{\pi}{4}, \frac{\pi}{2}$
4. $x(t) = 1 + 3\cos(t)$, $y(t) = 2 + 2\sin(t)$; $t = 0, \frac{\pi}{4}, \frac{\pi}{2}$
5. $x(t) = \sin(t)$, $y(t) = \cos(t)$; $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, 17.3$
6. $x(t) = 3 + \sin(t)$, $y(t) = 2 + \sin(t)$; $t = 0, \frac{\pi}{4}, \frac{\pi}{2}$
7. $x(t) = \ln(t)$, $y(t) = 1 - t^2$; $t = 1, 2, e$
8. $x(t) = \arctan(t)$, $y(t) = e^t$; $t = 0, 1, 2$

In Problems 9–12, use the given graphs of $x(t)$ and $y(t)$ to estimate (a) the slope of the line tangent to the parametric graph at $t = 0, 1, 2$ and 3, and (b) the points (x, y) at which $\frac{dy}{dx}$ is either 0 or undefined.

9. 

10. 

11. 

12. 

For Problems 13–20, use the given locations $x(t)$ and $y(t)$ of an object at time t seconds (measured in feet) to find the speed of the object at the given times.

13. $x(t) = t - t^2$, $y(t) = 2t + 1$; $t = 0, 1, 2$
14. $x(t) = t^3 + t$, $y(t) = t^2$; $t = 0, 1, 2$
15. $x(t) = 1 + \cos(t)$, $y(t) = 2 + \sin(t)$; $t = 0, \frac{\pi}{4}, \frac{\pi}{2}$
16. $x(t) = 1 + 3\cos(t)$, $y(t) = 2 + 2\sin(t)$; $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi$
17. $x(t)$ and $y(t)$ from Problem 9 at $t = 0, 1, 2, 3$ and 4
18. $x(t)$ and $y(t)$ from Problem 10 at $t = 0, 1, 2$ and 3
19. $x(t)$ and $y(t)$ from Problem 11 at $t = 0, 1, 2$ and 3
20. $x(t)$ and $y(t)$ from Problem 12 at $t = 0, 1, 2$ and 3
21. An object travels along a cycloidal path so that its location is given by $x(t) = R(t - \sin(t))$ and $y(t) = R(1 - \cos(t))$ after t seconds (with distances measured in feet).
    (a) Find the speed of the object at time t.
    (b) At what time is the object traveling fastest?
    (c) Where is the object on its cycloidal path when it is traveling fastest?
22. At time t seconds an object is located at $x(t) = 5\cos(t)$ and $y(t) = 2\sin(t)$ (measured in feet).
    (a) Find the speed of the object at time t.
    (b) At what time is the object traveling fastest?
    (c) Where is the object on its elliptical path when it is traveling fastest?

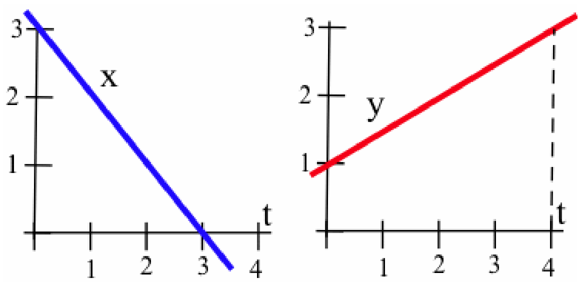
For 23–28 (a) represent the arclength of the parametric graph as a definite integral, and (b) evaluate the integral (using technology, if necessary).

23. $x(t) = t - t^2$, $y(t) = 2t + 1$ from $t = 0$ to 2
24. $x(t) = t^3 + t$, $y(t) = t^2$; $t = 0$ to 2

25. $x(t) = 1 + \cos(t)$, $y(t) = 2 + \sin(t)$; $t = 0$ to π

26. $x(t) = 1 + 3\cos(t)$, $y(t) = 2 + 2\sin(t)$; $t = 0$ to π

27. $x(t)$ and $y(t)$ given below; $t = 1$ to 3



28. $x(t)$ and $y(t)$ from Problem 12; $t = 0$ to 2

For 29–32 (a) represent the area of the region between the parametric graph and the x-axis as a definite integral, and (b) evaluate the integral.

29. $x(t) = t^2$, $y(t) = 4t^2 - t^4$ for $0 \le t \le 2$

30. $x(t) = 1 + \sin(t)$, $y(t) = 2 + \sin(t)$ for $0 \le t \le \pi$

31. $x(t) = t^2$, $y(t) = 1 + \cos(t)$ for $0 \le t \le 2$

32. $x(t) = \cos(t)$, $y(t) = 2 - \sin(t)$ for $0 \le t \le \frac{\pi}{2}$

33. **"Cycloid" with a square wheel**: Find the area under one "arch" of the path of a point on the corner of a "rolling" square with side length R.

34. Find the area of the region between the x-axis and the curate cycloid $x(t) = R \cdot t - r \cdot \sin(t)$, $y(t) = R - r \cdot \cos(t)$ for $0 \le t \le 2\pi$.

11.4 Practice Answers

1. When $t = 3$, $x = 28$, $y = 12$ and (using results from Example 1):

$$\frac{dy}{dx} = \frac{2t+1}{3t^2} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{t=3} = \frac{7}{27}$$

so an equation for the tangent line is $y = 12 + \frac{7}{27}(x - 28)$.

2. (a) When $t = 2$, $\frac{dy}{dx} \approx 0$; when $t = 5$, $\frac{dy}{dx} \approx -1$.
   (b) When $t \approx 2$ and $t \approx 4$ (according to the $y(t)$ graph).
   (c) A minimum occurs when $t \approx 2$ and a maximum when $t \approx 4$.
   (d) If the parametric graph has a maximum or minimum at $t = t^*$, then $\frac{dy}{dt}$ equals 0 or is undefined when $t = t^*$.

3. When $t = 1$:

$$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \approx \sqrt{1^2 + (-1)^2} = \sqrt{2} \approx 1.4\ \frac{\text{ft}}{\text{sec}}$$

When $t = 2$: speed $\approx \sqrt{(-1)^2 + 0^2} = 1\ \frac{\text{ft}}{\text{sec}}$

When $t = 3$: speed $\approx \sqrt{1^2 + 1^2} = \sqrt{2} \approx 1.4\ \frac{\text{ft}}{\text{sec}}$

4. Length $= \int_0^{2\pi} \sqrt{(-3\sin(t))^2 + (2\cos(t))^2}\, dt \approx 15.87$

5. (a) Area $= \int_0^2 t \cdot (4 - 2t)\ dt$ (b) $\int_0^2 \left[4t - 2t^2\right]\ dt = \left[2t^2 - \frac{2}{3}t^2\right]_0^2 = \frac{16}{3}$
   (c) No, it represents the area under the curve for $0 \le t \le 2$ minus the area under the curve for $2 \le t \le 3$.