# 2.5 Applications of the Chain Rule

The Chain Rule can help us determine the derivatives of logarithmic functions like  $f(x) = \ln(x)$  and general exponential functions like  $f(x) = a^x$ . We will also use it to answer some applied questions and to find slopes of graphs given by parametric equations.

#### Derivatives of Logarithms

You know from precalculus that the natural logarithm  $\ln(x)$  is defined as the inverse of the exponential function  $e^x$ :  $e^{\ln(x)} = x$  for x > 0. We can use this identity along with the Chain Rule to determine the derivative of the natural logarithm.

$$\mathbf{D}(\ln(x)) = \frac{1}{x}$$
 and  $\mathbf{D}(\ln(g(x))) = \frac{g'(x)}{g(x)}$ 

*Proof.* We know that  $\mathbf{D}(e^u) = e^u$ , so using the Chain Rule we have  $\mathbf{D}(e^{f(x)}) = e^{f(x)} \cdot f'(x)$ . Differentiating each side of the identity  $e^{\ln(x)} = x$ , we get:

$$\mathbf{D}\left(e^{\ln(x)}\right) = \mathbf{D}(x) \Rightarrow e^{\ln(x)} \cdot \mathbf{D}(\ln(x)) = 1$$
$$\Rightarrow x \cdot \mathbf{D}(\ln(x)) = 1 \Rightarrow \mathbf{D}(\ln(x)) = \frac{1}{x}$$

The function  $\ln(g(x))$  is the composition of  $f(x) = \ln(x)$  with g(x) so the Chain Rule says:

$$\mathbf{D}(\ln(g(x)) = \mathbf{D}(f(g(x))) = f'(g(x)) \cdot g'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

Graph  $f(x) = \ln(x)$  along with  $f'(x) = \frac{1}{x}$  and compare the behavior of the function at various points with the values of its derivative at those points. Does  $y = \frac{1}{x}$  possess the properties you would expect to see from the derivative of  $f(x) = \ln(x)$ ?

**Example 1.** Find  $D(\ln(\sin(x)))$  and  $D(\ln(x^2+3))$ .

**Solution.** Using the pattern  $\mathbf{D}(\ln(g(x)) = \frac{g'(x)}{g(x)}$  with  $g(x) = \sin(x)$ :

$$\mathbf{D}(\ln(\sin(x))) = \frac{g'(x)}{g(x)} = \frac{\mathbf{D}(\sin(x))}{\sin(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

With 
$$g(x) = x^2 + 3$$
,  $\mathbf{D}(\ln(x^2 + 3)) = \frac{g'(x)}{g(x)} = \frac{2x}{x^2 + 3}$ .

You can remember the differentiation pattern for the the natural logarithm in words as: "one over the inside times the the derivative of the inside." We can use the Change of Base Formula from precalculus to rewrite any logarithm as a natural logarithm, and then we can differentiate the resulting natural logarithm.

> Change of Base Formula for Logarithms:  $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$  for all positive *a*, *b* and *x*.

**Example 2.** Use the Change of Base formula and your calculator to find  $\log_{\pi}(7)$  and  $\log_{2}(8)$ .

**Solution.**  $\log_{\pi}(7) = \frac{\ln(7)}{\ln(\pi)} \approx \frac{1.946}{1.145} \approx 1.700$ . (Check that  $\pi^{1.7} \approx 7$ .) Likewise,  $\log_2(8) = \frac{\ln(8)}{\ln(2)} = 3$ .

**Practice 1.** Find the values of  $\log_9 20$ ,  $\log_3 20$  and  $\log_{\pi} e$ .

Putting b = e in the Change of Base Formula,  $\log_a(x) = \frac{\log_e(x)}{\log_e(a)} =$ 

 $\frac{\ln(x)}{\ln(a)}$ , so any logarithm can be written as a natural logarithm divided by a constant. This makes any logarithmic function easy to differentiate.

$$\mathbf{D}(\log_a(x)) = \frac{1}{x \ln(a)}$$
 and  $\mathbf{D}(\log_a(f(x))) = \frac{f'(x)}{f(x)} \cdot \frac{1}{\ln(a)}$ 

*Proof.*  $\mathbf{D}(\log_a(x)) = \mathbf{D}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln(a)} \cdot \mathbf{D}(\ln x) = \frac{1}{\ln(a)} \cdot \frac{1}{x} = \frac{1}{x \ln(a)}$ . The second differentiation formula follows from the Chain Rule.

**Practice 2.** Calculate  $\mathbf{D}(\log_{10}(\sin(x)))$  and  $\mathbf{D}(\log_{\pi}(e^{x}))$ .

The number e might seem like an "unnatural" base for a natural logarithm, but of all the possible bases, the logarithm with base e has the nicest and easiest derivative. The natural logarithm is even related to the distribution of prime numbers. In 1896, the mathematicians Hadamard and Vallée-Poussin proved the following conjecture of Gauss (the Prime Number Theorem): For large values of N,

number of primes less than 
$$N \approx \frac{N}{\ln(N)}$$

## Derivative of $a^x$

Once we know the derivative of  $e^x$  and the Chain Rule, it is relatively easy to determine the derivative of  $a^x$  for any a > 0.

$$\mathbf{D}(a^x) = a^x \cdot \ln(a) \text{ for } a > 0.$$

Your calculator likely has two logarithm buttons: **In** for the natural logarithm (base *e*) and **log** for the common logarithm (base 10). Be careful, however, as more advanced mathematics texts (as well as the Web site Wolfram | Alpha) use log for the (base *e*) natural logarithm.

*Proof.* If 
$$a > 0$$
, then  $a^x > 0$  and  $a^x = e^{\ln(a^x)} = e^{x \cdot \ln(a)}$ , so we have:  
 $\mathbf{D}(a^x) = \mathbf{D}\left(e^{\ln(a^x)}\right) = \mathbf{D}\left(e^{x \cdot \ln(a)}\right) = e^{x \cdot \ln(a)} \cdot \mathbf{D}(x \cdot \ln(a)) = a^x \cdot \ln(a).$ 

**Example 3.** Calculate  $\mathbf{D}(7^x)$  and  $\frac{d}{dt} \left(2^{\sin(t)}\right)$ .

**Solution.**  $\mathbf{D}(7^x) = 7^x \cdot \ln(7) \approx (1.95)7^x$ . We can write  $y = 2^{\sin(t)}$  as  $y = 2^u$  with  $u = \sin(t)$ . Using the Chain Rule:  $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = 2^u \cdot \ln(2)\cos(t) = 2^{\sin(t)} \cdot \ln(2) \cdot \cos(t)$ .

**Practice 3.** Calculate  $\mathbf{D}(\sin(2^x))$  and  $\frac{d}{dt}(3^{t^2})$ .

# Some Applied Problems

Let's examine some applications involving more complicated functions.

**Example 4.** A ball at the end of a rubber band (see margin) is oscillating up and down, and its height (in feet) above the floor at time *t* seconds is  $h(t) = 5 + 2 \sin\left(\frac{t}{2}\right)$  (with *t* in radians).

- (a) How fast is the ball traveling after 2 seconds? After 4 seconds? After 60 seconds?
- (b) Is the ball moving up or down after 2 seconds? After 4 seconds? After 60 seconds?
- (c) Is the vertical velocity of the ball ever 0?

**Solution.** (a)  $v(t) = h'(t) = \mathbf{D}\left(5 + 2\sin\left(\frac{t}{2}\right)\right) = 2\cos\left(\frac{t}{2}\right) \cdot \frac{1}{2}$  so  $v(t) = \cos\left(\frac{t}{2}\right)$  feet/second:  $v(2) = \cos\left(\frac{2}{2}\right) \approx 0.540$  ft/s,  $v(4) = \cos\left(\frac{4}{2}\right) \approx -0.416$  ft/s, and  $v(60) = \cos\left(\frac{60}{2}\right) \approx 0.154$  ft/s.

- (b) The ball is moving up at t = 2 and t = 60, down when t = 4.
- (c)  $v(t) = \cos\left(\frac{t}{2}\right) = 0$  when  $\frac{t}{2} = \frac{\pi}{2} \pm k \cdot \pi \Rightarrow t = \pi \pm 2\pi k$  for any integer *k*.

**Example 5.** If 2,400 people now have a disease, and the number of people with the disease appears to double every 3 years, then the number of people expected to have the disease in *t* years is  $y = 2400 \cdot 2^{\frac{t}{3}}$ .

- (a) How many people are expected to have the disease in 2 years?
- (b) When are 50,000 people expected to have the disease?
- (c) How fast is the number of people with the disease growing now? How fast is it expected to be growing 2 years from now?



**Solution.** (a) In 2 years,  $y = 2400 \cdot 2^{\frac{2}{3}} \approx 3,810$  people.

- (b) We know y = 50000 and need to solve  $50000 = 2400 \cdot 2^{\frac{t}{3}}$  for t. Taking logarithms of each side of the equation:  $\ln(50000) = \ln\left(2400 \cdot 2^{\frac{2}{3}}\right) = \ln(2400) + \frac{t}{3} \cdot \ln(2)$  so  $10.819 \approx 7.783 + 0.231t$  and  $t \approx 13.14$  years. We expect 50,000 people to have the disease about 13 years from now.
- (c) This question asks for  $\frac{dy}{dt}$  when t = 0 and t = 2.

$$\frac{dy}{dt} = \frac{d}{dt} \left( 2400 \cdot 2^{\frac{t}{3}} \right) = 2400 \cdot 2^{\frac{t}{3}} \cdot \ln(2) \cdot \frac{1}{3} \approx 554.5 \cdot 2^{\frac{t}{3}}$$

Now, at t = 0, the rate of growth of the disease is approximately  $554.5 \cdot 2^0 \approx 554.5$  people/year. In 2 years, the rate of growth will be approximately  $554.5 \cdot 2^{\frac{2}{3}} \approx 880$  people/year.

**Example 6.** You are riding in a balloon, and at time *t* (in minutes) you are  $h(t) = t + \sin(t)$  thousand feet above sea level. If the temperature at an elevation *h* is  $T(h) = \frac{72}{1+h}$  degrees Fahrenheit, then how fast is the temperature changing when t = 5 minutes?

**Solution.** As *t* changes, your elevation will change. And, as your elevation changes, so will the temperature. It is not difficult to write the temperature as a function of time, and then we could calculate  $\frac{dT}{dt} = T'(t)$  and evaluate T'(5). Or we could use the Chain Rule:

$$\frac{dT}{dt} = \frac{dT}{dh} \cdot \frac{dh}{dt} = -\frac{72}{(1+h)^2} \cdot (1+\cos(t))$$

At t = 5,  $h(5) = 5 + \sin(5) \approx 4.04$  so  $T'(5) \approx -\frac{72}{(1+4.04)^2} \cdot (1+0.284) \approx -3.64 \circ / \text{minute.}$ 

**Practice 4.** Write the temperature *T* in the previous example as a function of the variable *t* alone and then differentiate *T* to determine the value of  $\frac{dT}{dt}$  when t = 5 minutes.

**Example 7.** A scientist has determined that, under optimum conditions, an initial population of 40 bacteria will grow "exponentially" to  $f(t) = 40 \cdot e^{\frac{t}{5}}$  bacteria after *t* hours.

- (a) Graph y = f(t) for  $0 \le t \le 15$ . Calculate f(0), f(5) and f(10).
- (b) How fast is the population increasing at time *t*? (Find f'(t).)
- (c) Show that the rate of population increase, f'(t), is proportional to the population, f(t), at any time *t*. (Show  $f'(t) = K \cdot f(t)$  for some constant *K*.)



- **Solution.** (a) The graph of y = f(t) appears in the margin.  $f(0) = 40 \cdot e^{\frac{0}{5}} = 40$  bacteria,  $f(5) = 40 \cdot e^{\frac{5}{5}} = 40e \approx 109$  bacteria and  $f(10) = 40 \cdot e^{\frac{10}{5}} \approx 296$  bacteria.
- (b)  $f'(t) = \frac{d}{dt}(f(t)) = \frac{d}{dt}\left(40 \cdot e^{\frac{t}{5}}\right) = 40 \cdot e^{\frac{t}{5}} \cdot \frac{d}{dt}\left(\frac{t}{5}\right) = 40 \cdot e^{\frac{t}{5}} \cdot \frac{1}{5} = 8 \cdot e^{\frac{t}{5}}$ bacteria/hour.
- (c)  $f'(t) = 8 \cdot e^{\frac{t}{5}} = \frac{1}{5} \cdot 40e^{\frac{t}{5}} = \frac{1}{5}f(t)$  so  $f'(t) = K \cdot f(t)$  with  $K = \frac{1}{5}$ . The rate of change of the population is proportional to its size.

#### Parametric Equations

Suppose a robot has been programmed to move in the *xy*-plane so at time *t* its *x*-coordinate will be sin(t) and its *y*-coordinate will be  $t^2$ . Both *x* and *y* are functions of the independent parameter *t*: x(t) = sin(t) and  $y(t) = t^2$ . The path of the robot (see margin) can be found by plotting (x, y) = (x(t), y(t)) for lots of values of *t*.

t	$x(t) = \sin(t)$	$y(t) = t^2$	point
0	0	0	(0,0)
0.5	0.48	0.25	(0.48, 0.25)
1.0	0.84	1	(0.84, 1)
1.5	1.00	2.25	(1,2.25)
2.0	0.91	4	(0.91, 4)

Typically we know x(t) and y(t) and need to find  $\frac{dy}{dx}$ , the slope of the tangent line to the graph of (x(t), y(t)). The Chain Rule says:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

so , algebraically solving for  $\frac{dy}{dx}$ , we get:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

If we can calculate  $\frac{dy}{dt}$  and  $\frac{dx}{dt}$ , the derivatives of *y* and *x* with respect to the parameter *t*, then we can determine  $\frac{dy}{dx}$ , the rate of change of *y* with respect to *x*.

If 
$$x = x(t)$$
 and  $y = y(t)$  are differentiable  
with respect to  $t$  and  $\frac{dx}{dt} \neq 0$   
then  $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ .





**Example 8.** Find the slope of the tangent line to the graph of  $(x, y) = (\sin(t), t^2)$  when t = 2.

**Solution.**  $\frac{dx}{dt} = \cos(t)$  and  $\frac{dy}{dt} = 2t$ . When t = 2, the object is at the point  $(\sin(2), 2^2) \approx (0.91, 4)$  and the slope of the tangent line is:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\cos(t)} = \frac{2\cdot 2}{\cos(2)} \approx \frac{4}{-0.42} \approx -9.61$$

Notice in the figure that the slope of the tangent line to the curve at (0.91, 4) is negative and very steep.

**Practice 5.** Graph  $(x, y) = (3\cos(t), 2\sin(t))$  and find the slope of the tangent line when  $t = \frac{\pi}{2}$ .

When we calculated  $\frac{dy}{dx}$ , the slope of the tangent line to the graph of (x(t), y(t)), we used the derivatives  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ . Each of these also has a geometric meaning:  $\frac{dx}{dt}$  measures the rate of change of x(t) with respect to t: it tells us whether the x-coordinate is increasing or decreasing as the t-variable increases (and how fast it is changing), while  $\frac{dy}{dt}$  measures the rate of change of y(t) with respect to t.

**Example 9.** For the parametric graph in the margin, determine whether  $\frac{dx}{dt}$ ,  $\frac{dy}{dt}$  and  $\frac{dy}{dx}$  are positive or negative when t = 2.

**Solution.** As we move through the point *B* (where t = 2) in the direction of increasing values of *t*, we are moving to the left, so x(t) is decreasing and  $\frac{dx}{dt} < 0$ . The values of y(t) are increasing, so  $\frac{dy}{dt} > 0$ . Finally, the slope of the tangent line,  $\frac{dy}{dx}$ , is negative.

As a check on the sign of  $\frac{dy}{dx}$  in the previous example:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\text{positive}}{\text{negative}} = \text{negative}$$

**Practice 6.** For the parametric graph in the previous example, tell whether  $\frac{dx}{dt}$ ,  $\frac{dy}{dt}$  and  $\frac{dy}{dx}$  are positive or negative at t = 1 and t = 3.

### Speed

If we know the position of an object at any time, then we can determine its speed. The formula for speed comes from the distance formula and looks a lot like it, but involves derivatives.



If x = x(t) and y = y(t) give the location of an object at time t and both are differentiable functions of tthe speed of the object is then

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

*Proof.* The speed of an object is the limit, as  $\Delta t \rightarrow 0$ , of (see margin):

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} = \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta t)^2}}$$
$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \to \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$
as  $\Delta t \to 0$ .

**Example 10.** Find the speed of the object whose location at time *t* is  $(x, y) = (\sin(t), t^2)$  when t = 0 and t = 1.

Solution. 
$$\frac{dx}{dt} = \cos(t)$$
 and  $\frac{dy}{dt} = 2t$  so:  
speed  $= \sqrt{(\cos(t))^2 + (2t)^2} = \sqrt{\cos^2(t) + 4t^2}$ 

When t = 0, speed =  $\sqrt{\cos^2(0) + 4(0)^2} = \sqrt{1+0} = 1$ . When t = 1, speed =  $\sqrt{\cos^2(1) + 4(1)^2} \approx \sqrt{0.29 + 4} \approx 2.07.$ ◄

**Practice 7.** Show that an object located at  $(x, y) = (3\sin(t), 3\cos(t))$  at time *t* has a constant speed. (This object is moving on a circular path.)

**Practice 8.** Is the object at  $(x, y) = (3\cos(t), 2\sin(t))$  at time *t* traveling faster at the top of the ellipse  $(t = \frac{\pi}{2})$  or at the right edge (t = 0)?

## 2.5 Problems

- (- )

In Problems 1–27, differentiate the given function. 9.  $\ln(\sin(x))$ 

- ( )

1. 
$$\ln(5x)$$
2.  $\ln(x^2)$ 11.  $\log_2(\sin(x))$ 12.  $\ln(e^x)$ 3.  $\ln(x^k)$ 4.  $\ln(x^x) = x \cdot \ln(x)$ 13.  $\log_5(5^x)$ 14.  $\ln(e^{f(x)})$ 5.  $\ln(\cos(x))$ 6.  $\cos(\ln(x))$ 15.  $x \cdot \ln(3x)$ 16.  $e^x \cdot \ln(x)$ 

17.  $\frac{\ln(x)}{x}$ 18.  $\sqrt{x + \ln(3x)}$ 8.  $\log_2(kx)$ 7.  $\log_2(5x)$ 



10.  $\ln(kx)$ 

19. ln	(	5x - 3	) 20.	ln	(cos(	(t)	))	
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21.  $\cos(\ln(w))$  22.  $\ln(ax+b)$ 

23.  $\ln(\sqrt{t+1})$  24.  $3^x$ 

25.  $5^{\sin(x)}$  26.  $x \cdot \ln(x) - x$ 

27.  $\ln(\sec(x) + \tan(x))$ 

- 28. Find the slope of the line tangent to  $f(x) = \ln(x)$  at the point (e, 1). Find the slope of the line tangent to  $g(x) = e^x$  at the point (1, e). How are the slopes of f and g at these points related?
- 29. Find a point *P* on the graph of  $f(x) = \ln(x)$  so the tangent line to *f* at *P* goes through the origin.
- 30. You are moving from left to right along the graph of  $y = \ln(x)$  (see figure below).
  - (a) If the *x*-coordinate of your location at time *t* seconds is x(t) = 3t + 2, then how fast is your elevation increasing?
  - (b) If the *x*-coordinate of your location at time *t* seconds is *x*(*t*) = *e<sup>t</sup>*, then how fast is your elevation increasing?



- 31. The percent of a population, p(t), who have heard a rumor by time *t* is often modeled by  $p(t) = \frac{100}{1 + Ae^{-t}} = 100 (1 + Ae^{-t})^{-1}$  for some positive constant *A*. Calculate p'(t), the rate at which the rumor is spreading.
- 32. If we start with *A* atoms of a radioactive material that has a "half-life" (the time it takes for half of the material to decay) of 500 years, then the number of radioactive atoms left after *t* years is  $r(t) = A \cdot e^{-Kt}$  where  $K = \frac{\ln(2)}{500}$ . Calculate r'(t) and show that r'(t) is proportional to r(t) (that is,  $r'(t) = b \cdot r(t)$  for some constant *b*).

In 33-41, find a function with the given derivative.

33. 
$$f'(x) = \frac{8}{x}$$
  
34.  $h'(x) = \frac{3}{3x+5}$   
35.  $f'(x) = \frac{\cos(x)}{3+\sin(x)}$   
36.  $g'(x) = \frac{x}{1+x^2}$ 

37. 
$$g'(x) = 3e^{5x}$$
 38.  $h'(x) = e^2$ 

39. 
$$f'(x) = 2x \cdot e^{x^2}$$
 40.  $g'(x) = \cos(x)e^{\sin(x)}$ 

41. 
$$h'(x) = \cot(x) = \frac{\cos(x)}{\sin(x)}$$

- 42. Define A(x) to be the **area** bounded between the *t*-axis, the graph of y = f(t) and a vertical line at t = x (see figure below). The area under each "hump" of *f* is 2 square inches.
  - (a) Graph A(x) for  $0 \le x \le 9$ .
  - (b) Graph A'(x) for  $0 \le x \le 9$ .



Problems 43–48 involve parametric equations.

- 43. At time *t* minutes, robot A is at (t, 2t + 1) and robot *B* is at  $(t^2, 2t^2 + 1)$ .
  - (a) Where is each robot when t = 0 and t = 1?
  - (b) Sketch the path each robot follows during the first minute.
  - (c) Find the slope of the tangent line,  $\frac{dy}{dx}$ , to the path of each robot at t = 1 minute.
  - (d) Find the speed of each robot at t = 1 minute.
  - (e) Discuss the motion of a robot that follows the path  $(\sin(t), 2\sin(t) + 1)$  for 20 minutes.
- 44. Let x(t) = t + 1 and  $y(t) = t^2$ .

(a) Graph 
$$(x(t), y(t))$$
 for  $-1 \le t \le 4$ 

(b) Find  $\frac{dx}{dt}$ ,  $\frac{dy}{dt}$ , the tangent slope  $\frac{dy}{dx}$ , and speed when t = 1 and t = 4.

45. For the parametric graph shown below, determine whether  $\frac{dx}{dt}$ ,  $\frac{dy}{dt}$  and  $\frac{dy}{dx}$  are positive, negative or 0 when t = 1 and t = 3.



46. For the parametric graph shown below, determine whether  $\frac{dx}{dt}$ ,  $\frac{dy}{dt}$  and  $\frac{dy}{dx}$  are positive, negative or 0 when t = 1 and t = 3.



- 47. The parametric graph (x(t), y(t)) defined by  $x(t) = R \cdot (t \sin(t))$  and  $y(t) = R \cdot (1 \cos(t))$  is called a **cycloid**, the path of a light attached to the edge of a rolling wheel with radius *R*.
  - (a) Graph (x(t), y(t)) for  $0 \le t \le 4\pi$ .
  - (b) Find  $\frac{dx}{dt}$ ,  $\frac{dy}{dt}$ , the tangent slope  $\frac{dy}{dx}$ , and speed when  $t = \frac{\pi}{2}$  and  $t = \pi$ .
- 48. Describe the motion of particles whose locations at time *t* are  $(\cos(t), \sin(t))$  and  $(\cos(t), -\sin(t))$ .
- 49. (a) Describe the path of a robot whose location at time *t* is  $(3 \cdot \cos(t), 5 \cdot \sin(t))$ .
  - (b) Describe the path of a robot whose location at time *t* is  $(A \cdot \cos(t), B \cdot \sin(t))$ .
  - (c) Give parametric equations so the robot will move along the same path as in part (a) but in the opposite direction.

- 50. After *t* seconds, a projectile hurled with initial velocity *v* and angle  $\theta$  will be at  $x(t) = v \cdot \cos(\theta) \cdot t$  feet and  $y(t) = v \cdot \sin(\theta) \cdot t 16t^2$  feet (see figure below). (This formula neglects air resistance.)
  - (a) For an initial velocity of 80 feet/second and an angle of π/4, find T > 0 so that y(T) = 0. What does this value for t represent physically? Evaluate x(T).
  - (b) For *v* and  $\theta$  in part (a), calculate  $\frac{dy}{dx}$ . Find *T* so that  $\frac{dy}{dx} = 0$  at t = T, and evaluate x(T). What does x(T) represent physically?
  - (c) What initial velocity is needed so a ball hit at an angle of  $\frac{\pi}{4} \approx 0.7854$  will go over a 40-foothigh fence 350 feet away?
  - (d) What initial velocity is needed so a ball hit at an angle of 0.7 radians will go over a 40-foothigh fence 350 feet away?

initial speed = v



- 51. Use the method from the proof that  $\mathbf{D}(\ln(x)) = \frac{1}{x}$  to compute the derivative  $\mathbf{D}(\arctan(x))$ :
  - (a) Rewrite  $y = \arctan(x)$  as  $\tan(y) = x$ .
  - (b) Differentiate both sides using the Chain Rule and solve for *y*'.
  - (c) Use the identity  $1 + \tan^2(\theta) = \sec^2(\theta)$  and the fact that  $\tan(y) = x$  to show that  $y' = \frac{1}{1 + x^2}$ .
- 52. Use the method from the proof that  $\mathbf{D}(\ln(x)) = \frac{1}{x}$  to compute the derivative  $\mathbf{D}(\arcsin(x))$ :
  - (a) Rewrite  $y = \arcsin(x)$  as  $\sin(y) = x$ .
  - (b) Differentiate both sides using the Chain Rule and solve for *y*'.
  - (c) Use the identity  $\cos^2(\theta) + \sin^2(\theta) = 1$  and the fact that  $\sin(y) = x$  to show that  $y' = \frac{1}{\sqrt{1 x^2}}$ .

## 2.5 Practice Answers

1

1. 
$$\log_{9}(20) = \frac{\log(20)}{\log(9)} \approx 1.3634165 \approx \frac{\ln(20)}{\ln(9)}$$
  
 $\log_{3}(20) = \frac{\log(2)}{\log(3)} \approx 2.726833 \approx \frac{\ln(20)}{\ln(3)}$   
 $\log_{\pi}(e) = \frac{\log(e)}{\log(\pi)} \approx 0.8735685 \approx \frac{\ln(e)}{\ln(\pi)} = \frac{1}{\ln(\pi)}$   
2.  $\mathbf{D}(\log_{10}(\sin(x))) = \frac{1}{\sin(x) \cdot \ln(10)} \mathbf{D}(\sin(x)) = \frac{\cos(x)}{\sin(x) \cdot \ln(10)}$   
 $\mathbf{D}(\log_{\pi}(e^{x})) = \frac{1}{e^{x} \cdot \ln(\pi)} \mathbf{D}(e^{x}) = \frac{e^{x}}{e^{x} \cdot \ln(\pi)} = \frac{1}{\ln(\pi)}$   
3.  $\mathbf{D}(\sin(2^{x})) = \cos(2^{x}) \mathbf{D}(2^{x}) = \cos(2^{x}) \cdot 2^{x} \cdot \ln(2)$   
 $\frac{d}{dt} (3^{t^{2}}) = 3^{t^{2}} \ln(3) \mathbf{D}(t^{2}) = 3^{t^{2}} \ln(3) \cdot 2t$   
4.  $T = \frac{72}{1+h} = \frac{72}{1+t+\sin(t)} \Rightarrow$   
 $\frac{dT}{dt} = \frac{(1+t+\sin(t)) \cdot 0 - 72 \cdot \mathbf{D}(1+t+\sin(t))}{(1+t+\sin(t))^{2}} = \frac{-72(1+\cos(t))}{(1+t+\sin(t))^{2}}$   
When  $t = 5$ ,  $\frac{dT}{dt} = \frac{-72(1+\cos(5))}{(1+5+\sin(5))^{2}} \approx -3.63695$ .  
5.  $x(t) = 3\cos(t) \Rightarrow \frac{dx}{dt} = -3\sin(t)$ ,  $y(t) = 2\sin(t) \Rightarrow \frac{dy}{dt} = 2\cos(t)$ :  
 $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos(t)}{-3\sin(t)} \Rightarrow \frac{dy}{dx}\Big|_{t=\frac{\pi}{2}} = \frac{2\cos(\frac{\pi}{2})}{-3\sin(\frac{\pi}{2})} = \frac{2 \cdot 0}{-3 \cdot 1} = 0$ 

(See margin for graph.)

- 6. x = 1: positive, positive, positive. x = 3: positive, negative, negative.
- 7.  $x(t) = 3\sin(t) \Rightarrow \frac{dx}{dt} = 3\cos(t)$  and  $y(t) = 3\cos(t) \Rightarrow \frac{dy}{dt} =$  $-3\sin(t)$ . So:

speed = 
$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3\cos(t))^2 + (-3\sin(t))^2}$$
  
=  $\sqrt{9 \cdot \cos^2(t) + 9 \cdot \sin^2(t)} = \sqrt{9} = 3$  (a constant)

8.  $x(t) = 3\cos(t) \Rightarrow \frac{dx}{dt} = -3\sin(t)$  and  $y(t) = 2\sin(t) \Rightarrow \frac{dy}{dt} =$  $2\cos(t)$  so:

speed = 
$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-3\sin(t))^2 + (2\cos(t))^2}$$
  
=  $\sqrt{9 \cdot \sin^2(t) + 4 \cdot \cos^2(t)}$ 

When t = 0, the speed is  $\sqrt{9 \cdot 0^2 + 4 \cdot 1^2} = 2$ . When  $t = \frac{\pi}{2}$ , the speed is  $\sqrt{9 \cdot 1^2 + 4 \cdot 0^2} = 3$  (faster).

