

2.9 Implicit and Logarithmic Differentiation

This short section presents two more differentiation techniques, both more specialized than the ones we have already seen—and consequently used on a smaller class of functions. For some functions, however, one of these techniques may be the only method that works. The idea of each method is straightforward, but actually using each of them requires that you proceed carefully and practice.

Implicit Differentiation

In our work up until now, the functions we needed to differentiate were either given **explicitly** as a function of x , such as $y = f(x) = x^2 + \sin(x)$, or it was fairly straightforward to find an explicit formula, such as solving $y^3 - 3x^2 = 5$ to get $y = \sqrt[3]{5 + 3x^2}$. Sometimes, however, we will have an equation relating x and y that is either difficult or impossible to solve explicitly for y , such as $y^2 + 2y = \sin(x) + 4$ (difficult) or $y + \sin(y) = x^3 - x$ (impossible). In each case, we can still find $y' = f'(x)$ by using **implicit differentiation**.

The key idea behind implicit differentiation is to *assume* that y is a function of x even if we cannot explicitly solve for y . This assumption does not require any work, but we need to be very careful to treat y as a function when we differentiate and to use the Chain Rule or the Power Rule for Functions.

Example 1. Assume y is a function of x and compute each derivative:

$$(a) \mathbf{D}(y^3) \qquad (b) \frac{d}{dx} (x^3 y^2) \qquad (c) (\sin(y))'$$

Solution. (a) We need the Power Rule for Functions because y is a function of x :

$$\mathbf{D}(y^3) = 3y^2 \cdot \mathbf{D}(y) = 3y^2 \cdot y'$$

(b) We need to use the Product Rule and the Chain Rule:

$$\frac{d}{dx} (x^3 y^2) = x^3 \cdot \frac{d}{dx} (y^2) + y^2 \cdot \frac{d}{dx} (x^3) = x^3 \cdot 2y \cdot \frac{dy}{dx} + y^2 \cdot 3x^2$$

(c) We just need to remember that $\mathbf{D}(\sin(u)) = \cos(u)$ and then use the Chain Rule: $(\sin(y))' = \cos(y) \cdot y'$. ◀

Practice 1. Assume that y is a function of x . Calculate:

$$(a) \mathbf{D} (x^2 + y^2) \qquad (b) \frac{d}{dx} (\sin(2 + 3y)).$$

Implicit Differentiation:

To determine y' , differentiate each side of the defining equation, treating y as a function of x , and then algebraically solve for y' .

Example 2. Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$ with and without implicit differentiation.

Solution. Explicitly: We can solve $x^2 + y^2 = 25$ for y : $y = \pm\sqrt{25 - x^2}$ but because the point $(3, 4)$ is on the top half of the circle, we just need $y = \sqrt{25 - x^2}$ so:

$$D(y) = D\left(\sqrt{25 - x^2}\right) = \frac{1}{2} (25 - x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{25 - x^2}}$$

Replacing x with 3, we have $y' = \frac{-3}{\sqrt{25-3^2}} = -\frac{3}{4}$.

Implicitly: We differentiate each side of the equation $x^2 + y^2 = 25$ treating y as a function of x and then solve for y' :

$$D(x^2 + y^2) = D(25) \Rightarrow 2x + 2y \cdot y' = 0 \Rightarrow y' = \frac{-2x}{2y} = -\frac{x}{y}$$

so at the point $(3, 4)$, $y' = -\frac{3}{4}$, the same answer we found explicitly. ◀

Practice 2. Find the slope of the tangent line to $y^3 - 3x^2 = 15$ at the point $(2, 3)$ with and without implicit differentiation.

In the previous Example and Practice problem, it was easy to explicitly solve for y , and then we could differentiate y to get y' . Because we could explicitly solve for y , we had a choice of methods for calculating y' . Sometimes, however, we cannot explicitly solve for y and the only way to determine y' is with implicit differentiation.

Example 3. Determine y' at $(0, 2)$ for $y^2 + 2y = \sin(x) + 8$.

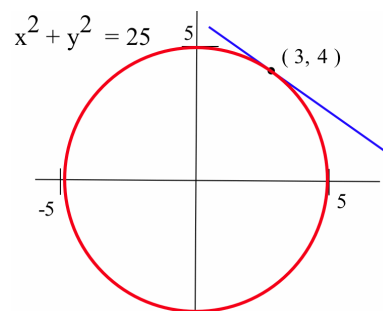
Solution. Assuming that y is a function of x and differentiating each side of the equation, we get:

$$\begin{aligned} D(y^2 + 2y) &= D(\sin(x) + 8) \Rightarrow 2y \cdot y' + 2y' = \cos(x) \\ \Rightarrow (2y + 2)y' &= \cos(x) \Rightarrow y' = \frac{\cos(x)}{2y + 2} \end{aligned}$$

so, at the point $(0, 2)$, $y' = \frac{\cos(0)}{2(2) + 2} = \frac{1}{6}$. ◀

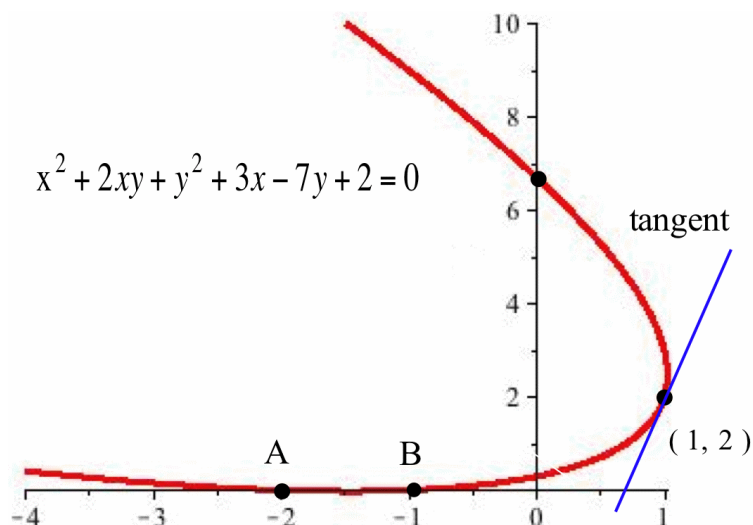
Practice 3. Determine y' at $(1, 0)$ for $y + \sin(y) = x^3 - x$.

In practice, the equations may be rather complicated, but if you proceed carefully and step by step, implicit differentiation is not difficult. Just remember that y **must be treated as a function** so every time you differentiate a term containing a y you should use the Chain Rule and get something that has a y' . The algebra needed to solve for y' is always easy—if you differentiated correctly, the resulting equation will be a linear equation in the variable y' .



We could have first solved the equation explicitly for y using the quadratic formula. Do you see how? Would that make the problem easier or harder than using implicit differentiation?

Example 4. Find an equation of the tangent line L to the “tilted” parabola graphed below at the point $(1, 2)$.



Solution. The line goes through the point $(1, 2)$ so we need to find the slope there. Differentiating each side of the equation, we get:

$$D(x^2 + 2xy + y^2 + 3x - 7y + 2) = D(0)$$

which yields:

$$\begin{aligned} 2x + 2x \cdot y' + 2y + 2y \cdot y' + 3 - 7y' &= 0 \\ \Rightarrow (2x + 2y - 7)y' &= -2x - 2y - 3 \\ \Rightarrow y' &= \frac{-2x - 2y - 3}{2x + 2y - 7} \end{aligned}$$

so the slope at $(1, 2)$ is $m = y' = \frac{-2 - 4 - 3}{2 + 4 - 7} = 9$. Finally, an equation for the line is $y - 2 = 9(x - 1)$ so $y = 9x - 7$. ◀

Practice 4. Find the points where the parabola graphed above crosses the y -axis, and find the slopes of the tangent lines at those points.

Implicit differentiation provides an alternate method for differentiating equations that can be solved explicitly for the function we want, and it is the *only* method for finding the derivative of a function that we cannot describe explicitly.

Logarithmic Differentiation

In Section 2.5 we saw that $D(\ln(f(x))) = \frac{f'(x)}{f(x)}$. If we simply multiply each side by $f(x)$, we have: $f'(x) = f(x) \cdot D(\ln(f(x)))$. When the logarithm of a function is simpler than the function itself, it is often easier to differentiate the logarithm of f than to differentiate f itself.

Logarithmic Differentiation:

$$f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x)))$$

In words: The derivative of f is f times the derivative of the natural logarithm of f . Usually it is easiest to proceed in three steps:

- Calculate $\ln(f(x))$ and simplify.
- Calculate $\mathbf{D}(\ln(f(x)))$ and simplify
- Multiply the result in the previous step by $f(x)$.

Let's examine what happens when we use this process on an "easy" function, $f(x) = x^2$, and a "hard" one, $f(x) = 2^x$. Certainly we don't need to use logarithmic differentiation to find the derivative of $f(x) = x^2$, but sometimes it is instructive to try a new algorithm on a familiar function. Logarithmic differentiation is the easiest way to find the derivative of $f(x) = 2^x$ (if we don't remember the pattern for differentiating a^x from Section 2.5).

$$\begin{aligned} f(x) &= x^2 \\ \ln(f(x)) &= \ln(x^2) = 2 \cdot \ln(x) \\ \mathbf{D}(\ln(f(x))) &= \mathbf{D}(2 \cdot \ln(x)) = \frac{2}{x} \\ f'(x) &= f(x) \cdot \mathbf{D}(\ln(f(x))) = x^2 \cdot \frac{2}{x} = 2x \end{aligned}$$

$$\begin{aligned} f(x) &= 2^x \\ \ln(f(x)) &= \ln(2^x) = x \cdot \ln(2) \\ \mathbf{D}(\ln(f(x))) &= \mathbf{D}(x \cdot \ln(2)) = \ln(2) \\ f'(x) &= f(x) \cdot \mathbf{D}(\ln(f(x))) = 2^x \cdot \ln(2) \end{aligned}$$

Example 5. Use the pattern $f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x)))$ to find the derivative of $f(x) = (3x + 7)^5 \sin(2x)$.

Solution. Apply the natural logarithm to both sides and rewrite:

$$\ln(f(x)) = \ln((3x + 7)^5 \cdot \sin(2x)) = 5 \ln(3x + 7) + \ln(\sin(2x))$$

so:

$$\begin{aligned} \mathbf{D}(\ln(f(x))) &= \mathbf{D}(5 \ln(3x + 7) + \ln(\sin(2x))) \\ &= 5 \cdot \frac{3}{3x + 7} + 2 \cdot \frac{\cos(2x)}{\sin(2x)} \end{aligned}$$

Then:

$$\begin{aligned} f'(x) &= f(x) \cdot \mathbf{D}(\ln(f(x))) \\ &= (3x + 7)^5 \sin(2x) \left(\frac{15}{3x + 7} + 2 \cdot \frac{\cos(2x)}{\sin(2x)} \right) \\ &= 15(3x + 7)^4 \sin(2x) + 2(3x + 7)^5 \cos(2x) \end{aligned}$$

the same result we would obtain using the Product Rule. ◀

Practice 5. Use logarithmic differentiation to find the derivative of $f(x) = (2x + 1)^3(3x^2 - 4)^7(x + 7)^4$.

We could have differentiated the functions in the previous Example and Practice problem without logarithmic differentiation. There are, however, functions for which logarithmic differentiation is the **only** method we can use. We know how to differentiate x raised to a constant power, $\mathbf{D}(x^p) = p \cdot x^{p-1}$, and a constant to a variable power, $\mathbf{D}(b^x) = b^x \ln(b)$, but the function $f(x) = x^x$ has both a variable base and a variable power, so neither differentiation rule applies. We need to use logarithmic differentiation.

Example 6. Find $\mathbf{D}(x^x)$, assuming that $x > 0$.

Solution. Apply the natural logarithm to both sides and rewrite:

$$\ln(f(x)) = \ln(x^x) = x \cdot \ln(x)$$

so:

$$\begin{aligned}\mathbf{D}(\ln(f(x))) &= \mathbf{D}(x \cdot \ln(x)) = x \cdot \mathbf{D}(\ln(x)) + \ln(x) \cdot \mathbf{D}(x) \\ &= x \cdot \frac{1}{x} + \ln(x) \cdot 1 = 1 + \ln(x)\end{aligned}$$

Then $\mathbf{D}(x^x) = f'(x) = f(x) \mathbf{D}(\ln(f(x))) = x^x(1 + \ln(x))$. ◀

Practice 6. Find $\mathbf{D}(x^{\sin(x)})$ assuming that $x > 0$.

Logarithmic differentiation is an alternate method for differentiating some functions such as products and quotients, and it is the only method we've seen for differentiating some other functions such as variable bases to variable exponents.

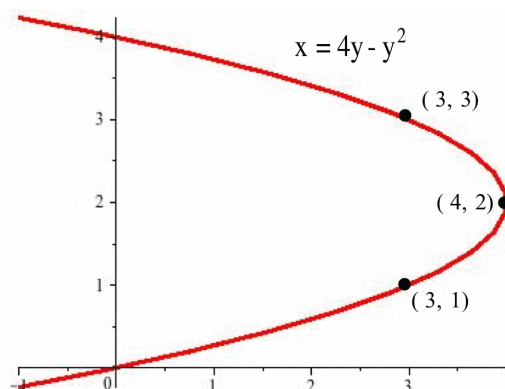
2.9 Problems

In Problems 1–10 find $\frac{dy}{dx}$ in two ways: (a) by differentiating implicitly and (b) by explicitly solving for y and then differentiating. Then find the value of $\frac{dy}{dx}$ at the given point using your results from both the implicit and the explicit differentiation.

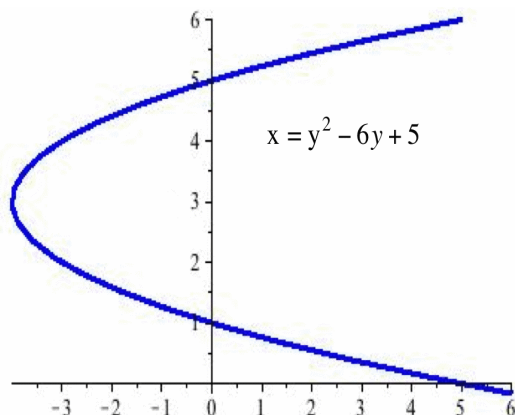
- $x^2 + y^2 = 100$, point: $(6, 8)$
- $x^2 + 5y^2 = 45$, point: $(5, 2)$
- $x^2 - 3xy + 7y = 5$, point: $(2, 1)$
- $\sqrt{x} + \sqrt{y} = 5$, point: $(4, 9)$

- $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point: $(0, 4)$
- $\frac{x^2}{9} + \frac{y^2}{16} = 1$, point: $(3, 0)$
- $\ln(y) + 3x - 7 = 0$, point: $(2, e)$
- $x^2 - y^2 = 16$, point: $(5, 3)$
- $x^2 - y^2 = 16$, point: $(5, -3)$
- $y^2 + 7x^3 - 3x = 8$, point: $(1, 2)$

11. Find the slopes of the lines tangent to the graph below at the points $(3, 1)$, $(3, 3)$ and $(4, 2)$.



12. Find the slopes of the lines tangent to the graph in the figure above at the points where the graph crosses the y -axis.
13. Find the slopes of the lines tangent to the graph below at the points $(5, 0)$, $(5, 6)$ and $(-4, 3)$.



14. Find the slopes of the lines tangent to the graph in the figure above at the points where the graph crosses the y -axis.

In Problems 15–22, find $\frac{dy}{dx}$ using implicit differentiation and then find the slope of the line tangent to the graph of the equation at the given point.

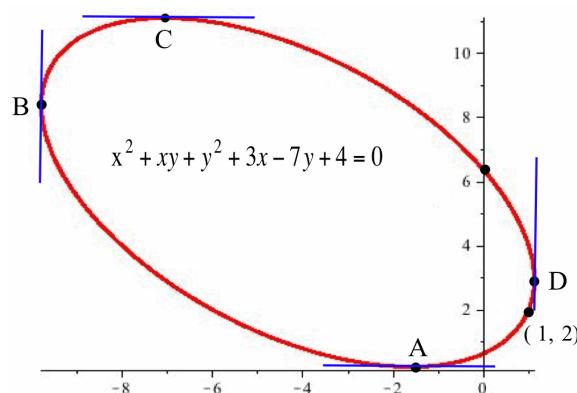
15. $y^3 - 5y = 5x^2 + 7$, point: $(1, 3)$
 16. $y^2 - 5xy + x^2 + 21 = 0$, point: $(2, 5)$
 17. $y^2 + \sin(y) = 2x - 6$, point: $(3, 0)$
 18. $y + 2x^2y^3 = 4x + 7$, point: $(3, 1)$
 19. $e^y + \sin(y) = x^2 - 3$, point: $(2, 0)$

20. $(x^2 + y^2 + 1)^2 - 4x^2 = 81$, point: $(0, 2\sqrt{2})$

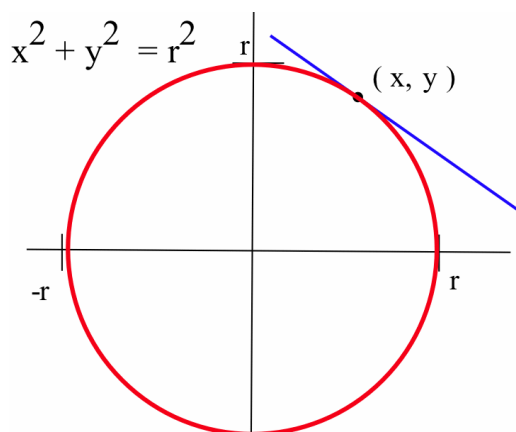
21. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$, point: $(8, 1)$

22. $x + \cos(xy) = y + 3$, point: $(2, 0)$

23. Find the slope of the line tangent to the ellipse shown in the figure below at the point $(1, 2)$.



24. Find the slopes of the tangent lines at the points where the ellipse shown above crosses the y -axis.
25. Find y' for $y = Ax^2 + Bx + C$ and for the equation $x = Ay^2 + By + C$.
26. Find y' for $y = Ax^3 + B$ and for $x = Ay^3 + B$.
27. Find y' for $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.
28. In Chapter 1 we assumed that the tangent line to a circle at a point was perpendicular to the radial line passing through that point and the center of the circle. Use implicit differentiation to prove that the line tangent to the circle $x^2 + y^2 = r^2$ (see below) at an arbitrary point (x, y) is perpendicular to the line passing through $(0, 0)$ and (x, y) .



Problems 29–31 use the figure from Problems 23–24.

29. Find the coordinates of point A where the tangent line to the ellipse is horizontal.
30. Find the coordinates of point B where the tangent line to the ellipse is vertical.
31. Find the coordinates of points C and D.

In 32–40, find $\frac{dy}{dx}$ in two ways: (a) by using the “usual” differentiation patterns and (b) by using logarithmic differentiation.

32. $y = x \cdot \sin(3x)$ 33. $y = (x^2 + 5)^7(x^3 - 1)^4$

34. $y = \frac{\sin(3x - 1)}{x + 7}$ 35. $y = x^5 \cdot (3x + 2)^4$

36. $y = 7^x$ 37. $y = e^{\sin(x)}$

38. $y = \cos^7(2x + 5)$ 39. $y = \sqrt{25 - x^2}$

40. $y = \frac{x \cdot \cos(x)}{x^2 + 1}$

In 41–46, use logarithmic differentiation to find $\frac{dy}{dx}$.

41. $y = x^{\cos(x)}$

42. $y = (\cos(x))^x$

43. $y = x^4 \cdot (x - 2)^7 \cdot \sin(3x)$

44. $y = \frac{\sqrt{x + 10}}{(2x + 3)^3 \cdot (5x - 1)^7}$

45. $y = (3 + \sin(x))^x$

46. $y = \sqrt{\frac{x^2 + 1}{x^2 - 1}}$

In 47–50, use the values in each table to calculate the values of the derivative in the last column.

	x	$f(x)$	$\ln(f(x))$	$\mathbf{D}(\ln(f(x)))$	$f'(x)$
47.	1	1	0.0	1.2	
	2	9	2.2	1.8	
	3	64	4.2	2.1	

	x	$g(x)$	$\ln(g(x))$	$\mathbf{D}(\ln(g(x)))$	$g'(x)$
48.	1	5	1.6	0.6	
	2	10	2.3	0.7	
	3	20	3.0	0.8	

	x	$f(x)$	$\ln(f(x))$	$\mathbf{D}(\ln(f(x)))$	$f'(x)$
49.	1	5	1.6	−1	
	2	2	0.7	0	
	3	7	1.9	2	
	x	$g(x)$	$\ln(g(x))$	$\mathbf{D}(\ln(g(x)))$	$g'(x)$
50.	2	1.4	0.3	1.2	
	3	3.3	1.2	0.6	
	7	13.6	2.6	0.2	

Problems 51–55 illustrate how logarithmic differentiation can be used to verify some differentiation patterns we already know (51–52, 54) and to derive some new patterns (53, 55). Assume that all of the functions are differentiable and that the function combinations are defined.

51. Use logarithmic differentiation on $f \cdot g$ to re-derive the Product Rule: $\mathbf{D}(f \cdot g) = f \cdot g' + g \cdot f'$.

52. Use logarithmic differentiation on $\frac{f}{g}$ to re-derive the quotient rule: $\mathbf{D}\left(\frac{f}{g}\right) = \frac{g \cdot f' - f \cdot g'}{g^2}$.

53. Use logarithmic differentiation to obtain a product rule for three functions: $\mathbf{D}(f \cdot g \cdot h) = ?$.

54. Use logarithmic differentiation on the exponential function a^x (with $a > 0$) to show that its derivative is $a^x \ln(a)$.

55. Use logarithmic differentiation to determine a pattern for the derivative of f^g : $\mathbf{D}(f^g) = ?$.

56. In Section 2.1 we proved the Power Rule $\mathbf{D}(x^n) = n \cdot x^{n-1}$ for any positive integer n .

(a) Why does this formula hold for $n = 0$?

(b) Use the Quotient Rule to prove that $\mathbf{D}(x^{-m}) = -m \cdot x^{-m-1}$ for any positive integer m and conclude that the Power Rule holds for all integers.

(c) Now let $y = x^{\frac{p}{q}}$ where p and q are integers so that $y^q = x^p$. Use implicit differentiation to show that the Power Rule holds for all rational exponents. (We still have not considered the case where $y = x^a$ with a an irrational number, because we haven't actually *defined* what x^a means for a irrational. We will take care of that—and the extension of the Power Rule to all real exponents—in Chapter 7.)

2.9 Practice Answers

1. $\mathbf{D}(x^2 + y^2) = 2x + 2y \cdot y'$
 $\frac{d}{dx}(\sin(2 + 3y)) = \cos(2 + 3y) \cdot \mathbf{D}(2 + 3y) = \cos(2 + 3y) \cdot 3y'$
2. Explicitly: $y' = \frac{1}{3} (3x^2 + 15)^{-\frac{2}{3}} \mathbf{D}(3x^2 + 15) = \frac{1}{3} (3x^2 + 15)^{-\frac{2}{3}} (6x)$.
 When $(x, y) = (2, 3)$, $y' = \frac{1}{3} (3(2)^2 + 15)^{-\frac{2}{3}} (6 \cdot 2) = 4 (27)^{-\frac{2}{3}} = \frac{4}{9}$.
 Implicitly: $\mathbf{D}(y^3 - 3x^2) = \mathbf{D}(15) \Rightarrow 3y^2 \cdot y' - 6x = 0$ so $y' = \frac{2x}{y^2}$.
 When $(x, y) = (2, 3)$, $y' = \frac{2 \cdot 2}{3^2} = \frac{4}{9}$.
3. $\mathbf{D}(y + \sin(y)) = \mathbf{D}(x^3 - x) \Rightarrow y' + \cos(y) \cdot y' = 3x^2 - 1 \Rightarrow y' \cdot (1 + \cos(y)) = 3x^2 - 1$, so we have $y' = \frac{3x^2 - 1}{1 + \cos(y)}$. When $(x, y) = (1, 0)$, $y' = \frac{3(1)^2 - 1}{1 + \cos(0)} = 1$.
4. To find where the parabola crosses the y -axis, we can set $x = 0$ and solve for the values of y : $y^2 - 7y + 2 = 0 \Rightarrow y = \frac{7 \pm \sqrt{(-7)^2 - 4(1)(2)}}{2(1)} = \frac{7 \pm \sqrt{41}}{2} \approx 0.3$ and 6.7 . The parabola crosses the y -axis (approximately) at the points $(0, 0.3)$ and $(0, 6.7)$. From Example 4, we know that $y' = \frac{-2x - 2y - 3}{2x + 2y - 7}$, so at the point $(0, 0.3)$, the slope is approximately $\frac{0 - 0.6 - 3}{0 + 0.6 - 7} \approx 0.56$, and at the point $(0, 6.7)$, the slope is approximately $\frac{0 - 13.4 - 3}{0 + 13.4 - 7} \approx -2.56$.
5. Applying the formula $f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x)))$ to the function $f(x) = (2x + 1)^3(3x^2 - 4)^7(x + 7)^4$, we have:

$$\ln(f(x)) = 3 \cdot \ln(2x + 1) + 7 \cdot \ln(3x^2 - 4) + 4 \cdot \ln(x + 7)$$

so:

$$\mathbf{D}(\ln(f(x))) = \frac{3}{2x + 1}(2) + \frac{7}{3x^2 - 4}(6x) + \frac{4}{x + 7}(1)$$

and thus:

$$f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x))) = (2x + 1)^3(3x^2 - 4)^7(x + 7)^4 \cdot \left[\frac{6}{2x + 1} + \frac{42x}{3x^2 - 4} + \frac{4}{x + 7} \right]$$

6. Using $f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x)))$ with $f(x) = x^{\sin(x)}$:

$$\ln(f(x)) = \ln(x^{\sin(x)}) = \sin(x) \cdot \ln(x)$$

so:

$$\mathbf{D}(\ln(f(x))) = \mathbf{D}(\sin(x) \cdot \ln(x)) = \sin(x) \cdot \mathbf{D}(\ln(x)) + \ln(x) \cdot \mathbf{D}(\sin(x)) = \sin(x) \cdot \frac{1}{x} + \ln(x) \cdot \cos(x)$$

and thus:

$$f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x))) = x^{\sin(x)} \cdot \left[\frac{\sin(x)}{x} + \ln(x) \cdot \cos(x) \right]$$