

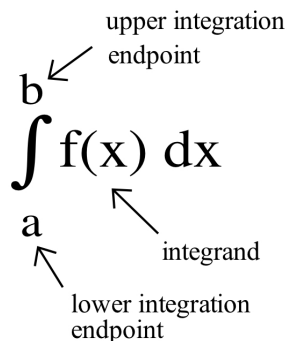
4.2 The Definite Integral

Each particular Riemann sum depends on several things: the function f , the interval $[a, b]$, the partition \mathcal{P} of that interval, and the chosen values c_k from each subinterval of that partition. Fortunately — for most of the functions needed for applications — as the approximating rectangles get thinner (and as the meshes of the partitions \mathcal{P} approach 0 and the number of subintervals n in those partitions approaches ∞) the values of the Riemann sums approach the same number, independent of the particular partitions \mathcal{P} and the chosen points c_k in the subintervals of those partitions.

This limit of the Riemann sums will become the next big topic in calculus: the **definite integral**. Integrals arise throughout the rest of this book and in applications in almost every field that uses mathematics.

Here we use the notation $\|\mathcal{P}\|$ to mean “the mesh of \mathcal{P} .”

The dx is a differential (see Section 3.6), the limit of the discrete quantity Δx in the Riemann sum.



You may have noticed that we did not precisely define what $\lim_{\|\mathcal{P}\| \rightarrow 0}$ means or how to compute this limit. Providing a definition turns out to be more complicated than the limits we have encountered so far, and in practice we will rarely need to compute such a limit, so a formal definition is left to more advanced textbooks.

Definition of the Definite Integral:

If $\lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n f(c_k) \cdot \Delta x_k \right)$ equals a finite number I , where each \mathcal{P} is a partition of the interval $[a, b]$, then we say f is **integrable** on the interval $[a, b]$ and call the number I the **definite integral** of f on $[a, b]$ and write it as $\int_a^b f(x) dx$.

We read the symbol $\int_a^b f(x) dx$ as “the integral from a to b of ‘eff’ of x ‘dee’ x ” or “the integral from a to b of $f(x)$ with respect to x .” Furthermore, we call $f(x)$ the **integrand**, a the **lower endpoint of integration** and b the **upper endpoint of integration**. (We will sometimes also call a and b the **upper and lower limits** of integration.)

Example 1. Describe the area between the graph of $f(x) = \frac{1}{x}$, the x -axis, and the vertical lines at $x = 1$ and $x = 5$ as a limit of Riemann sums and as a definite integral.

Solution. Here $f(x) = \frac{1}{x}$, $a = 1$ and $b = 5$, so:

$$\text{area} = \lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n \frac{1}{c_k} \cdot \Delta x_k \right) = \int_1^5 \frac{1}{x} dx$$

which, according to estimations made in Section 4.1, is approximately equal to 1.609. ◀

Practice 1. Describe the area between the graph of $f(x) = \sin(x)$, the x -axis, and the vertical lines at $x = 0$ and $x = \pi$ as a limit of Riemann sums and as a definite integral.

Example 2. Using the concept of area, determine the values of:

(a) $\lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n (1 + c_k) \cdot \Delta x_k \right)$ on the interval $[1, 3]$

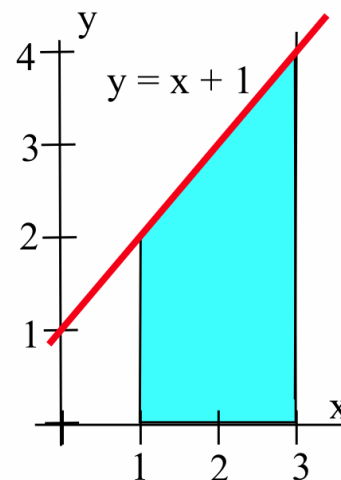
(b) $\int_0^4 (5 - x) dx$

(c) $\int_{-1}^1 \sqrt{1 - x^2} dx$

Solution. (a) The limit of the Riemann sums represents the area between the graph of $f(x) = 1 + x$, the x -axis, and the vertical lines at $x = 1$ and $x = 3$ (see margin); this area equals 6 square units.

(b) The definite integral represents the area between $f(x) = 5 - x$, the x -axis and the vertical lines at $x = 0$ and $x = 4$, which is a trapezoid with area 12 square units.

(c) The definite integral represents the area of the upper half of the circle $x^2 + y^2 = 1$, which has radius 1 and center at $(0, 0)$. The area of this semicircle is $\frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi \cdot 1^2 = \frac{\pi}{2}$. ◀



Practice 2. Using the concept of area, determine the values of:

(a) $\lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n (2c_k) \cdot \Delta x_k \right)$ on the interval $[1, 3]$ (b) $\int_3^8 4 dx$

Example 3. Represent each limit of Riemann sums as a definite integral.

(a) $\lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n (3 + c_k) \Delta x_k \right)$ on $[1, 4]$ (b) $\lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n \sqrt{c_k} \Delta x_k \right)$ on $[0, 9]$

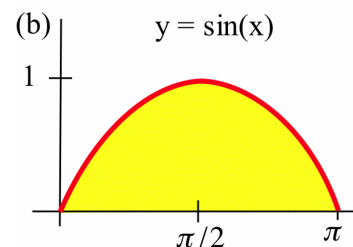
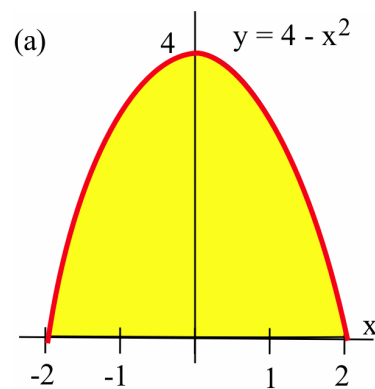
Solution. (a) $\int_1^4 (3 + x) dx$ (b) $\int_0^9 \sqrt{x} dx$ ◀

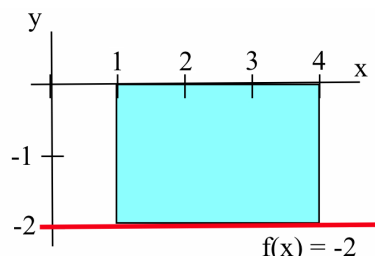
Example 4. Represent each shaded area in the margin figure as a definite integral. (Do not attempt to evaluate the definite integral, just translate the picture into symbols.)

Solution. (a) $\int_{-2}^2 (4 - x^2) dx$ (b) $\int_0^{\pi} \sin(x) dx$ ◀

The value of a definite integral $\int_a^b f(x) dx$ depends only on the function f being integrated and on the interval $[a, b]$. Replacing the variable x that appears in $\int_a^b f(x) dx$, sometimes called a “dummy variable,” does not change the value of the integral. The following definite integrals each represent the integral of the function f on the interval $[a, b]$, and they are all equal:

$$\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du = \int_a^b f(w) dw$$





Definite Integrals of Negative Functions

A definite integral is a limit of Riemann sums, and you can form Riemann sums using any integrand function f , positive or negative (or both), continuous or discontinuous. The definite integral of an integrable function will still have a geometric meaning even if the function is sometimes (or always) negative, and definite integrals of negative functions also have meaningful interpretations in applications.

Example 5. Find the definite integral of $f(x) = -2$ on $[1, 4]$.

Solution. Writing a Riemann sum for $f(x) = -2$ on the interval $[1, 4]$:

$$\sum_{k=1}^n f(c_k) \cdot \Delta x_k = \sum_{k=1}^n (-2) \cdot \Delta x_k = -2 \cdot \sum_{k=1}^n \Delta x_k = -2(3) = -6$$

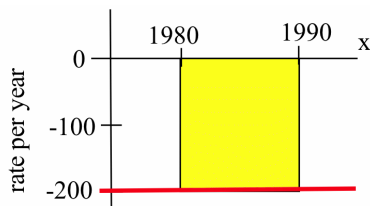
for every partition \mathcal{P} and every choice of values for c_k so:

$$\int_1^4 -2 dx = \lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n (-2) \cdot \Delta x_k \right) = \lim_{\|\mathcal{P}\| \rightarrow 0} -6 = -6$$

The **area** of the region in the margin figure is 6 units, but because the region is below the x -axis, the value of the **integral** is -6 . ◀

If the graph of $f(x)$ is below the x -axis for $a \leq x \leq b$ (f is negative) then $\int_a^b f(x) dx$ is -1 times the area of the region below the x -axis and above the graph of $f(x)$ between $x = a$ and $x = b$.

If $f(t)$ represents the rate of population change (people per year) for a town, then negative values of f for a given time interval would indicate that the population of the town was getting smaller, and the definite integral (now a negative number) would represent the change in the population—a decrease—during that time interval.



Example 6. In 1980 there were 12,000 ducks nesting around a lake. The **rate** of population change is shown in the margin. Write a definite integral to represent the **total change** in the duck population between 1980 and 1990, then estimate the population in 1990.

Solution. The total change in population is given by $\int_{1980}^{1990} f(t) dt$ and this definite integral is equal to -1 times the area of the rectangle in the margin figure:

$$-200 \frac{\text{ducks}}{\text{year}} \cdot 10 \text{ years} = -2000 \text{ ducks}$$

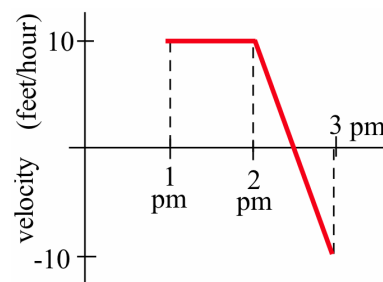
so:

$$\begin{aligned} [\text{1990 population}] &= [\text{1980 population}] + [\text{change from 1980 to 1990}] \\ &= 12000 + (-2000) = 10000 \end{aligned}$$

Approximately 10,000 ducks were nesting around the lake in 1990. ◀

If $f(t)$ represents the velocity of a car (in miles per hour) moving in the positive direction along a straight line at time t , then negative values of f indicate that the car was travelling in the negative direction (that is, backwards). The definite integral of f (the integral will be a negative number) represents the change in position of the car during the time interval: how far the car travelled in the negative direction.

Practice 3. A bug starts at the location $x = 12$ on the x -axis at 1:00 p.m. and walks along the axis with the velocity shown in the margin figure. How far does the bug travel between 1:00 p.m. and 3:00 p.m.? Where is the bug at 3:00 p.m.?



Frequently an integrand function will be positive some of the time and negative some of the time. If f represents a rate of population increase, then the integral of the positive parts of f will give the increase in population and the integral of the negative parts of f will give the decrease in population. Altogether, the integral of f over the entire time interval will give the **total (net) change** in the population.

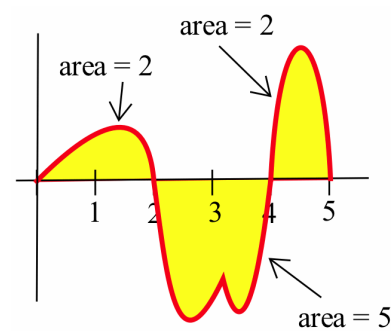
Geometrically, we can now interpret a definite integral as a difference of areas of the region(s) between the graph of f and the horizontal axis:

$$\int_a^b f(x) dx = [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}]$$

Example 7. Use the margin figure to calculate $\int_0^2 f(x) dx$, $\int_2^4 f(x) dx$, $\int_4^5 f(x) dx$ and $\int_0^5 f(x) dx$.

Solution. Using the areas indicated in the figure, $\int_0^2 f(x) dx = 2$, $\int_2^4 f(x) dx = -5$ and $\int_4^5 f(x) dx = 2$, while

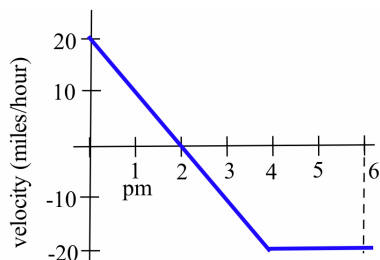
$$\begin{aligned} \int_0^5 f(x) dx &= [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}] \\ &= [2 + 2] - [5] = -1 \end{aligned}$$



where we added the areas of the regions above the x -axis and subtracted the area of the region below the x -axis. ◀

Practice 4. Use geometric reasoning to evaluate $\int_0^{2\pi} \sin(x) dx$.

If f represents a velocity, then integrals on the intervals where f is positive measure distances moved in the forward direction and integrals on the intervals where f is negative measure distances moved in the backward direction. The integral over the whole time interval gives the **total (net) change** in position: the distance moved forward minus the distance moved backward.



Practice 5. A car travels west with the velocity shown in the margin.

- How far does the car travel between noon and 6:00 p.m.?
- At 6:00 p.m., where is the car relative to its position at noon?

Units for the Definite Integral

We have already seen that the “area” under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. For example, if x measures time in “seconds” and $f(x)$ gives a velocity with units “feet per second,” then Δx has the units “seconds” and $f(x) \cdot \Delta x$ has units:

$$\left(\frac{\text{feet}}{\text{second}} \right) (\text{seconds}) = \text{feet}$$

which is a measure of distance. Because each Riemann sum $\sum f(x) \cdot \Delta x$ is a sum of “feet” and the definite integral is a limit of these Riemann sums, the definite integral has the same units, “feet.”

If the units of $f(x)$ are “square feet” and the units of x are “feet,” then $\int_a^b f(x) dx$ is a number with the units $(\text{feet}^2) \cdot (\text{feet}) = \text{feet}^3$, or cubic feet, a measure of volume. If $f(x)$ represents a force in pounds and x is a distance in feet, then $\int_a^b f(x) dx$ is a number with the units foot-pounds, a measure of work.

In general, the units for $\int_a^b f(x) dx$ are $(\text{units for } f(x)) \cdot (\text{units for } x)$. A quick check of the units when computing a definite integral can help avoid errors when solving an applied problem.

4.2 Problems

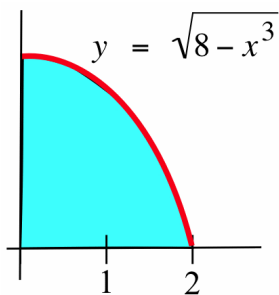
In Problems 1–4, rewrite each limit of Riemann sums as a definite integral.

- $\lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n (2 + 3c_k) \cdot \Delta x_k \right)$ on $[0, 4]$
- $\lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n (c_k)^3 \cdot \Delta x_k \right)$ on $[0, 11]$
- $\lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n \cos(5c_k) \cdot \Delta x_k \right)$ on $[2, 5]$
- $\lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n \sqrt{c_k} \cdot \Delta x_k \right)$ on $[1, 4]$

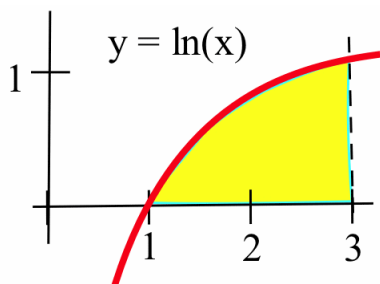
In Problems 5–10, represent the area of each bounded region as a definite integral. (Do not attempt to evaluate the integral, just translate the area into an integral.)

- The region bounded by $y = x^3$, the x -axis, and the lines $x = 1$ and $x = 5$.
- The region bounded by $y = \sqrt{x}$, the x -axis and the line $x = 9$.
- The region bounded by $y = x \cdot \sin(x)$, the x -axis, and the lines $x = \frac{1}{2}$ and $x = 2$.

8. The shaded region shown below:



9. The shaded region shown below:



10. The shaded region shown above for $2 \leq x \leq 3$.

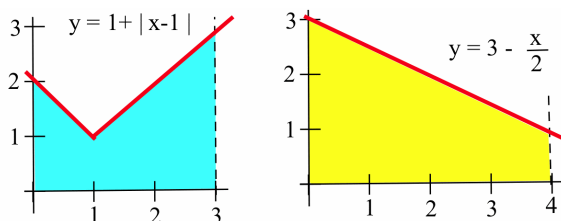
In Problems 11–15, represent the area of each bounded region as a definite integral, then use geometry to determine the value of that definite integral.

11. The region bounded by $y = 2x$, the x -axis, and the lines $x = 1$ and $x = 3$.

12. The region bounded by $y = 4 - 2x$, the x -axis and the y -axis.

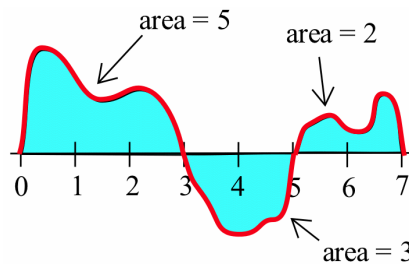
13. The region bounded by $y = |x|$, the x -axis and the line $x = -1$.

14. The shaded region shown below left.



15. The shaded region shown above right.

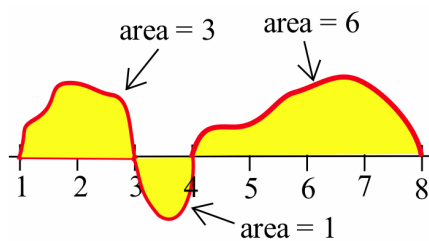
16. Evaluate each integral using the figure below showing the graph of f and various areas.



(a) $\int_0^5 f(x) dx$ (b) $\int_3^5 f(x) dx$ (c) $\int_5^7 f(x) dx$

(d) $\int_0^5 |f(x)| dx$ (e) $\int_3^7 f(x) dx$

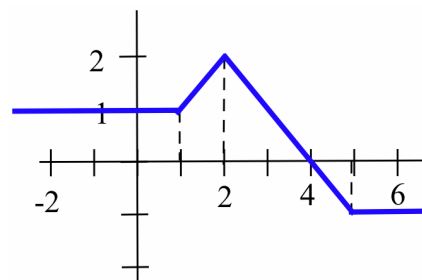
17. Evaluate each integral using the figure below showing the graph of g and various areas.



(a) $\int_1^3 g(x) dx$ (b) $\int_3^4 g(x) dx$ (c) $\int_4^8 g(x) dx$

(d) $\int_1^8 g(x) dx$ (e) $\int_3^8 |g(x)| dx$

18. Evaluate each integral using the figure below showing the graph of h .

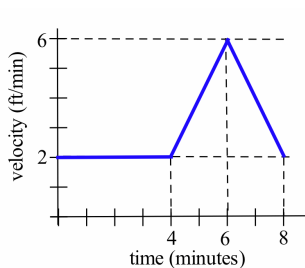


(a) $\int_{-1}^1 h(x) dx$ (b) $\int_4^6 h(x) dx$ (c) $\int_{-2}^6 h(x) dx$

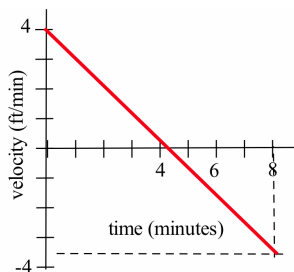
(d) $\int_{-2}^4 h(x) dx$ (e) $\int_{-2}^4 |h(x)| dx$

For Problems 19–20, the figure shows your velocity (in feet per minute) along a straight path. (a) Sketch a graph of your location. (b) How many feet did you walk in 8 minutes? (c) Where, relative to your starting location, are you after 8 minutes?

19. See figure below left.



20. See figure above right.



Problems 21–27 give the units for x and $f(x)$. Specify the units of the definite integral $\int_a^b f(x) dx$.

21. x is time in “seconds”; $f(x)$ is velocity in “meters per second”
22. x is time in “hours”; $f(x)$ is a flow rate in “gallons per hour”
23. x a position in “feet”; $f(x)$ area in “square feet”
24. x is a time in “days”; $f(x)$ is a temperature in “degrees Celsius”
25. x a height in “meters”; $f(x)$ force in “grams”
26. x is a position in “inches”; $f(x)$ is a density in “pounds per inch”
27. x is a time in “seconds”; $f(x)$ is an acceleration in “feet per second per second” $\left(\frac{\text{ft}}{\text{sec}^2}\right)$

The remaining problems use the summation formulas given at the end of Section 4.1, as demonstrated in the following Example.

Example 8. For $f(x) = x^2$, divide the interval $[0, 2]$ into n equally wide subintervals, evaluate the lower sum, and compute the limit of that lower sum as $n \rightarrow \infty$.

Solution. The width of the interval is $b - a = 2 - 0 = 2$ so each of the n subintervals should have width $\Delta x = \frac{b - a}{n} = \frac{2}{n}$. The endpoints of the k -th interval in the partition are therefore $(k - 1) \cdot \frac{2}{n}$ and $k \cdot \frac{2}{n}$ for $k = 1, 2, \dots, n$.

Because $f(x) = x^2$ is increasing on $[0, 2]$ the minimum value of the function on each subinterval occurs at the left endpoint of the subinterval, hence we need to choose $c_k = (k - 1) \cdot \frac{2}{n}$. So:

$$\begin{aligned} \text{LS} &= \sum_{k=1}^n f(c_k) \cdot \Delta x_k = \sum_{k=1}^n \left((k - 1) \cdot \frac{2}{n} \right)^2 \cdot \frac{2}{n} = \frac{8}{n^3} \cdot \sum_{k=1}^n (k - 1)^2 \\ &= \frac{8}{n^3} \cdot \sum_{k=1}^n (k^2 - 2k + 1) = \frac{8}{n^3} \left[\sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k + \sum_{k=1}^n 1 \right] \\ &= \frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} - 2 \cdot \frac{n(n+1)}{2} + n \right] = \frac{8}{n^3} \left[\frac{2n^3 - 3n^2 + n}{6} \right] \\ &= \frac{8}{6} \left[2 - \frac{3}{n} + \frac{1}{n^2} \right] \end{aligned}$$

As $n \rightarrow \infty$, $\text{LS} \rightarrow \frac{8}{6}(2) = \frac{8}{3}$ so we can be certain that $\int_0^2 x^2 dx \geq \frac{8}{3}$. ◀

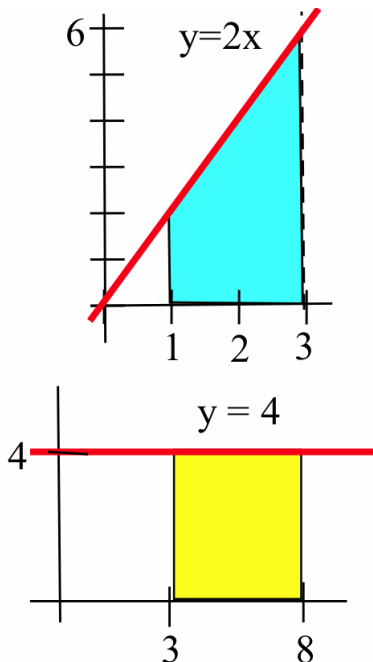
Practice 6. Redo Example 6 but find the upper Riemann sum for n equally wide partition intervals and show that the limit of these upper sums, as $n \rightarrow \infty$, is $\frac{8}{3}$.

From the previous Example and Practice problem, we know that

$$\frac{8}{3} \leq \int_0^2 x^2 dx \leq \frac{8}{3}$$

so we can conclude that $\int_0^2 x^2 = \frac{8}{3}$. We will discover a much easier method for evaluating this integral in Section 4.4.

28. For $f(x) = 3 + x$, partition the interval $[0, 2]$ into n equally wide subintervals of length $\Delta x = \frac{2}{n}$.
- Compute the lower sum for this function and partition, and calculate the limit of that lower sum as $n \rightarrow \infty$.
 - Compute the upper sum for this function and partition and find the limit of the upper sum as $n \rightarrow \infty$.
29. For $f(x) = x^3$, partition the interval $[0, 2]$ into n equally wide subintervals of length $\Delta x = \frac{2}{n}$.
- Compute the lower sum for this function and partition, and calculate the limit of that lower sum as $n \rightarrow \infty$.
 - Compute the upper sum for this function and partition and find the limit of the upper sum as $n \rightarrow \infty$.
30. For $f(x) = \sqrt{x}$, partition the interval $[0, 9]$ into n subintervals by taking $x_k = \frac{9}{n^2} \cdot k^2$ for $k = 1, 2, \dots, n$.
- Choose $c_k = x_k$ for each subinterval and compute the upper sum for this function and partition, then calculate the limit of that upper sum as $n \rightarrow \infty$.
 - Compute the lower sum for this function and partition and find the limit of the lower sum as $n \rightarrow \infty$.



4.2 Practice Answers

$$1. \text{ area} = \lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n \sin(c_k) \cdot \Delta x_k \right) = \int_0^{\pi} \sin(x) dx$$

$$2. \lim_{\|\mathcal{P}\| \rightarrow 0} \left(\sum_{k=1}^n (2c_k) \cdot \Delta x_k \right) = \text{area of trapezoid in margin} = 8$$

$$\int_3^8 4 dx = \text{area of rectangle in margin} = 20$$

3. (a) 12.5 feet forward and 2.5 feet backward = 15 feet total
 (b) The bug ends up 10 feet forward of its starting position at $x = 12$, so the bug's final location is at $x = 22$.

4. Between $x = 0$ and $x = 2\pi$, the graph of $y = \sin(x)$ has the same area above the x -axis as below the x -axis so the two areas cancel and the definite integral is 0: $\int_0^{2\pi} \sin(x) dx = 0$.

5. (a) 20 miles west (from noon to 2:00 p.m.) plus 60 miles east (from 2:00 p.m. to 6:00 p.m.) yields a total travel distance of 80 miles. (At 4:00 p.m. the driver is back at the starting position after driving 40 miles: 20 miles west and then 20 miles east.)

- (b) The car is 40 miles east of the starting location. (East is the "negative" of west.)

$$6. \Delta x = \frac{2-0}{n} = \frac{2}{n}, M_k = \frac{2}{n} \cdot k \text{ so } f(M_k) = \left(\frac{2}{n} \cdot k\right)^2 = \frac{4}{n^2} \cdot k^2. \text{ Then:}$$

$$\begin{aligned} \text{US} &= \sum_{k=1}^n f(M_k) \cdot \Delta x = \sum_{k=1}^n \frac{4}{n^2} \cdot k^2 \cdot \frac{2}{n} \\ &= \frac{8}{n^3} \sum_{k=1}^n k^2 = \frac{8}{n^3} \left[\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \right] = \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} \end{aligned}$$

so the limit of these upper sums as $n \rightarrow \infty$ is $\frac{8}{3}$.