2.5 *Applications of the Chain Rule*

The Chain Rule can help us determine the derivatives of logarithmic functions like $f(x) = \ln(x)$ and general exponential functions like $f(x) = a^x$. We will also use it to answer some applied questions and to find slopes of graphs given by parametric equations.

Derivatives of Logarithms

You know from precalculus that the natural logarithm $\ln(x)$ is defined as the inverse of the exponential function e^x: $e^{\ln(x)} = x$ for $x > 0$. We can use this identity along with the Chain Rule to determine the derivative of the natural logarithm.

$$\mathbf{D}(\ln(x)) = \frac{1}{x} \quad \text{and} \quad \mathbf{D}\left(\ln(g(x))\right) = \frac{g'(x)}{g(x)}$$

Proof. We know that $\mathbf{D}(e^u) = e^u$, so using the Chain Rule we have $\mathbf{D}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$. Differentiating each side of the identity $e^{\ln(x)} = x$, we get:

$$\mathbf{D}\left(e^{\ln(x)}\right) = \mathbf{D}(x) \Rightarrow e^{\ln(x)} \cdot \mathbf{D}(\ln(x)) = 1$$
$$\Rightarrow x \cdot \mathbf{D}(\ln(x)) = 1 \Rightarrow \mathbf{D}(\ln(x)) = \frac{1}{x}$$

The function $\ln(g(x))$ is the composition of $f(x) = \ln(x)$ with $g(x)$ so the Chain Rule says:

$$\mathbf{D}\left(\ln(g(x)\right) = \mathbf{D}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

You can remember the differentiation pattern for the the natural logarithm in words as: "one over the inside times the the derivative of the inside."

Graph $f(x) = \ln(x)$ along with $f'(x) = \dfrac{1}{x}$ and compare the behavior of the function at various points with the values of its derivative at those points. Does $y = \dfrac{1}{x}$ possess the properties you would expect to see from the derivative of $f(x) = \ln(x)$? □

Example 1. Find $\mathbf{D}(\ln(\sin(x)))$ and $\mathbf{D}(\ln(x^2 + 3))$.

Solution. Using the pattern $\mathbf{D}(\ln(g(x)) = \dfrac{g'(x)}{g(x)}$ with $g(x) = \sin(x)$:

$$\mathbf{D}(\ln(\sin(x))) = \frac{g'(x)}{g(x)} = \frac{\mathbf{D}(\sin(x))}{\sin(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

With $g(x) = x^2 + 3$, $\mathbf{D}(\ln(x^2 + 3)) = \dfrac{g'(x)}{g(x)} = \dfrac{2x}{x^2 + 3}$. ◀

We can use the Change of Base Formula from precalculus to rewrite any logarithm as a natural logarithm, and then we can differentiate the resulting natural logarithm.

Change of Base Formula for Logarithms:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \text{ for all positive } a, b \text{ and } x.$$

Your calculator likely has two logarithm buttons: **ln** for the natural logarithm (base e) and **log** for the common logarithm (base 10). Be careful, however, as more advanced mathematics texts (as well as the Web site Wolfram | Alpha) use log for the (base e) natural logarithm.

Example 2. Use the Change of Base formula and your calculator to find $\log_\pi(7)$ and $\log_2(8)$.

Solution. $\log_\pi(7) = \dfrac{\ln(7)}{\ln(\pi)} \approx \dfrac{1.946}{1.145} \approx 1.700$. (Check that $\pi^{1.7} \approx 7$.)
Likewise, $\log_2(8) = \dfrac{\ln(8)}{\ln(2)} = 3$. ◀

Practice 1. Find the values of $\log_9 20$, $\log_3 20$ and $\log_\pi e$.

Putting $b = e$ in the Change of Base Formula, $\log_a(x) = \dfrac{\log_e(x)}{\log_e(a)} = \dfrac{\ln(x)}{\ln(a)}$, so any logarithm can be written as a natural logarithm divided by a constant. This makes any logarithmic function easy to differentiate.

$$\mathbf{D}\left(\log_a(x)\right) = \frac{1}{x\ln(a)} \quad \text{and} \quad \mathbf{D}\left(\log_a(f(x))\right) = \frac{f'(x)}{f(x)} \cdot \frac{1}{\ln(a)}$$

Proof. $\mathbf{D}\left(\log_a(x)\right) = \mathbf{D}\left(\dfrac{\ln x}{\ln a}\right) = \dfrac{1}{\ln(a)} \cdot \mathbf{D}(\ln x) = \dfrac{1}{\ln(a)} \cdot \dfrac{1}{x} = \dfrac{1}{x\ln(a)}$.
The second differentiation formula follows from the Chain Rule. □

Practice 2. Calculate $\mathbf{D}\left(\log_{10}(\sin(x))\right)$ and $\mathbf{D}\left(\log_\pi(e^x)\right)$.

The number e might seem like an "unnatural" base for a natural logarithm, but of all the possible bases, the logarithm with base e has the nicest and easiest derivative. The natural logarithm is even related to the distribution of prime numbers. In 1896, the mathematicians Hadamard and Vallée-Poussin proved the following conjecture of Gauss (the Prime Number Theorem): For large values of N,

$$\text{number of primes less than } N \approx \frac{N}{\ln(N)}$$

Derivative of a^x

Once we know the derivative of e^x and the Chain Rule, it is relatively easy to determine the derivative of a^x for any $a > 0$.

$$\mathbf{D}(a^x) = a^x \cdot \ln(a) \text{ for } a > 0.$$

Proof. If $a > 0$, then $a^x > 0$ and $a^x = e^{\ln(a^x)} = e^{x\cdot\ln(a)}$, so we have: $\mathbf{D}(a^x) = \mathbf{D}\left(e^{\ln(a^x)}\right) = \mathbf{D}\left(e^{x\cdot\ln(a)}\right) = e^{x\cdot\ln(a)} \cdot \mathbf{D}(x \cdot \ln(a)) = a^x \cdot \ln(a)$. □

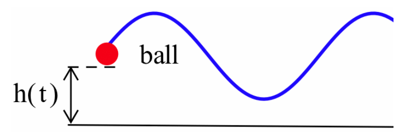
Example 3. Calculate $\mathbf{D}(7^x)$ and $\frac{d}{dt}\left(2^{\sin(t)}\right)$.

Solution. $\mathbf{D}(7^x) = 7^x \cdot \ln(7) \approx (1.95)7^x$. We can write $y = 2^{\sin(t)}$ as $y = 2^u$ with $u = \sin(t)$. Using the Chain Rule: $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = 2^u \cdot \ln(2)\cos(t) = 2^{\sin(t)} \cdot \ln(2) \cdot \cos(t)$. ◀

Practice 3. Calculate $\mathbf{D}\left(\sin(2^x)\right)$ and $\frac{d}{dt}\left(3^{t^2}\right)$.

Some Applied Problems

Let's examine some applications involving more complicated functions.



Example 4. A ball at the end of a rubber band (see margin) is oscillating up and down, and its height (in feet) above the floor at time t seconds is $h(t) = 5 + 2\sin\left(\frac{t}{2}\right)$ (with t in radians).

(a) How fast is the ball traveling after 2 seconds? After 4 seconds? After 60 seconds?

(b) Is the ball moving up or down after 2 seconds? After 4 seconds? After 60 seconds?

(c) Is the vertical velocity of the ball ever 0?

Solution. (a) $v(t) = h'(t) = \mathbf{D}\left(5 + 2\sin\left(\frac{t}{2}\right)\right) = 2\cos\left(\frac{t}{2}\right) \cdot \frac{1}{2}$ so $v(t) = \cos\left(\frac{t}{2}\right)$ feet/second: $v(2) = \cos(\frac{2}{2}) \approx 0.540$ ft/s, $v(4) = \cos(\frac{4}{2}) \approx -0.416$ ft/s, and $v(60) = \cos(\frac{60}{2}) \approx 0.154$ ft/s.

(b) The ball is moving up at $t = 2$ and $t = 60$, down when $t = 4$.

(c) $v(t) = \cos\left(\frac{t}{2}\right) = 0$ when $\frac{t}{2} = \frac{\pi}{2} \pm k \cdot \pi \Rightarrow t = \pi \pm 2\pi k$ for any integer k. ◀

Example 5. If 2,400 people now have a disease, and the number of people with the disease appears to double every 3 years, then the number of people expected to have the disease in t years is $y = 2400 \cdot 2^{\frac{t}{3}}$.

(a) How many people are expected to have the disease in 2 years?

(b) When are 50,000 people expected to have the disease?

(c) How fast is the number of people with the disease growing now? How fast is it expected to be growing 2 years from now?

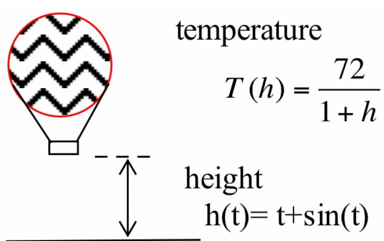
Solution. (a) In 2 years, $y = 2400 \cdot 2^{\frac{2}{3}} \approx 3{,}810$ people.

(b) We know $y = 50000$ and need to solve $50000 = 2400 \cdot 2^{\frac{t}{3}}$ for t. Taking logarithms of each side of the equation: $\ln(50000) = \ln\left(2400 \cdot 2^{\frac{2}{3}}\right) = \ln(2400) + \frac{t}{3} \cdot \ln(2)$ so $10.819 \approx 7.783 + 0.231t$ and $t \approx 13.14$ years. We expect 50,000 people to have the disease about 13 years from now.

(c) This question asks for $\frac{dy}{dt}$ when $t = 0$ and $t = 2$.

$$\frac{dy}{dt} = \frac{d}{dt}\left(2400 \cdot 2^{\frac{t}{3}}\right) = 2400 \cdot 2^{\frac{t}{3}} \cdot \ln(2) \cdot \frac{1}{3} \approx 554.5 \cdot 2^{\frac{t}{3}}$$

Now, at $t = 0$, the rate of growth of the disease is approximately $554.5 \cdot 2^0 \approx 554.5$ people/year. In 2 years, the rate of growth will be approximately $554.5 \cdot 2^{\frac{2}{3}} \approx 880$ people/year. ◀



Example 6. You are riding in a balloon, and at time t (in minutes) you are $h(t) = t + \sin(t)$ thousand feet above sea level. If the temperature at an elevation h is $T(h) = \frac{72}{1+h}$ degrees Fahrenheit, then how fast is the temperature changing when $t = 5$ minutes?

Solution. As t changes, your elevation will change. And, as your elevation changes, so will the temperature. It is not difficult to write the temperature as a function of time, and then we could calculate $\frac{dT}{dt} = T'(t)$ and evaluate $T'(5)$. Or we could use the Chain Rule:

$$\frac{dT}{dt} = \frac{dT}{dh} \cdot \frac{dh}{dt} = -\frac{72}{(1+h)^2} \cdot (1 + \cos(t))$$

At $t = 5$, $h(5) = 5 + \sin(5) \approx 4.04$ so $T'(5) \approx -\frac{72}{(1+4.04)^2} \cdot (1 + 0.284) \approx -3.64\ {}^\circ$/minute. ◀

Practice 4. Write the temperature T in the previous example as a function of the variable t alone and then differentiate T to determine the value of $\frac{dT}{dt}$ when $t = 5$ minutes.

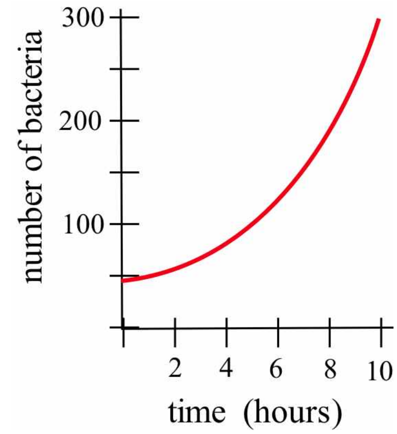
Example 7. A scientist has determined that, under optimum conditions, an initial population of 40 bacteria will grow "exponentially" to $f(t) = 40 \cdot e^{\frac{t}{5}}$ bacteria after t hours.

(a) Graph $y = f(t)$ for $0 \le t \le 15$. Calculate $f(0)$, $f(5)$ and $f(10)$.

(b) How fast is the population increasing at time t? (Find $f'(t)$.)

(c) Show that the rate of population increase, $f'(t)$, is proportional to the population, $f(t)$, at any time t. (Show $f'(t) = K \cdot f(t)$ for some constant K.)

Solution. (a) The graph of $y = f(t)$ appears in the margin. $f(0) = 40 \cdot e^{\frac{0}{5}} = 40$ bacteria, $f(5) = 40 \cdot e^{\frac{5}{5}} = 40e \approx 109$ bacteria and $f(10) = 40 \cdot e^{\frac{10}{5}} \approx 296$ bacteria.

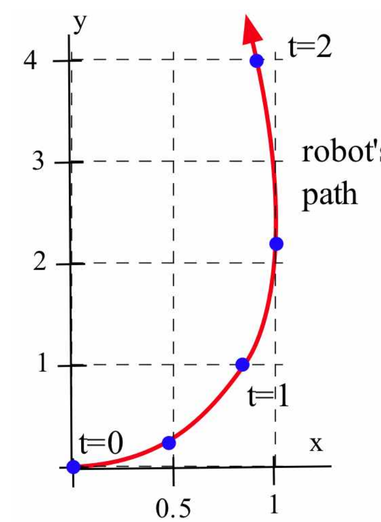


(b) $f'(t) = \frac{d}{dt}(f(t)) = \frac{d}{dt}\left(40 \cdot e^{\frac{t}{5}}\right) = 40 \cdot e^{\frac{t}{5}} \cdot \frac{d}{dt}\left(\frac{t}{5}\right) = 40 \cdot e^{\frac{t}{5}} \cdot \frac{1}{5} = 8 \cdot e^{\frac{t}{5}}$ bacteria/hour.

(c) $f'(t) = 8 \cdot e^{\frac{t}{5}} = \frac{1}{5} \cdot 40e^{\frac{t}{5}} = \frac{1}{5} f(t)$ so $f'(t) = K \cdot f(t)$ with $K = \frac{1}{5}$. The rate of change of the population is proportional to its size. ◀

Parametric Equations

Suppose a robot has been programmed to move in the xy-plane so at time t its x-coordinate will be $\sin(t)$ and its y-coordinate will be t^2. Both x and y are functions of the independent parameter t: $x(t) = \sin(t)$ and $y(t) = t^2$. The path of the robot (see margin) can be found by plotting $(x, y) = (x(t), y(t))$ for lots of values of t.



t	$x(t) = \sin(t)$	$y(t) = t^2$	point
0	0	0	$(0, 0)$
0.5	0.48	0.25	$(0.48, 0.25)$
1.0	0.84	1	$(0.84, 1)$
1.5	1.00	2.25	$(1, 2.25)$
2.0	0.91	4	$(0.91, 4)$

Typically we know $x(t)$ and $y(t)$ and need to find $\frac{dy}{dx}$, the slope of the tangent line to the graph of $(x(t), y(t))$. The Chain Rule says:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

so , algebraically solving for $\frac{dy}{dx}$, we get:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

If we can calculate $\frac{dy}{dt}$ and $\frac{dx}{dt}$, the derivatives of y and x with respect to the parameter t, then we can determine $\frac{dy}{dx}$, the rate of change of y with respect to x.

> If $x = x(t)$ and $y = y(t)$ are differentiable with respect to t and $\frac{dx}{dt} \neq 0$
>
> then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Example 8. Find the slope of the tangent line to the graph of $(x, y) = (\sin(t), t^2)$ when $t = 2$.

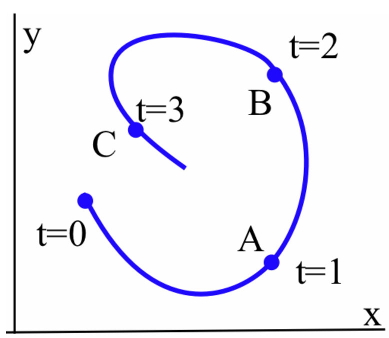
Solution. $\frac{dx}{dt} = \cos(t)$ and $\frac{dy}{dt} = 2t$. When $t = 2$, the object is at the point $(\sin(2), 2^2) \approx (0.91, 4)$ and the slope of the tangent line is:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\cos(t)} = \frac{2 \cdot 2}{\cos(2)} \approx \frac{4}{-0.42} \approx -9.61$$

Notice in the figure that the slope of the tangent line to the curve at $(0.91, 4)$ is negative and very steep. ◀

Practice 5. Graph $(x, y) = (3\cos(t), 2\sin(t))$ and find the slope of the tangent line when $t = \frac{\pi}{2}$.

When we calculated $\frac{dy}{dx}$, the slope of the tangent line to the graph of $(x(t), y(t))$, we used the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Each of these also has a geometric meaning: $\frac{dx}{dt}$ measures the rate of change of $x(t)$ with respect to t: it tells us whether the x-coordinate is increasing or decreasing as the t-variable increases (and how fast it is changing), while $\frac{dy}{dt}$ measures the rate of change of $y(t)$ with respect to t.



Example 9. For the parametric graph in the margin, determine whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive or negative when $t = 2$.

Solution. As we move through the point B (where $t = 2$) in the direction of increasing values of t, we are moving to the left, so $x(t)$ is decreasing and $\frac{dx}{dt} < 0$. The values of $y(t)$ are increasing, so $\frac{dy}{dt} > 0$. Finally, the slope of the tangent line, $\frac{dy}{dx}$, is negative. ◀

As a check on the sign of $\frac{dy}{dx}$ in the previous example:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\text{positive}}{\text{negative}} = \text{negative}$$

Practice 6. For the parametric graph in the previous example, tell whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive or negative at $t = 1$ and $t = 3$.

Speed

If we know the position of an object at any time, then we can determine its speed. The formula for speed comes from the distance formula and looks a lot like it, but involves derivatives.

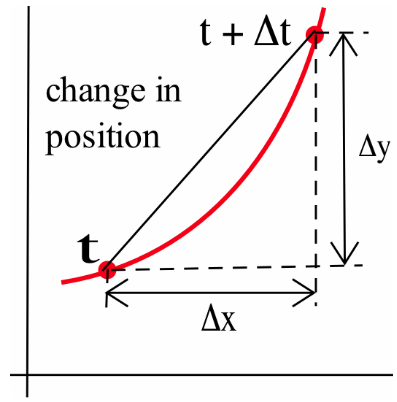
If	$x = x(t)$ and $y = y(t)$ give the location of an object at time t and both are differentiable functions of t
then	the speed of the object is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Proof. The speed of an object is the limit, as $\Delta t \to 0$, of (see margin):

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} = \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta t)^2}}$$
$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \to \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

as $\Delta t \to 0$. □



Example 10. Find the speed of the object whose location at time t is $(x, y) = \left(\sin(t), t^2\right)$ when $t = 0$ and $t = 1$.

Solution. $\frac{dx}{dt} = \cos(t)$ and $\frac{dy}{dt} = 2t$ so:

$$\text{speed } = \sqrt{(\cos(t))^2 + (2t)^2} = \sqrt{\cos^2(t) + 4t^2}$$

When $t = 0$, speed $= \sqrt{\cos^2(0) + 4(0)^2} = \sqrt{1+0} = 1$. When $t = 1$, speed $= \sqrt{\cos^2(1) + 4(1)^2} \approx \sqrt{0.29 + 4} \approx 2.07$. ◀

Practice 7. Show that an object located at $(x, y) = (3\sin(t), 3\cos(t))$ at time t has a constant speed. (This object is moving on a circular path.)

Practice 8. Is the object at $(x, y) = (3\cos(t), 2\sin(t))$ at time t traveling faster at the top of the ellipse ($t = \frac{\pi}{2}$) or at the right edge ($t = 0$)?

2.5 Problems

In Problems 1–27, differentiate the given function.

1. $\ln(5x)$
2. $\ln(x^2)$
3. $\ln(x^k)$
4. $\ln(x^x) = x \cdot \ln(x)$
5. $\ln(\cos(x))$
6. $\cos(\ln(x))$
7. $\log_2(5x)$
8. $\log_2(kx)$
9. $\ln(\sin(x))$
10. $\ln(kx)$
11. $\log_2(\sin(x))$
12. $\ln(e^x)$
13. $\log_5(5^x)$
14. $\ln\left(e^{f(x)}\right)$
15. $x \cdot \ln(3x)$
16. $e^x \cdot \ln(x)$
17. $\dfrac{\ln(x)}{x}$
18. $\sqrt{x + \ln(3x)}$

19. $\ln\left(\sqrt{5x-3}\right)$

20. $\ln(\cos(t))$

21. $\cos(\ln(w))$

22. $\ln(ax+b)$

23. $\ln\left(\sqrt{t+1}\right)$

24. 3^x

25. $5^{\sin(x)}$

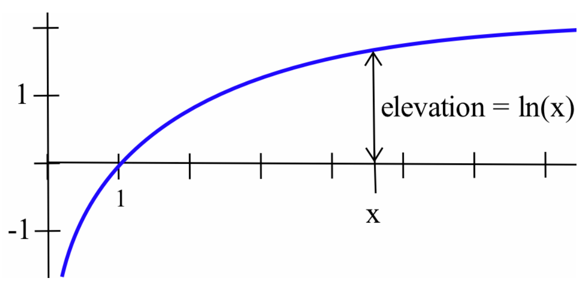
26. $x \cdot \ln(x) - x$

27. $\ln(\sec(x)+\tan(x))$

28. Find the slope of the line tangent to $f(x) = \ln(x)$ at the point $(e, 1)$. Find the slope of the line tangent to $g(x) = e^x$ at the point $(1, e)$. How are the slopes of f and g at these points related?

29. Find a point P on the graph of $f(x) = \ln(x)$ so the tangent line to f at P goes through the origin.

30. You are moving from left to right along the graph of $y = \ln(x)$ (see figure below).
    (a) If the x-coordinate of your location at time t seconds is $x(t) = 3t+2$, then how fast is your elevation increasing?
    (b) If the x-coordinate of your location at time t seconds is $x(t) = e^t$, then how fast is your elevation increasing?



31. The percent of a population, $p(t)$, who have heard a rumor by time t is often modeled by $p(t) = \dfrac{100}{1+Ae^{-t}} = 100\left(1+Ae^{-t}\right)^{-1}$ for some positive constant A. Calculate $p'(t)$, the rate at which the rumor is spreading.

32. If we start with A atoms of a radioactive material that has a "half-life" (the time it takes for half of the material to decay) of 500 years, then the number of radioactive atoms left after t years is $r(t) = A \cdot e^{-Kt}$ where $K = \dfrac{\ln(2)}{500}$. Calculate $r'(t)$ and show that $r'(t)$ is proportional to $r(t)$ (that is, $r'(t) = b \cdot r(t)$ for some constant b).

In 33–41, find a function with the given derivative.

33. $f'(x) = \dfrac{8}{x}$

34. $h'(x) = \dfrac{3}{3x+5}$

35. $f'(x) = \dfrac{\cos(x)}{3+\sin(x)}$

36. $g'(x) = \dfrac{x}{1+x^2}$

37. $g'(x) = 3e^{5x}$

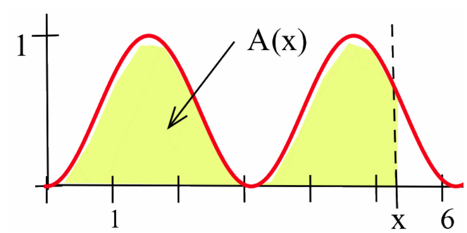
38. $h'(x) = e^2$

39. $f'(x) = 2x \cdot e^{x^2}$

40. $g'(x) = \cos(x)e^{\sin(x)}$

41. $h'(x) = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$

42. Define $A(x)$ to be the **area** bounded between the t-axis, the graph of $y = f(t)$ and a vertical line at $t = x$ (see figure below). The area under each "hump" of f is 2 square inches.
    (a) Graph $A(x)$ for $0 \le x \le 9$.
    (b) Graph $A'(x)$ for $0 \le x \le 9$.

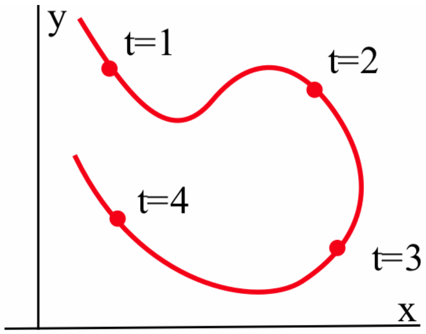


Problems 43–48 involve parametric equations.

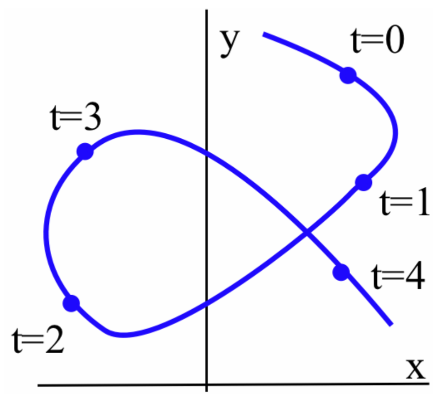
43. At time t minutes, robot A is at $(t, 2t+1)$ and robot B is at $(t^2, 2t^2+1)$.
    (a) Where is each robot when $t = 0$ and $t = 1$?
    (b) Sketch the path each robot follows during the first minute.
    (c) Find the slope of the tangent line, $\dfrac{dy}{dx}$, to the path of each robot at $t = 1$ minute.
    (d) Find the speed of each robot at $t = 1$ minute.
    (e) Discuss the motion of a robot that follows the path $(\sin(t), 2\sin(t)+1)$ for 20 minutes.

44. Let $x(t) = t+1$ and $y(t) = t^2$.
    (a) Graph $(x(t), y(t))$ for $-1 \le t \le 4$.
    (b) Find $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, the tangent slope $\dfrac{dy}{dx}$, and speed when $t = 1$ and $t = 4$.

45. For the parametric graph shown below, determine whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive, negative or 0 when $t = 1$ and $t = 3$.



46. For the parametric graph shown below, determine whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive, negative or 0 when $t = 1$ and $t = 3$.

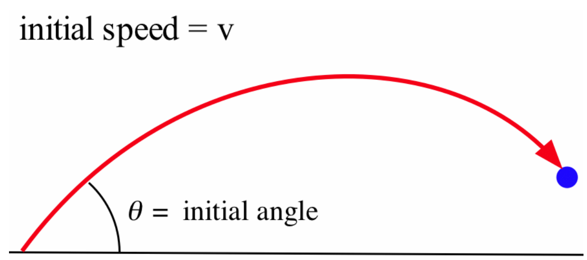


47. The parametric graph $(x(t), y(t))$ defined by $x(t) = R \cdot (t - \sin(t))$ and $y(t) = R \cdot (1 - \cos(t))$ is called a **cycloid**, the path of a light attached to the edge of a rolling wheel with radius R.
   (a) Graph $(x(t), y(t))$ for $0 \le t \le 4\pi$.
   (b) Find $\frac{dx}{dt}$, $\frac{dy}{dt}$, the tangent slope $\frac{dy}{dx}$, and speed when $t = \frac{\pi}{2}$ and $t = \pi$.

48. Describe the motion of particles whose locations at time t are $(\cos(t), \sin(t))$ and $(\cos(t), -\sin(t))$.

49. (a) Describe the path of a robot whose location at time t is $(3 \cdot \cos(t), 5 \cdot \sin(t))$.
   (b) Describe the path of a robot whose location at time t is $(A \cdot \cos(t), B \cdot \sin(t))$.
   (c) Give parametric equations so the robot will move along the same path as in part (a) but in the opposite direction.

50. After t seconds, a projectile hurled with initial velocity v and angle θ will be at $x(t) = v \cdot \cos(\theta) \cdot t$ feet and $y(t) = v \cdot \sin(\theta) \cdot t - 16t^2$ feet (see figure below). (This formula neglects air resistance.)
   (a) For an initial velocity of 80 feet/second and an angle of $\frac{\pi}{4}$, find $T > 0$ so that $y(T) = 0$. What does this value for t represent physically? Evaluate $x(T)$.
   (b) For v and θ in part (a), calculate $\frac{dy}{dx}$. Find T so that $\frac{dy}{dx} = 0$ at $t = T$, and evaluate $x(T)$. What does $x(T)$ represent physically?
   (c) What initial velocity is needed so a ball hit at an angle of $\frac{\pi}{4} \approx 0.7854$ will go over a 40-foot-high fence 350 feet away?
   (d) What initial velocity is needed so a ball hit at an angle of 0.7 radians will go over a 40-foot-high fence 350 feet away?



51. Use the method from the proof that $\mathbf{D}(\ln(x)) = \frac{1}{x}$ to compute the derivative $\mathbf{D}(\arctan(x))$:
   (a) Rewrite $y = \arctan(x)$ as $\tan(y) = x$.
   (b) Differentiate both sides using the Chain Rule and solve for y'.
   (c) Use the identity $1 + \tan^2(\theta) = \sec^2(\theta)$ and the fact that $\tan(y) = x$ to show that $y' = \frac{1}{1 + x^2}$.

52. Use the method from the proof that $\mathbf{D}(\ln(x)) = \frac{1}{x}$ to compute the derivative $\mathbf{D}(\arcsin(x))$:
   (a) Rewrite $y = \arcsin(x)$ as $\sin(y) = x$.
   (b) Differentiate both sides using the Chain Rule and solve for y'.
   (c) Use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ and the fact that $\sin(y) = x$ to show that $y' = \frac{1}{\sqrt{1 - x^2}}$.

2.5 Practice Answers

1. $\log_9(20) = \dfrac{\log(20)}{\log(9)} \approx 1.3634165 \approx \dfrac{\ln(20)}{\ln(9)}$

   $\log_3(20) = \dfrac{\log(20)}{\log(3)} \approx 2.726833 \approx \dfrac{\ln(20)}{\ln(3)}$

   $\log_\pi(e) = \dfrac{\log(e)}{\log(\pi)} \approx 0.8735685 \approx \dfrac{\ln(e)}{\ln(\pi)} = \dfrac{1}{\ln(\pi)}$

2. $\mathbf{D}\left(\log_{10}(\sin(x))\right) = \dfrac{1}{\sin(x)\cdot\ln(10)}\mathbf{D}(\sin(x)) = \dfrac{\cos(x)}{\sin(x)\cdot\ln(10)}$

   $\mathbf{D}\left(\log_\pi(e^x)\right) = \dfrac{1}{e^x\cdot\ln(\pi)}\mathbf{D}(e^x) = \dfrac{e^x}{e^x\cdot\ln(\pi)} = \dfrac{1}{\ln(\pi)}$

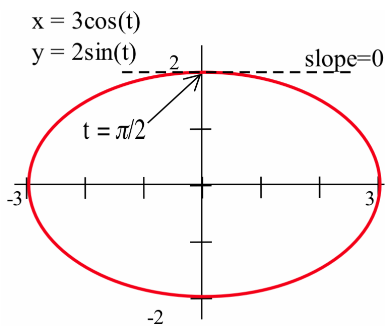
3. $\mathbf{D}\left(\sin(2^x)\right) = \cos(2^x)\,\mathbf{D}\left(2^x\right) = \cos(2^x)\cdot 2^x\cdot\ln(2)$

   $\dfrac{d}{dt}\left(3^{t^2}\right) = 3^{t^2}\ln(3)\,\mathbf{D}(t^2) = 3^{t^2}\ln(3)\cdot 2t$

4. $T = \dfrac{72}{1+h} = \dfrac{72}{1+t+\sin(t)} \Rightarrow$

   $\dfrac{dT}{dt} = \dfrac{(1+t+\sin(t))\cdot 0 - 72\cdot\mathbf{D}(1+t+\sin(t))}{(1+t+\sin(t))^2} = \dfrac{-72(1+\cos(t))}{(1+t+\sin(t))^2}$

   When $t=5$, $\dfrac{dT}{dt} = \dfrac{-72(1+\cos(5))}{(1+5+\sin(5))^2} \approx -3.63695$.



5. $x(t) = 3\cos(t) \Rightarrow \dfrac{dx}{dt} = -3\sin(t)$, $y(t) = 2\sin(t) \Rightarrow \frac{dy}{dt} = 2\cos(t)$:

   $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos(t)}{-3\sin(t)} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{2\cos(\frac{\pi}{2})}{-3\sin(\frac{\pi}{2})} = \frac{2\cdot 0}{-3\cdot 1} = 0$$

   (See margin for graph.)

6. $x=1$: positive, positive, positive. $x=3$: positive, negative, negative.

7. $x(t) = 3\sin(t) \Rightarrow \frac{dx}{dt} = 3\cos(t)$ and $y(t) = 3\cos(t) \Rightarrow \frac{dy}{dt} = -3\sin(t)$. So:

   $$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3\cos(t))^2 + (-3\sin(t))^2}$$
   $$= \sqrt{9\cdot\cos^2(t) + 9\cdot\sin^2(t)} = \sqrt{9} = 3 \quad \text{(a constant)}$$

8. $x(t) = 3\cos(t) \Rightarrow \frac{dx}{dt} = -3\sin(t)$ and $y(t) = 2\sin(t) \Rightarrow \frac{dy}{dt} = 2\cos(t)$ so:

   $$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-3\sin(t))^2 + (2\cos(t))^2}$$
   $$= \sqrt{9\cdot\sin^2(t) + 4\cdot\cos^2(t)}$$

   When $t=0$, the speed is $\sqrt{9\cdot 0^2 + 4\cdot 1^2} = 2$.
   When $t=\frac{\pi}{2}$, the speed is $\sqrt{9\cdot 1^2 + 4\cdot 0^2} = 3$ (faster).