3.3 The First Derivative and the Shape of f

This section examines some of the interplay between the shape of the graph of a function f and the behavior of its derivative, f'. If we have a graph of f, we will investigate what we can conclude about the values of f'. And if we know values of f', we will investigate what we can conclude about the graph of f.

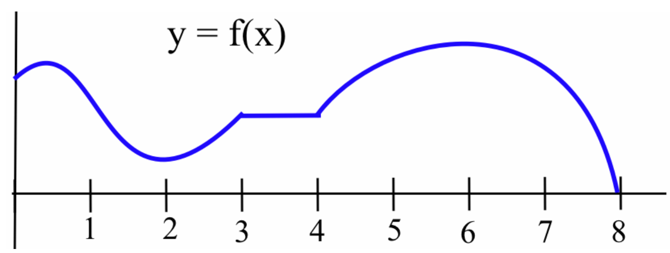
In this definition, I can be of the form (a,b), $[a,b)$, $(a,b]$, $[a,b]$, $(-\infty,b)$, $(-\infty,b]$, (a,∞), $[a,\infty)$ or $(-\infty,\infty)$, where $a < b$.

Definitions: Given any interval I, a function f is . . .
increasing on I if, for all x_1 and x_2 in I, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
decreasing on I if, for all x_1 and x_2 in I, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
monotonic on I if f is increasing or decreasing on I

Graphically, f is increasing (decreasing) if, as we move from left to right along the graph of f, the height of the graph increases (decreases).

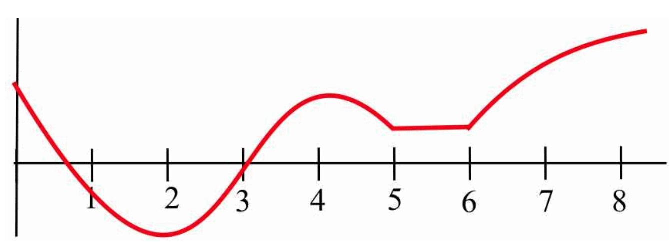
These same ideas make sense if we consider $h(t)$ to be the height (in feet) of a rocket at time t seconds. We naturally say that the rocket is rising or that its height is increasing if the height $h(t)$ increases over a period of time, as t increases.

Example 1. List the intervals on which the function graphed below is increasing or decreasing.



Solution. f is increasing on the intervals $[0, 0.3]$ (approximately), $[2, 3]$ and $[4, 6]$. f is decreasing on (approximately) $[0.3, 2]$ and $[6, 8]$. On the interval $[3, 4]$ the function is not increasing or decreasing — it is **constant**. It is also valid to say that f is increasing on the intervals $[0.5, 0.8]$ and $(0.5, 0.8)$ as well as many others, but we usually talk about the longest intervals on which f is monotonic. ◀

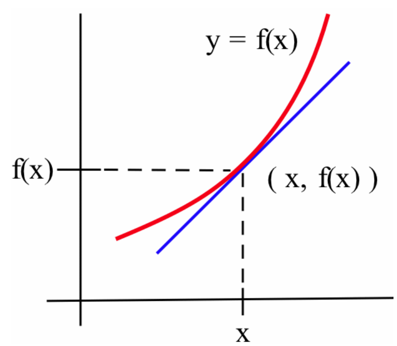
Practice 1. List the intervals on which the function graphed below is increasing or decreasing.



If we have an accurate graph of a function, then it is relatively easy to determine where f is monotonic, but if the function is defined by a formula, then a little more work is required. The next two theorems relate the values of the derivative of f to the monotonicity of f. The first theorem says that if we know where f is monotonic, then we also know something about the values of f'. The second theorem says that if we know about the values of f' then we can draw conclusions about where f is monotonic.

First Shape Theorem:
For a function f that is differentiable on an interval (a, b):

- if f is increasing on (a, b) then $f'(x) \geq 0$ for all x in (a, b)
- if f is decreasing on (a, b) then $f'(x) \leq 0$ for all x in (a, b)
- if f is constant on (a, b), then $f'(x) = 0$ for all x in (a, b)



Proof. Most people find a picture such as the one in the margin to be a convincing justification of this theorem: if the graph of f increases near a point $(x, f(x))$, then the tangent line is also increasing, and the slope of the tangent line is positive (or perhaps zero at a few places). A more precise proof, however, requires that we use the definitions of the derivative of f and of "increasing" (given above).

Case I: Assume that f is increasing on (a, b). We know that f is differentiable, so if x is any number in the interval (a, b) then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

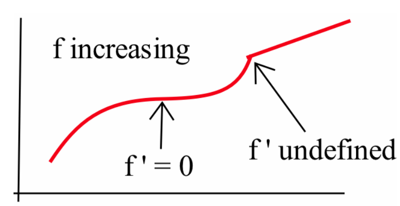
and this limit exists and is a finite value. If h is any small enough **positive** number so that $x + h$ is also in the interval (a, b), then $x < x + h \Rightarrow f(x) < f(x+h)$ (by the definition of "increasing"). We know that the numerator, $f(x+h) - f(x)$, and the denominator, h, are both positive, so the limiting value, $f'(x)$, must be positive or zero: $f'(x) \geq 0$.

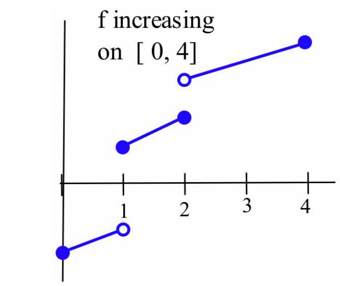
Case II: Assume that f is decreasing on (a, b). If $x < x + h$, then $f(x) > f(x+h)$ (by the definition of "decreasing"). So the numerator of the limit, $f(x+h) - f(x)$, will be negative but the denominator, h, will still be positive, so the limiting value, $f'(x)$, must be negative or zero: $f'(x) \leq 0$.

The proof of this part is very similar to the "increasing" proof.

Case III: The derivative of a constant is 0, so if f is constant on (a, b) then $f'(x) = 0$ for all x in (a, b). $\square$

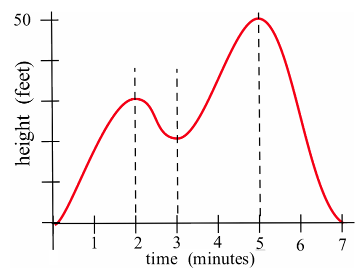
The previous theorem is easy to understand, but you need to pay attention to exactly what it says and what it does **not** say. It is possible for a differentiable function that is increasing on an interval to have horizontal tangent lines at some places in the interval (see margin). It is



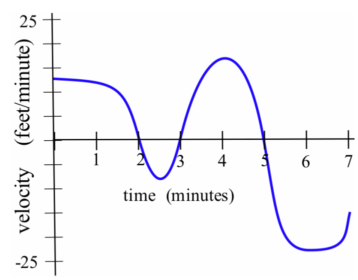


also possible for a continuous function that is increasing on an interval to have an undefined derivative at some places in the interval. Finally, it is possible for a function that is increasing on an interval to fail to be continuous at some places in the interval (see margin).

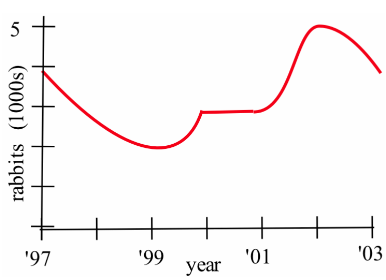
The First Shape Theorem has a natural interpretation in terms of the height $h(t)$ and upward velocity $h'(t)$ of a helicopter at time t. If the height of the helicopter is increasing ($h(t)$ is an increasing function), then the helicopter has a positive or zero upward velocity: $h'(t) \geq 0$. If the height of the helicopter is not changing, then its upward velocity is 0: $h'(t) = 0$.



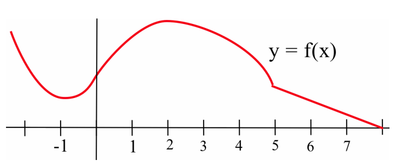
Example 2. A figure in the margin shows the height of a helicopter during a period of time. Sketch the graph of the upward velocity of the helicopter, $\frac{dh}{dt}$.



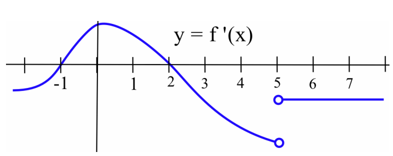
Solution. The graph of $v(t) = \frac{dh}{dt}$ appears in the margin. Notice that $h(t)$ has a local maximum when $t = 2$ and $t = 5$, and that $v(2) = 0$ and $v(5) = 0$. Similarly, $h(t)$ has a local minimum when $t = 3$, and $v(3) = 0$. When h is increasing, v is positive. When h is decreasing, v is negative. ◀



Practice 2. A figure in the margin shows the population of rabbits on an island during a 6-year period. Sketch the graph of the rate of population change, $\frac{dR}{dt}$, during those years.



Example 3. A graph of f appears in the margin; sketch a graph of f'.

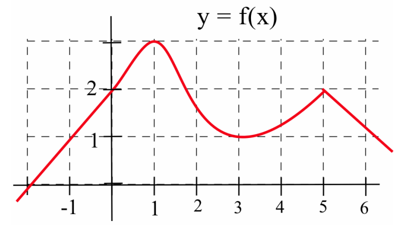


Solution. It is a good idea to look first for the points where $f'(x) = 0$ or where f is not differentiable (the critical points of f). These locations are usually easy to spot, and they naturally break the problem into several smaller pieces. The only numbers at which $f'(x) = 0$ are $x = -1$ and $x = 2$, so the only places the graph of $f'(x)$ will cross the x-axis are at $x = -1$ and $x = 2$: we can therefore plot the points $(-1, 0)$ and $(2, 0)$ on the graph of f'. The only place where f is not differentiable is at the "corner" above $x = 5$, so the graph of f' will not be defined for $x = 5$. The rest of the graph of f is relatively easy to sketch:

- if $x < -1$ then $f(x)$ is decreasing so $f'(x)$ is negative
- if $-1 < x < 2$ then $f(x)$ is increasing so $f'(x)$ is positive
- if $2 < x < 5$ then $f(x)$ is decreasing so $f'(x)$ is negative
- if $5 < x$ then $f(x)$ is decreasing so $f'(x)$ is negative

A graph of f' appears on the previous page: $f(x)$ is continuous at $x = 5$, but not differentiable at $x = 5$ (indicated by the "hole"). ◀

Practice 3. A graph of f appears in the margin. Sketch a graph of f'. (The graph of f has a "corner" at $x = 5$.)



The next theorem is almost the converse of the First Shape Theorem and explains the relationship between the values of the derivative and the graph of a function from a different perspective. It says that if we know something about the values of f', then we can draw some conclusions about the shape of the graph of f.

Second Shape Theorem:
For a function f that is differentiable on an interval I:

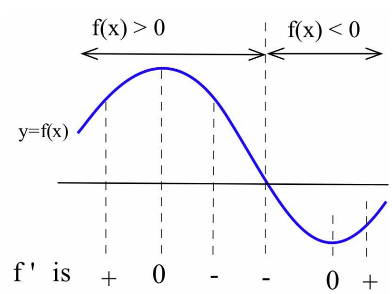
- if $f'(x) > 0$ for all x in the interval I, then f is increasing on I
- if $f'(x) < 0$ for all x in the interval I, then f is decreasing on I
- if $f'(x) = 0$ for all x in the interval I, then f is constant on I

Proof. This theorem follows directly from the Mean Value Theorem, and the last part is just a restatement of the First Corollary of the Mean Value Theorem.

Case I: Assume that $f'(x) > 0$ for all x in I and pick any points a and b in I with $a < b$. Then, by the Mean Value Theorem, there is a point c between a and b so that $\dfrac{f(b) - f(a)}{b - a} = f'(c) > 0$ and we can conclude that $f(b) - f(a) > 0$, which means that $f(b) > f(a)$. Because $a < b \Rightarrow f(a) < f(b)$, we know that f is increasing on I.

Case II: Assume that $f'(x) < 0$ for all x in I and pick any points a and b in I with $a < b$. Then there is a point c between a and b so that $\dfrac{f(b) - f(a)}{b - a} = f'(c) < 0$, and we can conclude that $f(b) - f(a) = (b - a)f'(c) < 0$ so $f(b) < f(a)$. Because $a < b \Rightarrow f(a) > f(b)$, we know f is decreasing on I. □

Practice 4. Rewrite the Second Shape Theorem as a statement about the height $h(t)$ and upward velocity $h'(t)$ of a helicopter at time t seconds.



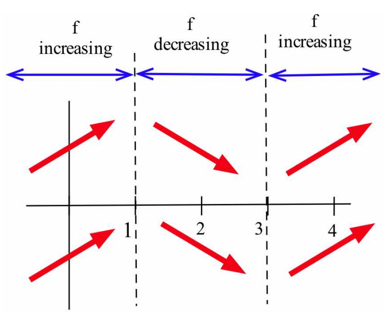
The value of a function f at a number x tells us the height of the graph of f above or below the point $(x, 0)$ on the x-axis. The value of f' at a number x tells us whether the graph of f is increasing or decreasing (or neither) as the graph passes through the point $(x, f(x))$ on the graph of f. If $f(x)$ is positive, it is possible for $f'(x)$ to be positive, negative, zero or undefined: the value of $f(x)$ has absolutely nothing to do with the value of f'. The margin figure illustrates some of the possible combinations of values for f and f'.

Practice 5. Graph a continuous function that satisfies the conditions on f and f' given below:

x	-2	-1	0	1	2	3
$f(x)$	1	-1	-2	-1	0	2
$f'(x)$	-1	0	1	2	-1	1

The Second Shape Theorem can be particularly useful if we need to graph a function f defined by a formula. Between any two consecutive critical numbers of f, the graph of f is monotonic (why?). If we can find all of the critical numbers of f, then the domain of f will be broken naturally into a number of pieces on which f will be monotonic.

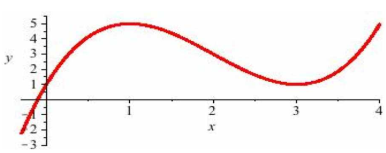
Example 4. Use information about the values of f' to help graph $f(x) = x^3 - 6x^2 + 9x + 1$.



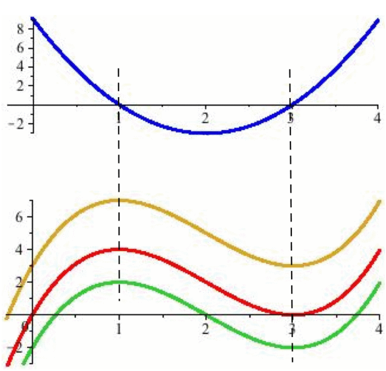
Solution. $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$ so $f'(x) = 0$ only when $x = 1$ or $x = 3$; f' is a polynomial, so it is always defined. The only critical numbers, $x = 1$ and $x = 3$, break the real number line into three pieces on which f is monotonic: $(-\infty, 1)$, $(1, 3)$ and $(3, \infty)$.

- $x < 1 \Rightarrow f'(x) = 3(\text{negative})(\text{negative}) > 0 \Rightarrow f$ increasing
- $1 < x < 3 \Rightarrow f'(x) = 3(\text{positive})(\text{negative}) < 0 \Rightarrow f$ is decreasing
- $3 < x \Rightarrow f'(x) = 3(\text{positive})(\text{positive}) > 0 \Rightarrow f$ is increasing

Although we don't yet know the value of f anywhere, we do know a lot about the shape of the graph of f: as we move from left to right along the x-axis, the graph of f increases until $x = 1$, then decreases until $x = 3$, after which the graph increases again (see margin). The graph of f "turns" when $x = 1$ and $x = 3$. To plot the graph of f, we still need to evaluate f at a few values of x, but only at a **very** few values: $f(1) = 5$, and $(1, 5)$ is a local maximum of f; $f(3) = 1$, and $(3, 1)$ is a local minimum of f. A graph of f appears in the margin. ◀

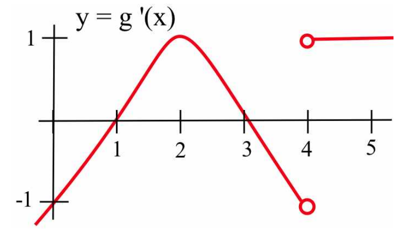


Practice 6. Use information about the values of f' to help graph the function $f(x) = x^3 - 3x^2 - 24x + 5$.

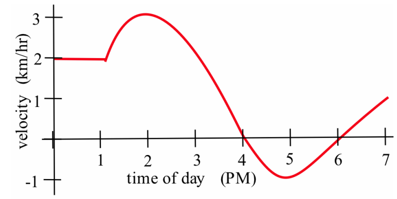


Example 5. Use the graph of f' in the margin to sketch the shape of the graph of f. Why isn't the graph of f' enough to completely determine the graph of f?

Solution. Several functions that have the derivative we want appear in the margin, and each provides a correct answer. By the Second Corollary to the Mean Value Theorem, we know there is a whole family of "parallel" functions that share the derivative we want, and each of these functions provides a correct answer. If we had additional information about the function — such as a point it passes through — then only one member of the family would satisfy the extra condition and there would be only one correct answer. ◀



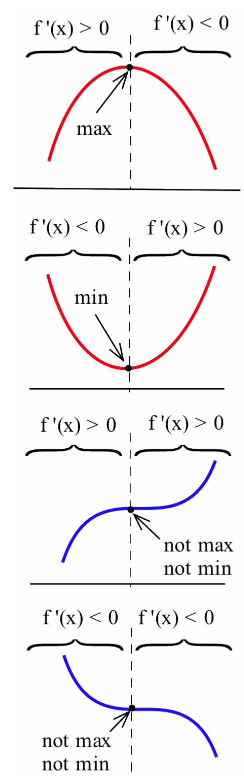
Practice 7. Use the graph of g' provided in the margin to sketch the shape of a graph of g.



Practice 8. A weather balloon is released from the ground and sends back its upward velocity measurements (see margin). Sketch a graph of the height of the balloon. When was the balloon highest?

Using the Derivative to Test for Extremes

The first derivative of a function tells about the general shape of the function, and we can use that shape information to determine whether an extreme point is a (local) maximum or minimum or neither.



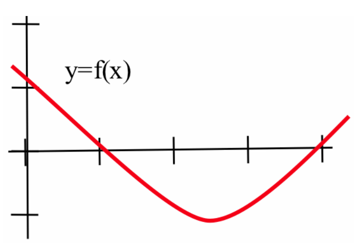
First Derivative Test for Local Extremes:
Let f be a continuous function with $f'(c) = 0$ or $f'(c)$ undefined.

- If $f'(\text{left of } c) > 0$ and $f'(\text{right of } c) < 0$ then $(c, f(c))$ is a local maximum.
- If $f'(\text{left of } c) < 0$ and $f'(\text{right of } c) > 0$ then $(c, f(c))$ is a local minimum.
- If $f'(\text{left of } c) > 0$ and $f'(\text{right of } c) > 0$ then $(c, f(c))$ is **not** a local extreme.
- If $f'(\text{left of } c) < 0$ and $f'(\text{right of } c) < 0$ then $(c, f(c))$ is **not** a local extreme.

Practice 9. Find all extremes of $f(x) = 3x^2 - 12x + 7$ and use the First Derivative Test to classify them as maximums, minimums or neither.

3.3 Problems

In Problems 1–3, sketch the graph of the **derivative** of each function.

1.


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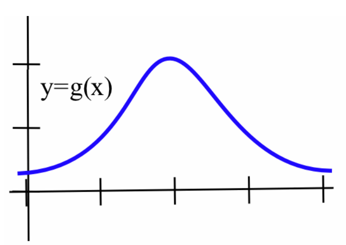

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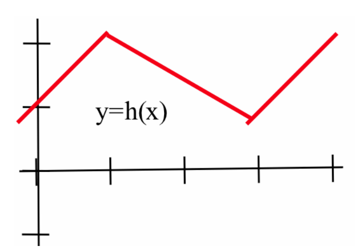
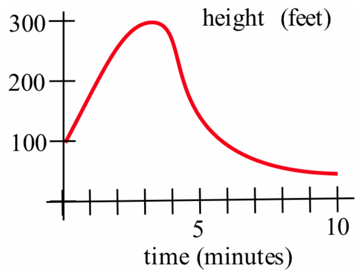
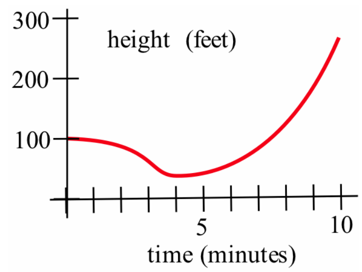
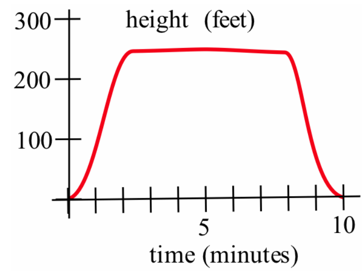


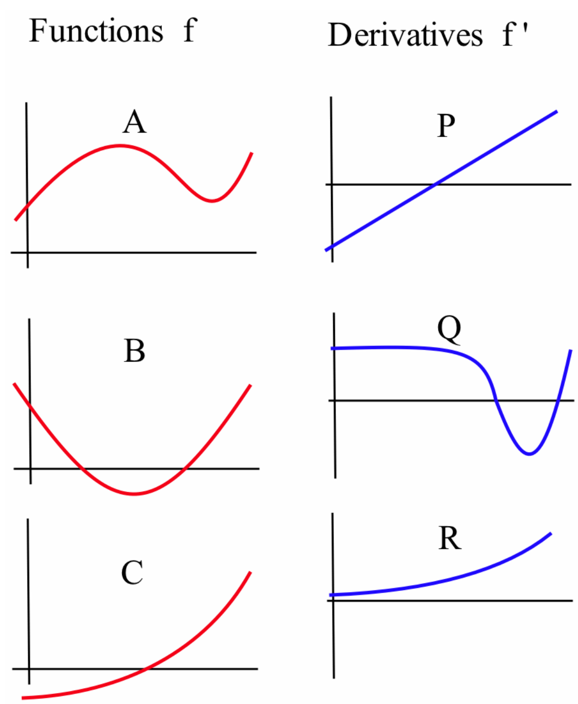
Problems 4–6 show the graph of the height of a helicopter; sketch a graph of its upward velocity.

4. 

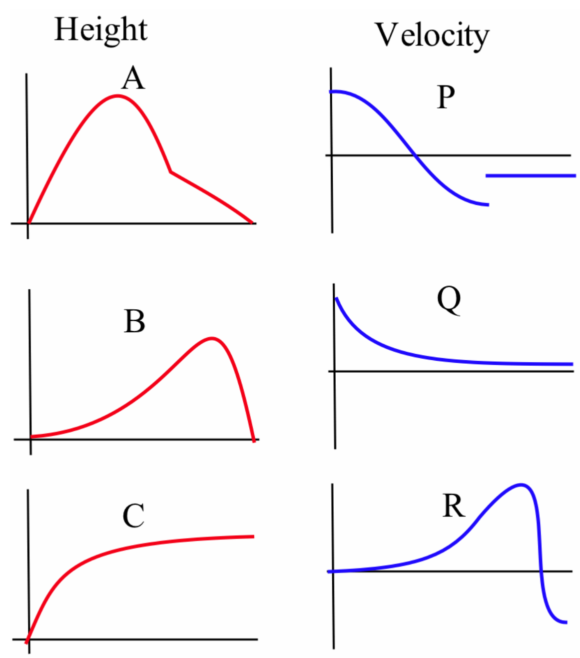
5. 

6. 

7. In the figure below, match the graphs of the functions with those of their derivatives.



8. Match the graphs showing the heights of rockets with those showing their velocities.

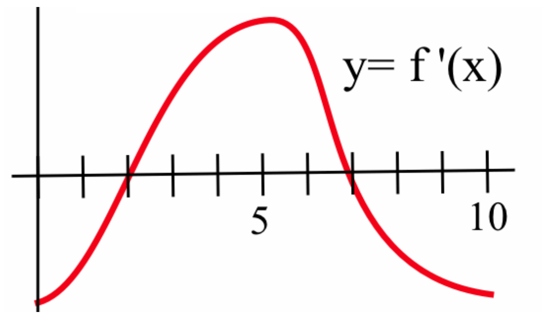


9. Use the Second Shape Theorem to show that $f(x) = \ln(x)$ is monotonic increasing on the interval $(0, \infty)$.

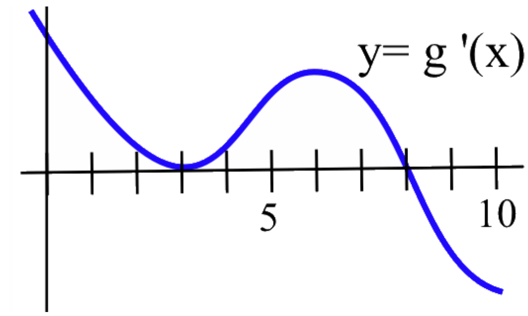
10. Use the Second Shape Theorem to show that $g(x) = e^x$ is monotonic increasing on the entire real number line.

11. A student is working with a complicated function f and has shown that the derivative of f is always positive. A minute later the student also claims that $f(x) = 2$ when $x = 1$ and when $x = \pi$. Without checking the student's work, how can you be certain that it contains an error?

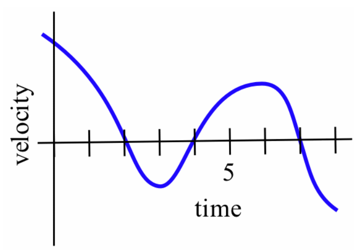
12. The figure below shows the graph of the **derivative** of a continuous function f.
    (a) List the critical numbers of f.
    (b) What values of x result in a local maximum?
    (c) What values of x result in a local minimum?



13. The figure below shows the graph of the **derivative** of a continuous function g.
    (a) List the critical numbers of g.
    (b) What values of x result in a local maximum?
    (c) What values of x result in a local minimum?



Problems 14–16 show the graphs of the upward velocities of three helicopters. Use the graphs to determine when each helicopter was at a (relative) maximum or minimum height.

14.


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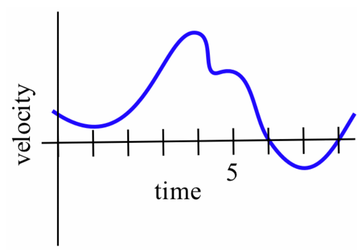
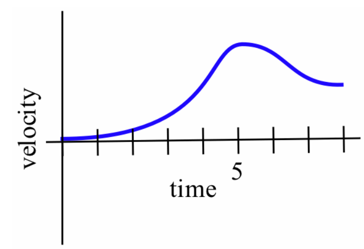


16.


In 17–22, use information from the derivative of each function to help you graph the function. Find all local maximums and minimums of each function.

17. $f(x) = x^3 - 3x^2 - 9x - 5$
18. $g(x) = 2x^3 - 15x^2 + 6$
19. $h(x) = x^4 - 8x^2 + 3$
20. $s(t) = t + \sin(t)$
21. $r(t) = \dfrac{2}{t^2 + 1}$
22. $f(x) = \dfrac{x^2 + 3}{x}$
23. $f(x) = 2x + \cos(x)$ so $f(0) = 1$. Without graphing the function, you can be certain that f has how many **positive** roots?
24. $g(x) = 2x - \cos(x)$ so $g(0) = -1$. Without graphing the function, you can be certain that g has how many **positive** roots?
25. $h(x) = x^3 + 9x - 10$ has a root at $x = 1$. Without graphing h, show that h has no other roots.
26. Sketch the graphs of monotonic decreasing functions that have exactly (a) no roots (b) one root and (c) two roots.
27. Each of the following statements is false. Give (or sketch) a counterexample for each statement.
    (a) If f is increasing on an interval I, then $f'(x) > 0$ for all x in I.
    (b) If f is increasing and differentiable on I, then $f'(x) > 0$ for all x in I.
    (c) If cars A and B always have the same speed, then they will always be the same distance apart.

28. (a) Find several different functions f that all have the same derivative $f'(x) = 2$.
    (b) Determine a function f with derivative $f'(x) = 2$ that also satisfies $f(1) = 5$.
    (c) Determine a function g with $g'(x) = 2$ for which the graph of g goes through $(2, 1)$.

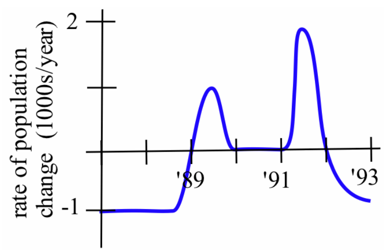
29. (a) Find several different functions h that all have the same derivative $h'(x) = 2x$.
    (b) Determine a function f with derivative $f'(x) = 2x$ that also satisfies $f(3) = 20$.
    (c) Determine a function g with $g'(x) = 2x$ for which the graph of g goes through $(2, 7)$.

30. Sketch functions with the given properties to help determine whether each statement is true or false.
    (a) If $f'(7) > 0$ and $f'(x) > 0$ for all x near 7, then $f(7)$ is a local maximum of f on $[1, 7]$.
    (b) If $g'(7) < 0$ and $g'(x) < 0$ for all x near 7, then $g(7)$ is a local minimum of g on $[1, 7]$.
    (c) If $h'(1) > 0$ and $h'(x) > 0$ for all x near 1, then $h(1)$ is a local minimum of h on $[1, 7]$.
    (d) If $r'(1) < 0$ and $r'(x) < 0$ for all x near 1, then $r(1)$ is a local maximum of r on $[1, 7]$.
    (e) If $s'(7) = 0$, then $s(7)$ is a local maximum of s on $[1, 7]$.

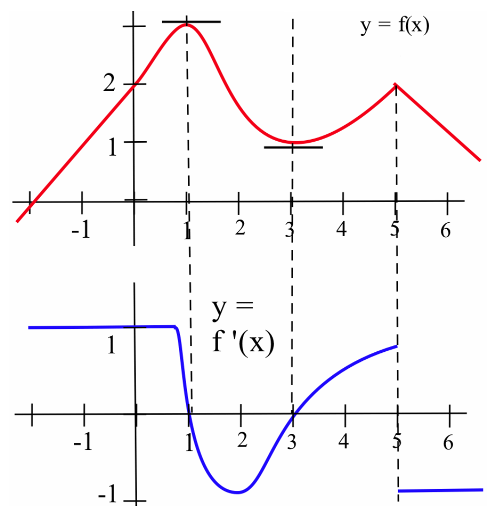
3.3 Practice Answers

1. g is increasing on $[2, 4]$ and $[6, 8]$; g is decreasing on $[0, 2]$ and $[4, 5]$; g is constant on $[5, 6]$.

2. The graph in the margin shows the rate of population change, $\frac{dR}{dt}$.



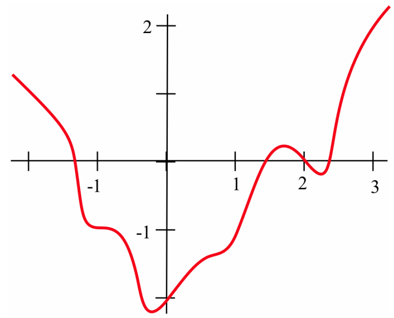
3. A graph of f' appears below. Notice how the graph of f' is 0 where f has a maximum or minimum.



4. The Second Shape Theorem for helicopters:
    - If the upward velocity h' is positive during time interval I then the height h is increasing during time interval I.
    - If the upward velocity h' is negative during time interval I then the height h is decreasing during time interval I.
    - If the upward velocity h' is zero during time interval I then the height h is constant during time interval I.

5. A graph satisfying the conditions in the table appears in the margin.

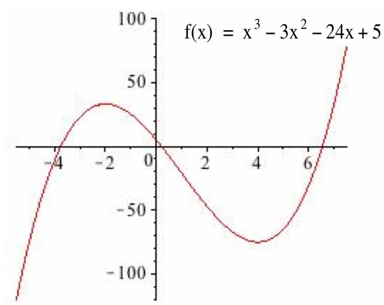
x	-2	-1	0	1	2	3
$f(x)$	1	-1	-2	-1	0	2
$f'(x)$	-1	0	1	2	-1	1



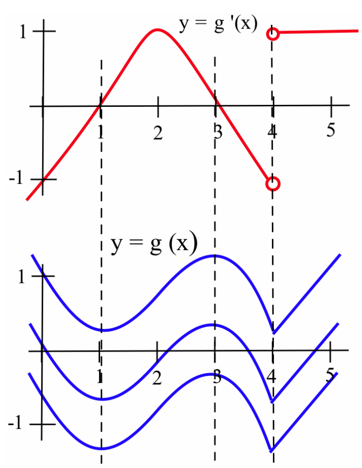
6. $f'(x) = 3x^2 - 6x - 24 = 3(x-4)(x+2)$ so $f'(x) = 0$ if $x = -2$ or $x = 4$.

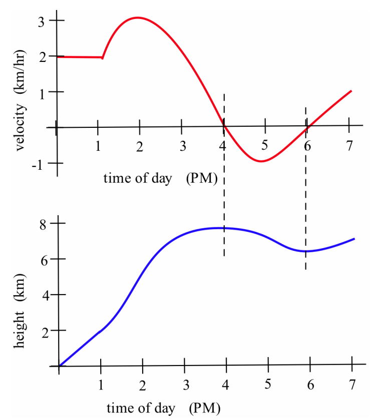
   - $x < -2 \Rightarrow f'(x) = 3(\text{negative})(\text{negative}) > 0 \Rightarrow f$ increasing
   - $-2 < x < 4 \Rightarrow f'(x) = 3(\text{negative})(\text{positive}) < 0 \Rightarrow f$ decreasing
   - $x > 4 \Rightarrow f'(x) = 3(\text{positive})(\text{positive}) > 0 \Rightarrow f$ increasing

   Thus f has a relative maximum at $x = -2$ and a relative minimum at $x = 4$. A graph of f appears in the margin.



7. The figure below left shows several possible graphs for g. Each has the correct shape to give the graph of g'. Notice that the graphs of g are "parallel" (differ by a constant).





8. The figure above right shows the height graph for the balloon. The balloon was highest at 4 p.m. and had a local minimum at 6 p.m.

9. $f'(x) = 6x - 12$ so $f'(x) = 0$ only if $x = 2$.

   - $x < 2 \Rightarrow f'(x) < 0 \Rightarrow f$ decreasing
   - $x > 2 \Rightarrow f'(x) > 0 \Rightarrow f$ increasing

   From this we can conclude that f has a minimum when $x = 2$ and has a shape similar to graph provided in the margin.

   We could also have noticed that the graph of the quadratic function $f(x) = 3x^2 - 12x + 7$ must be an upward-opening parabola.