3.7 *L'Hôpital's Rule*

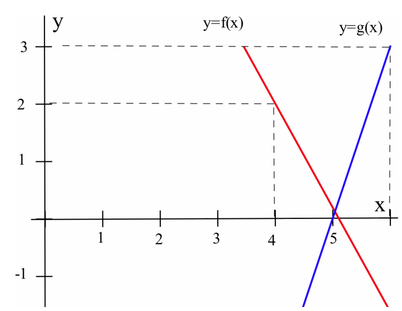
When taking limits of slopes of secant lines, $m_{\text{sec}} = \dfrac{f(x+h) - f(x)}{h}$ as $h \to 0$, we frequently encountered one difficulty: both the numerator and the denominator approached 0. And because the denominator approached 0, we could not apply the Main Limit Theorem. In many situations, however, we managed to get past this "$\frac{0}{0}$" difficulty by using algebra or geometry or trigonometry to rewrite the expression and then take the limit. But there was no common approach or pattern. The algebraic steps we used to evaluate $\displaystyle\lim_{h\to 0} \frac{(2+h)^2 - 4}{h}$ seem quite different from the trigonometric steps needed for $\displaystyle\lim_{h\to 0} \frac{\sin(2+h) - \sin(2)}{h}$.

In this section we consider a single technique, called l'Hôpital's Rule, that enables us to quickly and easily evaluate many limits of the form "$\frac{0}{0}$" as well as several other challenging indeterminate forms.

Although discovered by Johann Bernoulli, this rule was named for the Marquis de l'Hôpital (pronounced low-pee-TALL), who published it in his 1696 calculus textbook, *Analysis of the Infinitely Small for the Understanding of Curved Lines*.

A Linear Example

The graphs of two linear functions appear in the margin and we want to find $\displaystyle\lim_{x\to 5} \frac{f(x)}{g(x)}$. Unfortunately, $\displaystyle\lim_{x\to 5} f(x) = 0$ and $\displaystyle\lim_{x\to 5} g(x) = 0$ so we cannot apply the Main Limit Theorem. We do know, however, that f and g are linear, so we can calculate their slopes, and we know that they both lines go through the point $(5, 0)$ so we can find their equations: $f(x) = -2(x-5)$ and $g(x) = 3(x-5)$.



Now the limit is easier to compute:

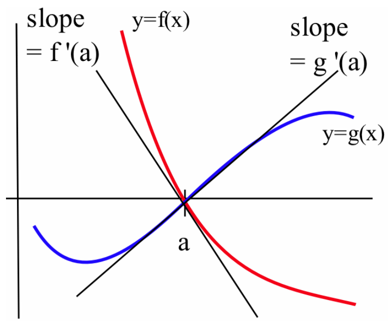
$$\lim_{x\to 5} \frac{f(x)}{g(x)} = \lim_{x\to 5} \frac{-2(x-5)}{3(x-5)} = \lim_{x\to 5} \frac{-2}{3} = -\frac{2}{3} = \frac{\text{slope of } f}{\text{slope of } g}$$

In fact, this pattern works for any two linear functions: If f and g are linear functions with slopes $m \neq 0$ and $n \neq 0$ and a common root at $x = a$, then $f(x) - f(a) = m(x-a)$ and $g(x) - g(a) = n(x-a)$ so $f(x) = m(x-a)$ and $g(x) = n(x-a)$. Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{m(x-a)}{n(x-a)} = \lim_{x\to a} \frac{m}{n} = \frac{m}{n} = \frac{\text{slope of } f}{\text{slope of } g}$$

A more powerful result—that the same pattern holds true for differentiable functions even if they are not linear—is called l'Hôpital's Rule.

> L'Hôpital's Rule ("$\frac{0}{0}$" Form)
>
> If f and g are differentiable at $x = a$,
> $f(a) = 0$, $g(a) = 0$ and $g'(a) \neq 0$
>
> then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \frac{\text{slope of } f \text{ at } a}{\text{slope of } g \text{ at } a}$



Unfortunately, we have ignored some subtle difficulties, such as $g(x)$ or $g'(x)$ possibly being 0 when x is close to, but not equal to, a. Because of these issues, a full-fledged proof of l'Hôpital's Rule is omitted.

Idea for a proof: Even though f and g may not be linear functions, they *are* differentiable. So at the point $x = a$ they are "almost linear" in the sense that we can approximate them quite well using their tangent lines at that point (see margin).

Because $f(a) = g(a) = 0$, $f(x) \approx f(a) + f'(a)(x-a) = f'(a)(x-a)$ and $g(x) \approx g(a) + g'(a)(x-a) = g'(a)(x-a)$. So:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \approx \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x\to a} \frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)}$$

Example 1. Determine $\lim_{x\to 0} \frac{x^2 + \sin(5x)}{3x}$ and $\lim_{x\to 1} \frac{\ln(x)}{e^x - e}$.

Solution. We could evaluate the first limit without l'Hôpital's Rule, but let's use it anyway. We can match the pattern of l'Hôpital's Rule by letting $a = 0$, $f(x) = x^2 + \sin(5x)$ and $g(x) = 3x$. Then $f(0) = 0$, $g(0) = 0$, and f and g are differentiable with $f'(x) = 2x + 5\cos(5x)$ and $g'(x) = 3$, so:

$$\lim_{x\to 0} \frac{x^2 + \sin(5x)}{3x} = \frac{f'(0)}{g'(0)} = \frac{2\cdot 0 + 5\cos(5\cdot 0)}{3} = \frac{5}{3}$$

For the second limit, let $a = 1$, $f(x) = \ln(x)$ and $g(x) = e^x - e$. Then $f(1) = 0$, $g(1) = 0$, f and g are differentiable for x near 1 (when $x > 0$), and $f'(x) = \frac{1}{x}$ and $g'(x) = e^x$. Then:

$$\lim_{x\to 1} \frac{\ln(x)}{e^x - e} = \frac{f'(1)}{g'(1)} = \frac{\frac{1}{1}}{e^1} = \frac{1}{e}$$

Here no simplification was possible, so we needed l'Hôpital's Rule. ◀

Practice 1. Evaluate $\lim_{x\to 0} \frac{1 - \cos(5x)}{3x}$ and $\lim_{x\to 2} \frac{x^2 + x - 6}{x^2 + 2x - 8}$.

Strong Version of l'Hôpital's Rule

We can strengthen L'Hôpital's Rule to include cases when $g'(a) = 0$, and the indeterminate form "$\frac{\infty}{\infty}$" when f and g increase without bound.

L'Hôpital's Rule (Strong "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" Forms)

If f and g are differentiable on an open interval I containing a, $g'(x) \neq 0$ on I except possibly at a, and $\lim_{x\to a} \frac{f(x)}{g(x)} = $ "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the limit on the right exists.

(Here "a" can represent a finite number or "∞.")

Example 2. Evaluate $\lim_{x\to\infty} \dfrac{e^{7x}}{5x}$.

Solution. As "$x \to \infty$," both e^{7x} and $5x$ increase without bound, so we have an "$\frac{\infty}{\infty}$" indeterminate form and can use the Strong Version of l'Hôpital's Rule: $\lim_{x\to\infty} \dfrac{e^{7x}}{5x} = \lim_{x\to\infty} \dfrac{7e^{7x}}{5} = \infty$. ◀

The limit of $\dfrac{f'}{g'}$ may also be an indeterminate form, in which case we can apply l'Hôpital's Rule again to the ratio $\dfrac{f'}{g'}$. We can continue using l'Hôpital's Rule at each stage as long as we have an indeterminate quotient.

Example 3. Compute $\lim_{x\to 0} \dfrac{x^3}{x-\sin(x)}$.

Solution. As $x \to 0$, $f(x) = x^3 \to 0$ and $g(x) = x - \sin(x) \to 0$ so:

$$\lim_{x\to 0} \frac{x^3}{x-\sin(x)} = \lim_{x\to 0} \frac{3x^2}{1-\cos(x)} = \lim_{x\to 0} \frac{6x}{\sin(x)} = \lim_{x\to 0} \frac{6}{\cos(x)} = 6$$

where we have used l'Hôpital's Rule three times in succession. (At each stage, you should verify the conditions for l'Hôpital's Rule hold.) ◀

Practice 2. Use l'Hôpital's Rule to find $\lim_{x\to\infty} \dfrac{x^2+e^x}{x^3+8x}$.

Which Function Grows Faster?

Sometimes we want to compare the asymptotic behavior of two systems or functions for large values of x. L'Hôpital's Rule can be useful in such situations. For example, if we have two algorithms for sorting names, and each algorithm takes longer and longer to sort larger collections of names, we may want to know which algorithm will accomplish the task more efficiently for really large collections of names.

Example 4. Algorithm A requires $n \cdot \ln(n)$ steps to sort n names and algorithm B requires $n^{1.5}$ steps. Which algorithm will be better for sorting very large collections of names?

Solution. We can compare the ratio of the number of steps each algorithm requires, $\dfrac{n \cdot \ln(n)}{n^{1.5}}$, and then take the limit of this ratio as n grows arbitrarily large: $\lim_{n\to\infty} \dfrac{n \cdot \ln(n)}{n^{1.5}}$.

If this limit is infinite, we say that $n \cdot \ln(n)$ "grows faster" than $n^{1.5}$. If the limit is 0, we say that $n^{1.5}$ grows faster than $n \cdot \ln(n)$.

Because $n \cdot \ln(n)$ and $n^{1.5}$ both grow arbitrarily large when n becomes large, we can simplify the ratio to $\dfrac{\ln(n)}{n^{0.5}}$ and then use l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\ln(n)}{n^{0.5}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{0.5n^{-0.5}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

We conclude that $n^{1.5}$ grows faster than $n \cdot \ln(n)$ so algorithm A requires fewer steps for really large sorts. ◀

Practice 3. Algorithm A requires e^n operations to find the shortest path connecting n towns, while algorithm B requires $100 \cdot \ln(n)$ operations for the same task and algorithm C requires n^5 operations. Which algorithm is best for finding the shortest path connecting a very large number of towns? The worst?

Other Indeterminate Forms

We call "$\frac{0}{0}$" an **indeterminate form** because knowing that f approaches 0 and g approaches 0 is not enough to determine the limit of $\frac{f}{g}$, even if that limit exists. The ratio of a "small" number divided by a "small" number can be almost anything as three simple "$\frac{0}{0}$" examples show:

$$\lim_{x\to 0} \frac{3x}{x} = 3 \quad \text{while} \quad \lim_{x\to 0} \frac{x^2}{x} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{5x}{x^3} = \infty$$

Similarly, "$\frac{\infty}{\infty}$" is an indeterminate form because knowing that f and g both grow arbitrarily large is not enough to determine the value of the limit of $\frac{f}{g}$ or even if the limit exists:

$$\lim_{x\to \infty} \frac{3x}{x} = 3 \quad \text{while} \quad \lim_{x\to \infty} \frac{x^2}{x} = \infty \quad \text{and} \quad \lim_{x\to \infty} \frac{5x}{x^3} = 0$$

In addition to the indeterminate quotient forms "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" there are several other "indeterminate forms." In each case, the resulting limit depends not only on each function's limit but also on how quickly each function approaches its limit.

- **Product**: If f approaches 0 and g grows arbitrarily large, the product $f \cdot g$ has the indeterminate form "$0 \cdot \infty$."
- **Exponent**: If f and g both approach 0, the function f^g has the indeterminate form "0^0."
- **Exponent**: If f approaches 1 and g grows arbitrarily large, the function f^g has the indeterminate form "1^∞."
- **Exponent**: If f grows arbitrarily large and g approaches 0, the function f^g has the indeterminate form "∞^0."

- **Difference**: If f and g both grow arbitrarily large, the function $f - g$ has the indeterminate form "$\infty - \infty$."

Unfortunately, l'Hôpital's Rule can only be used directly with an indeterminate quotient ($\frac{0}{0}$ or "$\frac{\infty}{\infty}$"), but we can algebraically manipulate these other forms into quotients and *then* apply l'Hôpital's Rule.

Example 5. Evaluate $\displaystyle\lim_{x\to 0^+} x \cdot \ln(x)$.

Solution. This limit involves an indeterminate product (of the form "$0 \cdot -\infty$") but we need a quotient in order to apply l'Hôpital's Rule. If we rewrite the product $x \cdot \ln(x)$ as a quotient:

$$\lim_{x\to 0^+} x \cdot \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

results from applying the "$\frac{\infty}{\infty}$" version of l'Hôpital's Rule. ◀

To use l'Hôpital's Rule on a product $f \cdot g$ with indeterminate form "$0 \cdot \infty$," first rewrite $f \cdot g$ as a quotient: $\dfrac{f}{\frac{1}{g}}$ or $\dfrac{g}{\frac{1}{f}}$. Then apply l'Hôpital's Rule.

Example 6. Evaluate $\displaystyle\lim_{x\to 0^+} x^x$.

Solution. This limit involves the indeterminate form 0^0. We can convert it to a product by recalling a property of exponential and logarithmic functions: for any positive number a, $a = e^{\ln(a)}$ so:

$$f^g = e^{\ln(f^g)} = e^{g \cdot \ln(f)}$$

Applying this to x^x:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = \lim_{x\to 0^+} e^{x \cdot \ln(x)}$$

This last limit involves the indeterminate product $x \cdot \ln(x)$. From the previous example we know that $\displaystyle\lim_{x\to 0^+} x \cdot \ln(x) = 0$ so we can conclude that:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x \cdot \ln(x)} = e^{\lim_{x\to 0^+} x \cdot \ln(x)} = e^0 = 1$$

because the function $f(u) = e^u$ is continuous everywhere. ◀

To use l'Hôpital's Rule on an expression involving exponents, f^g with the indeterminate form "0^0," "1^∞" or "∞^0," first convert it to an expression involving an indeterminate product by recognizing that $f^g = e^{g \cdot \ln(f)}$ and then determining the limit of $g \cdot \ln(f)$. The final result is $e^{\text{limit of } g \cdot \ln(f)}$.

Example 7. Evaluate $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$.

Solution. This expression has the form 1^∞ so we first use logarithms to convert the problem into a limit involving a product:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\infty} e^{x\cdot\ln\left(1+\frac{a}{x}\right)}$$

so now we need to compute $\lim_{x\to\infty} x\cdot\ln\left(1+\frac{a}{x}\right)$. This limit has the form "$\infty\cdot 0$" so we now convert the product to a quotient:

$$\lim_{x\to\infty} x\cdot\ln\left(1+\frac{a}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}$$

This last limit has the form "$\frac{0}{0}$" so we can finally apply l'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{\frac{-a}{x^2}}{1+\frac{a}{x}}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{a}{1+\frac{a}{x}} = \frac{a}{1} = a$$

and conclude that:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\infty} e^{x\cdot\ln\left(1+\frac{a}{x}\right)} = e^{\lim_{x\to\infty} x\cdot\ln\left(1+\frac{a}{x}\right)} = e^a$$

where we have again used the continuity of the function $f(u) = e^u$. ◀

3.7 Problems

In Problems 1–15, evaluate each limit. Be sure to justify any use of l'Hôpital's Rule.

1. $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$
2. $\lim_{x\to 2}\frac{x^4-16}{x^5-32}$
3. $\lim_{x\to 0}\frac{\ln(1+3x)}{5x}$
4. $\lim_{x\to\infty}\frac{e^x}{x^3}$
5. $\lim_{x\to 0}\frac{x\cdot e^x}{1-e^x}$
6. $\lim_{x\to 0}\frac{2^x-1}{x}$
7. $\lim_{x\to\infty}\frac{\ln(x)}{x}$
8. $\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}}$
9. $\lim_{x\to\infty}\frac{\ln(x)}{x^p}$ $(p>0)$
10. $\lim_{x\to 0}\frac{e^{3x}-e^{2x}}{4x}$
11. $\lim_{x\to 0}\frac{1-\cos(3x)}{x^2}$
12. $\lim_{x\to 0}\frac{1-\cos(2x)}{x}$
13. $\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}$
14. $\lim_{x\to 0}\frac{\cos(a+x)-\cos(a)}{x}$
15. $\lim_{x\to 0}\frac{1-\cos(x)}{x\cdot\cos(x)}$
16. Find a value for p so that $\lim_{x\to\infty}\frac{3x}{px+7}=2$.
17. Find a value for p so that $\lim_{x\to 0}\frac{e^{px}-1}{3x}=5$.
18. The limit $\lim_{x\to\infty}\frac{\sqrt{3x+5}}{\sqrt{2x-1}}$ has the indeterminate form "$\frac{\infty}{\infty}$." Why doesn't l'Hôpital's Rule work with this limit? (Hint: Apply l'Hôpital's Rule twice and see what happens.) Evaluate the limit without using l'Hôpital's Rule.
19. (a) Evaluate $\lim_{x\to\infty}\frac{e^x}{x}$, $\lim_{x\to\infty}\frac{e^x}{x^2}$ and $\lim_{x\to\infty}\frac{e^x}{x^5}$.
    (b) An algorithm is "exponential" if it requires $a\cdot e^{bn}$ steps (a, and b are positive constants). An algorithm is "polynomial" if it requires $c\cdot n^d$ steps. Show that polynomial algorithms require fewer steps than exponential ones for large values of n.

20. The problem $\lim_{x\to 0} \frac{x^2}{3x^2+x}$ appeared on a test. One student determined the limit was an indeterminate "$\frac{0}{0}$" form and applied l'Hôpital's Rule to get:

$$\lim_{x\to 0} \frac{x^2}{3x^2+x} = \lim_{x\to 0} \frac{2x}{6x+1} = \lim_{x\to 0} \frac{2}{6} = \frac{1}{3}$$

Another student also determined the limit was an indeterminate "$\frac{0}{0}$" form and wrote:

$$\lim_{x\to 0} \frac{x^2}{3x^2+x} = \lim_{x\to 0} \frac{2x}{6x+1} = \frac{0}{0+1} = 0$$

Which student is correct? Why?

In Problems 21–30, evaluate each limit. Be sure to justify any use of l'Hôpital's Rule.

21. $\lim_{x\to 0^+} \sin(x)\cdot\ln(x)$

22. $\lim_{x\to\infty} x^3 e^{-x}$

23. $\lim_{x\to 0^+} \sqrt{x}\cdot\ln(x)$

24. $\lim_{x\to 0^+} x^{\sin(x)}$

25. $\lim_{x\to\infty} \left(1-\frac{3}{x^2}\right)^x$

26. $\lim_{x\to 0} (1-\cos(3x))^x$

27. $\lim_{x\to 0} \left(\frac{1}{x}-\frac{1}{\sin(x)}\right)$

28. $\lim_{x\to\infty} [x-\ln(x)]$

29. $\lim_{x\to\infty} \left(\frac{x+5}{x}\right)^{\frac{1}{x}}$

30. $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{\frac{2}{x}}$

3.7 Practice Answers

1. Both numerator and denominator in the first limit are differentiable and both equal 0 when $x=0$, so we apply l'Hôpital's Rule:

$$\lim_{x\to 0} \frac{1-\cos(5x)}{3x} = \lim_{x\to 0} \frac{5\sin(5x)}{3} = \frac{0}{3} = 0$$

Both numerator and denominator in the second limit are differentiable and both equal 0 when $x=0$, so we apply l'Hôpital's Rule:

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2+2x-8} = \lim_{x\to 2} \frac{2x+1}{2x+2} = \frac{5}{6}$$

2. Both numerator and denominator are differentiable and both become arbitrarily large as $x\to\infty$, so we apply l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{x^2+e^x}{x^3+8x} = \lim_{x\to\infty} \frac{2x+e^x}{3x^2+8} = \lim_{x\to\infty} \frac{2+e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

Note that we needed to apply l'Hôpital's Rule three times and that each stage involved an "$\frac{\infty}{\infty}$" indeterminate form.

3. Comparing A with e^n operations to B with $100\cdot\ln(n)$ operations we can apply l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{e^n}{100\ln(n)} = \lim_{n\to\infty} \frac{e^n}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n\cdot e^n}{100} = \infty$$

to show that B requires fewer operations than A.

Comparing B with $100\ln(n)$ operations to C with n^5 operations, we again apply l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{100\ln(n)}{n^5} = \lim_{n\to\infty} \frac{\frac{100}{n}}{5n^4} = \lim_{n\to\infty} \frac{20}{n^5} = 0$$

to show that B requires fewer operations than C. So B requires the fewest operations of the three algorithms.

Comparing A to C we must apply l'Hôpital's Rule repeatedly:

$$\begin{aligned}\lim_{n\to\infty} \frac{e^n}{n^5} &= \lim_{n\to\infty} \frac{e^n}{5n^4} = \lim_{n\to\infty} \frac{e^n}{20n^3} = \lim_{n\to\infty} \frac{e^n}{60n^2} \\ &= \lim_{n\to\infty} \frac{e^n}{120n} = \lim_{n\to\infty} \frac{e^n}{120} = \infty\end{aligned}$$

So A requires more operations than C and thus A requires the most operations of the three algorithms.