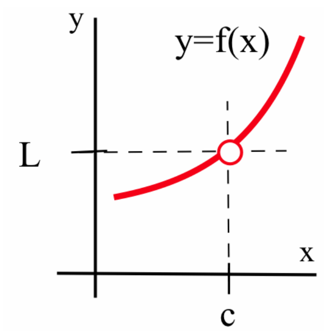
1.1 *The Limit of a Function*

Calculus has been called the study of continuous change, and the **limit** is the basic concept that allows us to describe and analyze such change. An understanding of limits is necessary to understand derivatives, integrals and other fundamental topics of calculus.



The symbol $\to$ means "approaches" or "gets very close to."

The Idea (Informally)

The limit of a function at a point describes the behavior of the function when the input variable is near — **but does not equal** — a specified number (see margin figure). If the values of $f(x)$ get closer and closer — as close as we want — to one number L as we take values of x very close to (but not equal to) a number c, then

> we say: "the limit of $f(x)$, as x approaches c, is L"
> and we write: $\lim_{x \to c} f(x) = L$

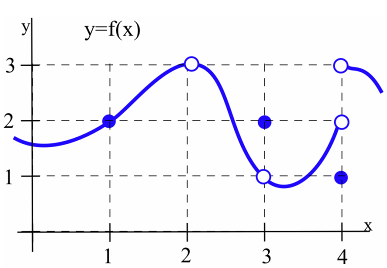
It is very important to note that:

> $f(c)$ is a single number that describes the behavior (value) of f **at** the point $x = c$

while:

> $\lim_{x \to c} f(x)$ is a single number that describes the behavior of f **near, but not at** the point $x = c$

If we have a graph of the function $f(x)$ near $x = c$, then it is usually easy to determine $\lim_{x \to c} f(x)$.



Example 1. Use the graph of $y = f(x)$ given in the margin to determine the following limits:

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 2} f(x)$ (c) $\lim_{x \to 3} f(x)$ (d) $\lim_{x \to 4} f(x)$

Solution. Each of these limits involves a different issue, as you may be able to tell from the graph.

(a) $\lim_{x \to 1} f(x) = 2$: When x is very close to 1, the values of $f(x)$ are very close to $y = 2$. In this example, it happens that $f(1) = 2$, but that is irrelevant for the limit. The only thing that matters is what happens for x *close to* 1 but with $x \neq 1$.

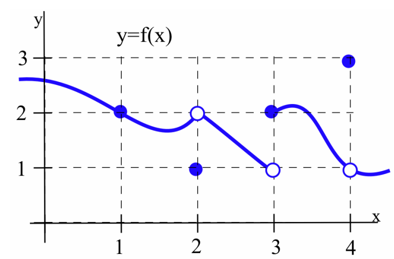
(b) $f(2)$ is undefined, but we only care about the behavior of $f(x)$ for x *close to* 2 and not equal to 2. When x is close to 2, the values of $f(x)$ are close to 3. If we restrict x close enough to 2, the values of y will be as close to 3 as we want, so $\lim_{x\to 2} f(x) = 3$.

(c) When x is close to 3, the values of $f(x)$ are close to 1, so $\lim_{x\to 3} f(x) = 1$. For this limit it is completely irrelevant that $f(3) = 2$: we only care about what happens to $f(x)$ for x close to and not equal to 3.

(d) This one is harder and we need to be careful. When x is close to 4 and slightly **less than** 4 (x is just to the left of 4 on the x-axis) then the values of $f(x)$ are close to 2. But if x is close to 4 and slightly **larger than** 4 then the values of $f(x)$ are close to 3.

If we know only that x is very close to 4, then we cannot say whether $y = f(x)$ will be close to 2 or close to 3—it depends on whether x is on the right or the left side of 4. In this situation, the $f(x)$ values are not close to a single number so we say $\lim_{x\to 4} f(x)$ **does not exist**.

In (d), it is irrelevant that $f(4) = 1$. The limit, as x approaches 4, would still be undefined if $f(4)$ was 3 or 2 or anything else. ◀



Practice 1. Use the graph of $y = f(x)$ in the margin to determine the following limits:

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{t\to 2} f(t)$ (c) $\lim_{x\to 3} f(x)$ (d) $\lim_{w\to 4} f(w)$

Example 2. Determine the value of $\lim_{x\to 3} \frac{2x^2 - x - 1}{x - 1}$.

Solution. We need to investigate the values of $f(x) = \frac{2x^2-x-1}{x-1}$ when x is close to 3. If the $f(x)$ values get arbitrarily close to—or even equal to—some number L, then L will be the limit.

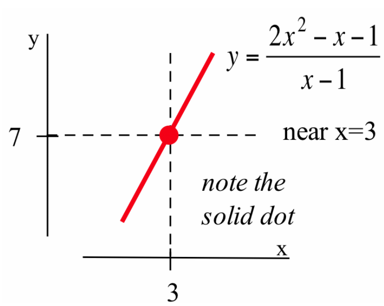
One way to keep track of both the x and the $f(x)$ values is to set up a table and to pick several x values that get closer and closer (but not equal) to 3.

We can pick some values of x that approach 3 from the left, say $x = 2.91$, 2.9997, 2.999993 and 2.9999999, and some values of x that approach 3 from the right, say $x = 3.1$, 3.004, 3.0001 and 3.000002. The only thing important about these particular values for x is that they get closer and closer to 3 without actually equaling 3. You should try some other values "close to 3" to see what happens. Our table of values is:

x	$f(x)$	x	$f(x)$
2.9	6.82	3.1	7.2
2.9997	6.9994	3.004	7.008
2.999993	6.999986	3.0001	7.0002
2.9999999	6.9999998	3.000002	7.000004
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
3	7	3	7

As the x values get closer and closer to 3, the $f(x)$ values are getting closer and closer to 7. In fact, we can get $f(x)$ as close to 7 as we want ("arbitrarily close") by taking the values of x very close ("sufficiently close") to 3. We write:

$$\lim_{x \to 3} \frac{2x^2 - x - 1}{x - 1} = 7$$



Instead of using a table of values, we could have graphed $y = f(x)$ for x close to 3 (see margin) and used the graph to answer the limit question. This graphical approach is easier, particularly if you have a calculator or computer do the graphing work for you, but it is really very similar to the "table of values" method: in each case you need to evaluate $y = f(x)$ at many values of x near 3. ◀

In the previous example, you might have noticed that if we just evaluate $f(3)$, then we get the correct answer, 7. That works for this particular problem, but it often fails. The next example (identical to the previous one, except $x \to 1$) illustrates one such difficulty.

Example 3. Find $\displaystyle\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1}$.

Solution. You might try to evaluate $f(x) = \frac{2x^2 - x - 1}{x - 1}$ at $x = 1$, but $f(1) = \frac{0}{0}$, so f is not defined at $x = 1$.

It is tempting — **but wrong** — to conclude that this function does not have a limit as x approaches 1.

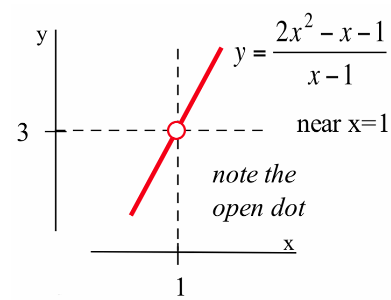
Table Method: Trying some "test" values for x that get closer and closer to 1 from both the left and the right, we get:

x	$f(x)$	x	$f(x)$
0.9	2.82	1.1	3.2
0.9998	2.9996	1.003	3.006
0.999994	2.999988	1.0001	3.0002
0.9999999	2.9999998	1.000007	3.000014
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
1	3	1	3

The function f is not defined at $x = 1$, but when x gets close to 1, the values of $f(x)$ get very close to 3. We can get $f(x)$ as close to 3 as we want by taking x very close to 1, so:

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = 3$$

Graph Method: We can graph $y = f(x) = \frac{2x^2-x-1}{x-1}$ for x close to 1 (see margin) and notice that whenever x is close to 1, the values of $y = f(x)$ are close to 3; f is not defined at $x = 1$, so the graph has a hole above $x = 1$, but we only care about what $f(x)$ is doing for x *close to* but **not equal to** 1.



Algebra Method: We could have found the same result by noting:

$$f(x) = \frac{2x^2 - x - 1}{x - 1} = \frac{(2x+1)(x-1)}{x - 1} = 2x + 1$$

as long as $x \neq 1$. The "$x \to 1$" part of the limit means that x is *close to* 1 but **not equal to** 1, so our division step is valid and:

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = \lim_{x \to 1} [2x + 1] = 3$$

which is the same answer we obtained using the first two methods. ◀

Three Methods for Evaluating Limits

The previous example utilized three different methods, each of which led us to the same answer for the limit.

The Algebra Method

The algebra method involves algebraically simplifying the function before trying to evaluate its limit. Often, this simplification just means factoring and dividing, but sometimes more complicated algebraic or even trigonometric steps are needed.

The Table Method

To evaluate a limit of a function $f(x)$ as x approaches c, the table method involves calculating the values of $f(x)$ for "enough" values of x very close to c so that we can "confidently" determine which value $f(x)$ is approaching. If $f(x)$ is well behaved, we may not need to use very many values for x. However, this method is usually used with complicated functions, and then we need to evaluate $f(x)$ for lots of values of x.

A computer or calculator can often make the function evaluations easier, but their calculations are subject to "round off" errors. The result of any computer calculation that involves both large and small numbers should be viewed with some suspicion. For example, the function

$$f(x) = \frac{((0.1)^x + 1) - 1}{(0.1)^x} = \frac{(0.1)^x}{(0.1)^x} = 1$$

for every value of x, and my calculator gives the correct answer for some values of x: $f(3) = 1$, and $f(8)$ and $f(9)$ both equal 1.

But my calculator says $\left((0.1)^{10} + 1\right) - 1 = 0$, so it evaluates $f(10)$ to be 0, definitely an incorrect value.

Your calculator may evaluate $f(10)$ correctly, but try $f(35)$ or $f(107)$.

Calculators are too handy to be ignored, but they are too prone to these types of errors to be believed uncritically. Be careful.

The Graph Method

The graph method is closely related to the table method, but we create a graph of the function instead of a table of values, and then we use the graph to determine which value $f(x)$ is approaching.

Which Method Should You Use?

In general, the algebraic method is preferred because it is precise and does not depend on which values of x we chose or the accuracy of our graph or precision of our calculator. **If you can evaluate a limit algebraically, you should do so.** Sometimes, however, it will be very difficult to evaluate a limit algebraically, and the table or graph methods offer worthwhile alternatives. Even when you can algebraically evaluate the limit of a function, it is still a good idea to graph the function or evaluate it at a few points just to verify your algebraic answer.

The table and graph methods have the same advantages and disadvantages. Both can be used on complicated functions that are difficult to handle algebraically or whose algebraic properties you don't know.

Often both methods can be easily programmed on a calculator or computer. However, these two methods are very time-consuming by hand and are prone to round-off errors on computers. You need to know how to use these methods when you can't figure out how to use the algebraic method, but you need to use these two methods warily.

Example 4. Evaluate each limit.

(a) $\displaystyle\lim_{x\to 0} \frac{x^2+5x+6}{x^2+3x+2}$ (b) $\displaystyle\lim_{x\to -2} \frac{x^2+5x+6}{x^2+3x+2}$

Solution. The function in each limit is the same but x is approaching a different number in each of them.

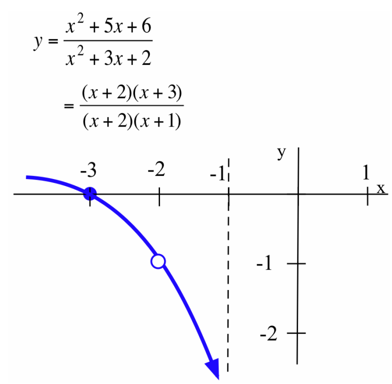
(a) Because $x \to 0$, we know that x is getting closer and closer to 0, so the values of the x^2, $5x$ and $3x$ terms get as close to 0 as we want. The numerator approaches 6 and the denominator approaches 2, so the values of the whole function get arbitrarily close to $\frac{6}{2} = 3$, the limit.

(b) As x approaches -2, the numerator and denominator approach 0, and a small number divided by a small number can be almost anything—the ratio depends on the size of the top compared to the size of the bottom. More investigation is needed.

Table Method: If we pick some values of x close to (but not equal to) -2, we get the table:

x	x^2+5x+6	x^2+3x+2	$\frac{x^2+5x+6}{x^2+3x+2}$
-1.97	0.0309	-0.0291	-1.061856
-2.005	-0.004975	0.005025	-0.990050
-1.9998	0.00020004	-0.00019996	-1.00040008
-2.00003	-0.00002999	0.0000300009	-0.9996666
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
-2	0	0	-1

Even though the numerator and denominator are each getting closer and closer to 0, their ratio is getting arbitrarily close to -1, which is the limit.



Graph Method: The graph of $y = f(x) = \frac{x^2+5x+6}{x^2+3x+2}$ in the margin shows that the values of $f(x)$ are very close to -1 when the x-values are close to -2.

Algebra Method: Factoring the numerator and denominator:

$$f(x) = \frac{x^2+5x+6}{x^2+3x+2} = \frac{(x+2)(x+3)}{(x+2)(x+1)}$$

We know $x \to -2$ so $x \neq -2$ and we can divide the top and bottom by $(x+2)$. Then

$$f(x) = \frac{(x+3)}{(x+1)} \to \frac{1}{-1} = -1$$

as $x \to -2$. ◀

You should remember the technique used in the previous example:

> If $\lim\limits_{x \to c} \frac{\text{polynomial}}{\text{another polynomial}} = \frac{0}{0}$,
> try dividing the top and bottom by $x - c$.

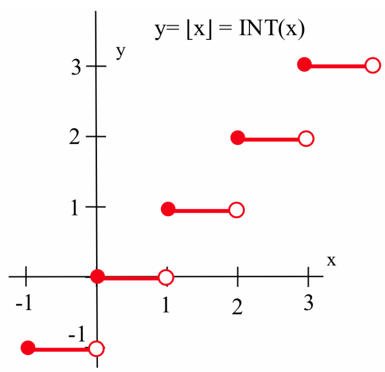
Practice 2. Evaluate each limit.

(a) $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$

(b) $\lim\limits_{t \to 0} \dfrac{t \cdot \sin(t)}{t^2 + 3t}$

(c) $\lim\limits_{w \to 2} \dfrac{w - 2}{\ln(\frac{w}{2})}$

One-Sided Limits



Sometimes, what happens to us at a place depends on the direction we use to approach that place. If we approach Niagara Falls from the upstream side, then we will be 182 feet higher and have different worries than if we approach from the downstream side. Similarly, the values of a function near a point may depend on the direction we use to approach that point.

If we let x approach 3 from the left (x is close to 3 and $x < 3$) then the values of $\lfloor x \rfloor = \text{INT}(x)$ equal 2 (see margin).

If we let x approach 3 from the right (x is close to 3 and $x > 3$) then the values of $\lfloor x \rfloor = \text{INT}(x)$ equal 3.

On the number line we can approach a point from the left or the right, and that leads to **one-sided limits**.

Definition of Left and Right Limits:
The **left limit** as x approaches c of $f(x)$ is L if the values of $f(x)$ get as close to L as we want when x is very close to but left of c ($x < c$):

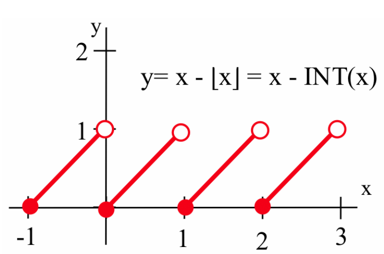
$$\lim_{x \to c^-} f(x) = L$$

The **right limit**, $\lim\limits_{x \to c^+} f(x)$, requires that x lie to the right of c ($x > c$).

Example 5. Evaluate $\lim\limits_{x \to 2^-} x - \lfloor x \rfloor$ and $\lim\limits_{x \to 2^+} x - \lfloor x \rfloor$.

Solution. The left-limit notation $x \to 2^-$ requires that x be close to 2 and that x be to the left of 2, so $x < 2$. If $1 < x < 2$, then $\lfloor x \rfloor = 1$ and:

$$\lim_{x \to 2^-} x - \lfloor x \rfloor = \lim_{x \to 2^-} x - 1 = 2 - 1 = 1$$



If x is close to 2 and is to the right of 2, then $2 < x < 3$, so $\lfloor x \rfloor = 2$ and:

$$\lim_{x \to 2^+} x - \lfloor x \rfloor = \lim_{x \to 2^+} x - 2 = 2 - 2 = 0$$

A graph of $f(x) = x - \lfloor x \rfloor$ appears in the margin. ◀

If the left and right limits of $f(x)$ have the same value at $x = c$:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

then the value of $f(x)$ is close to L whenever x is close to c, and it does not matter whether x is left or right of c, so

$$\lim_{x \to c} f(x) = L$$

Similarly, if:

$$\lim_{x \to c} f(x) = L$$

then $f(x)$ is close to L whenever x is close to c and less than c, and whenever x is close to c and greater than c, so:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

We can combine these two statements into a single theorem.

One-Sided Limit Theorem:

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$

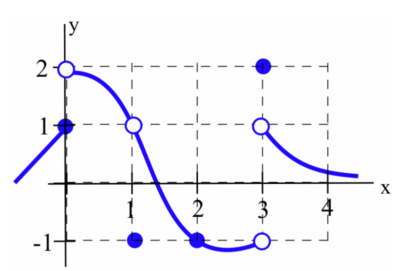
This theorem has an important corollary.

Corollary:

If $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, then $\lim_{x \to c} f(x)$ does not exist.

One-sided limits are particularly useful for describing the behavior of functions that have steps or jumps.

To determine the limit of a function involving the greatest integer or absolute value or a multiline definition, definitely consider both the left and right limits.



Practice 3. Use the graph in the margin to evaluate the one- and two-sided limits of f at $x = 0$, 1, 2 and 3.

Practice 4. Defining $f(x)$ as:

$$f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x < 3 \\ 2 & \text{if } 3 < x \end{cases}$$

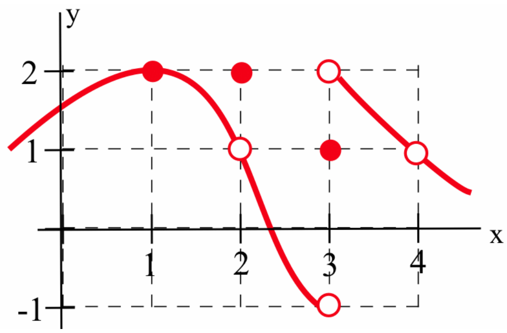
find the one- and two-sided limits of f at 1 and 3.

1.1 Problems

1. Use the graph below to determine the limits.

   (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$

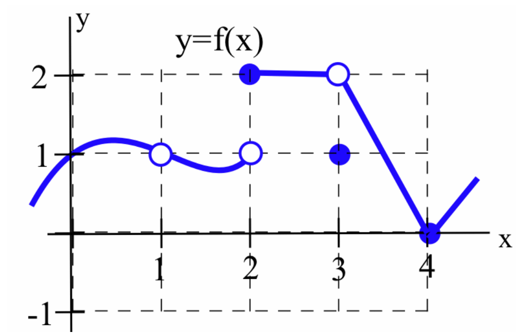
   (c) $\lim_{x\to 3} f(x)$ (d) $\lim_{x\to 4} f(x)$



2. Use the graph below to determine the limits.

   (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$

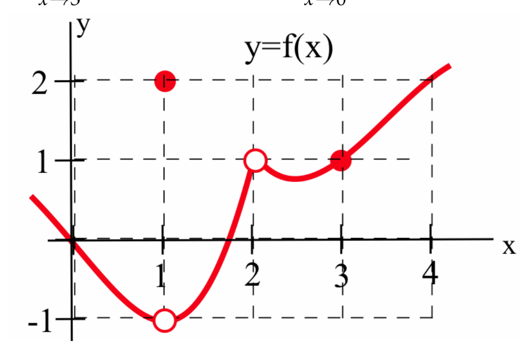
   (c) $\lim_{x\to 3} f(x)$ (d) $\lim_{x\to 4} f(x)$



3. Use the graph below to determine the limits.

   (a) $\lim_{x\to 1} f(2x)$ (b) $\lim_{x\to 2} f(x-1)$

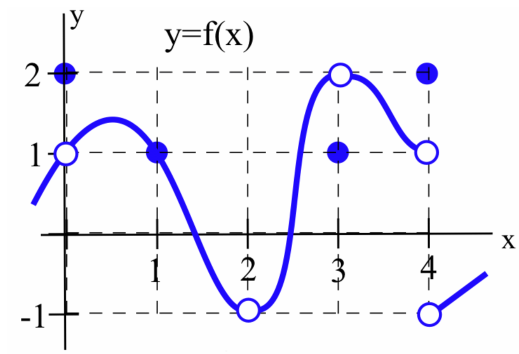
   (c) $\lim_{x\to 3} f(2x-5)$ (d) $\lim_{x\to 0} f(4+x)$



4. Use the graph below to determine the limits.

   (a) $\lim_{x\to 1} f(3x)$ (b) $\lim_{x\to 2} f(x+1)$

   (c) $\lim_{x\to 3} f(2x-4)$ (d) $\lim_{x\to 0} |f(4+x)|$



In Problems 5–11, evaluate each limit.

5. (a) $\displaystyle\lim_{x\to 1} \frac{x^2+3x+3}{x-2}$ (b) $\displaystyle\lim_{x\to 2} \frac{x^2+3x+3}{x-2}$

6. (a) $\displaystyle\lim_{x\to 0} \frac{x+7}{x^2+9x+14}$ (b) $\displaystyle\lim_{x\to 3} \frac{x+7}{x^2+9x+14}$

   (c) $\displaystyle\lim_{x\to -4} \frac{x+7}{x^2+9x+14}$ (d) $\displaystyle\lim_{x\to -7} \frac{x+7}{x^2+9x+14}$

7. (a) $\displaystyle\lim_{x\to 1} \frac{\cos(x)}{x}$ (b) $\displaystyle\lim_{x\to \pi} \frac{\cos(x)}{x}$

   (c) $\displaystyle\lim_{x\to -1} \frac{\cos(x)}{x}$

8. (a) $\displaystyle\lim_{x\to 7} \sqrt{x-3}$ (b) $\displaystyle\lim_{x\to 9} \sqrt{x}-3$

   (c) $\displaystyle\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

9. (a) $\displaystyle\lim_{x\to 0^-} |x|$ (b) $\displaystyle\lim_{x\to 0^+} |x|$

   (c) $\displaystyle\lim_{x\to 0} |x|$

10. (a) $\displaystyle\lim_{x\to 0^-} \frac{|x|}{x}$ (b) $\displaystyle\lim_{x\to 0^+} \frac{|x|}{x}$

    (c) $\displaystyle\lim_{x\to 0} \frac{|x|}{x}$

11. (a) $\lim_{x \to 5} |x-5|$ (b) $\lim_{x \to 3} \frac{|x-5|}{x-5}$

(c) $\lim_{x \to 5} \frac{|x-5|}{x-5}$

12. Find the one- and two-sided limits of:

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ \sin(x) & \text{if } 0 < x \leq 2 \\ 1 & \text{if } 2 < x \end{cases}$$

as $x \to 0$, 1 and 2.

13. Find the one- and two-sided limits of:

$$g(x) = \begin{cases} 1 & \text{if } x \leq 2 \\ \frac{8}{x} & \text{if } 2 < x < 4 \\ 6-x & \text{if } 4 < x \end{cases}$$

as $x \to 1$, 2, 4 and 5.

In 14–17, use a calculator or computer to get approximate answers accurate to 2 decimal places.

14. (a) $\lim_{x \to 0} \frac{2^x - 1}{x}$ (b) $\lim_{x \to 1} \frac{\log_{10}(x)}{x-1}$

15. (a) $\lim_{x \to 0} \frac{3^x - 1}{x}$ (b) $\lim_{x \to 1} \frac{\ln(x)}{x-1}$

16. (a) $\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x-5}$ (b) $\lim_{x \to 0} \frac{\sin(3x)}{5x}$

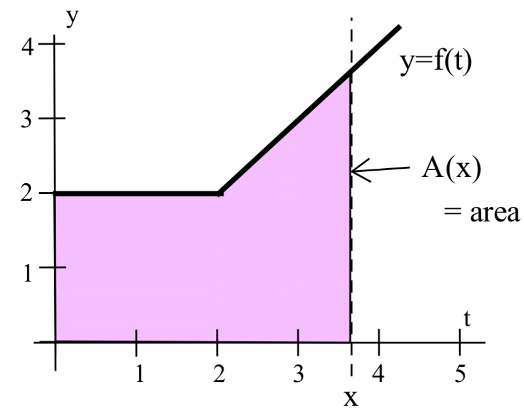
17. (a) $\lim_{x \to 16} \frac{\sqrt{x}-4}{x-16}$ (b) $\lim_{x \to 0} \frac{\sin(7x)}{2x}$

18. Define $A(x)$ to be the **area** bounded by the t- and y-axes, the "bent line" in the figure below, and the vertical line $t = x$. For example, $A(4) = 10$.

(a) Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.

(b) Graph $y = A(x)$ for $0 \leq x \leq 4$.

(c) What area does $A(3) - A(1)$ represent?

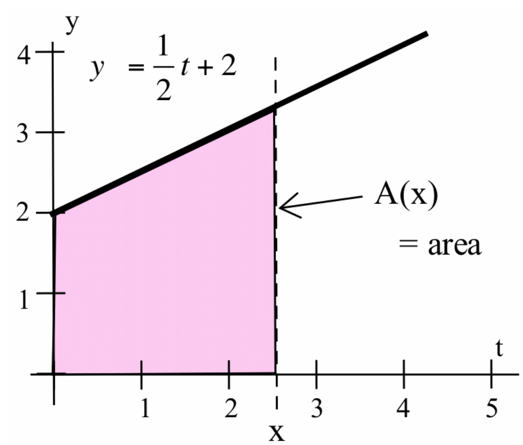


19. Define $A(x)$ to be the **area** bounded by the t- and y-axes, the line $y = \frac{1}{2}t + 2$ and the vertical line $t = x$ (See figure below). For example, $A(4) = 12$.

(a) Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.

(b) Graph $y = A(x)$ for $0 \leq x \leq 4$.

(c) What area does $A(3) - A(1)$ represent?



20. Sketch the graph of $f(t) = \sqrt{4t - t^2}$ for $0 \leq t \leq 4$ (you should get a semicircle). Define $A(x)$ to be the area bounded below by the t-axis, above by the graph $y = f(t)$ and on the right by the vertical line at $t = x$.

(a) Evaluate $A(0)$, $A(2)$ and $A(4)$.

(b) Sketch a graph $y = A(x)$ for $0 \leq x \leq 4$.

(c) What area does $A(3) - A(1)$ represent?

1.1 Practice Answers

1. (a) 2
   (b) 2
   (c) does not exist (no limit)
   (d) 1

2. (a) $\lim_{x \to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x \to 2} (x+1) = 3$
   (b) $\lim_{t \to 0} \frac{t \sin(t)}{t(t+3)} = \lim_{t \to 0} \frac{\sin(t)}{t+3} = \frac{0}{3} = 0$
   (c) $\lim_{w \to 2} \frac{w-2}{\ln(\frac{w}{2})} = 2$ To see this, make a graph or a table:

w	$\frac{w-2}{\ln(\frac{w}{2})}$	w	$\frac{w-2}{\ln(\frac{w}{2})}$
2.2	2.098411737	1.9	1.949572575
2.01	2.004995844	1.99	1.994995823
2.003	2.001499625	1.9992	1.999599973
2.0001	2.00005	1.9999	1.99995
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
2	2	2	2

3. $\lim_{x \to 0^-} f(x) = 1$ $\quad \lim_{x \to 0^+} f(x) = 2$ $\quad \lim_{x \to 0} f(x)$ DNE

   $\lim_{x \to 1^-} f(x) = 1$ $\quad \lim_{x \to 1^+} f(x) = 1$ $\quad \lim_{x \to 1} f(x) = 1$

   $\lim_{x \to 2^-} f(x) = -1$ $\quad \lim_{x \to 2^+} f(x) = -1$ $\quad \lim_{x \to 2} f(x) = -1$

   $\lim_{x \to 3^-} f(x) = -1$ $\quad \lim_{x \to 3^+} f(x) = 1$ $\quad \lim_{x \to 3} f(x)$ DNE

4. $\lim_{x \to 1^-} f(x) = 1$ $\quad \lim_{x \to 1^+} f(x) = 1$ $\quad \lim_{x \to 1} f(x) = 1$

   $\lim_{x \to 3^-} f(x) = 3$ $\quad \lim_{x \to 3^+} f(x) = 2$ $\quad \lim_{x \to 3} f(x)$ DNE