## 4.3 *Properties of the Definite Integral*

We have defined definite integrals as limits of Riemann sums, which can often be interpreted as "areas" of geometric regions. These two powerful concepts of the definite integral can help us understand integrals and use them in a variety of applications.

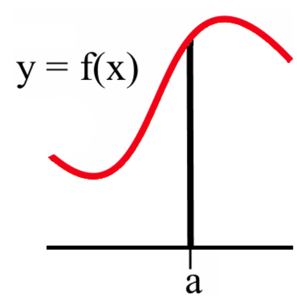
This section continues to emphasize this dual view of definite integrals and presents several properties of definite integrals. We will justify these properties using the properties of summations and the definition of a definite integral as a Riemann sum, but they also have natural interpretations as properties of areas of regions.

We will then use these properties to help understand functions that are defined by integrals, and later to help calculate the values of definite integrals.

### *Properties of the Definite Integral*

As you read each statement about definite integrals, draw a sketch or examine the accompanying figure to interpret the property as a statement about areas.

$$\int_a^a f(x)\,dx = 0$$




This property says that the definite integral of a function over an interval consisting of a single point must be 0. Geometrically, we can see that the area under the graph of a a function above a single point should be 0 because the "width" of a point is 0. In terms of Riemann sums, we can't partition a single point, so instead we must *define* the value of any definite integral over a non-existent "interval" to be 0.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

In words, this property says that if we reverse the limits of integration, we must multiply the value of the definite integral by $-1$.

Geometrically, if $a < b$ then the $x$-values in the first integral are moving "backwards" from $x = b$ to $x = a$, so it might seem reasonable that we should get a negative answer.

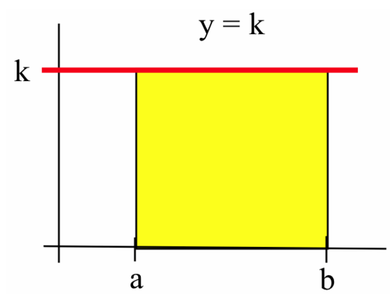
In terms of Riemann sums, if we move from right to left, each $\Delta x_k$ in any partition $\mathcal{P}$ will be negative:

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \sum_{k=1}^{n} f(c_k) \cdot (-|\Delta x_k|) = -1 \cdot \sum_{k=1}^{n} f(c_k) \cdot |\Delta x_k|$$

resulting in $-1$ times the Riemann sum we would use for $\int_a^b f(x)\,dx$.

Our definition of a Riemann sum only allows each $\Delta x_k$ to be positive, however, so we can simply treat this integral property as another definition.

$$\int_a^b k\,dx = k\,(b-a) \qquad (k \text{ is any constant})$$




Here we use the fact that the sum of the lengths of the subinterval of any partition of the interval $[a,b]$ is equal to the width of $[a,b]$, which is $b-a$.

Thinking geometrically, if $k > 0$ (see margin), then $\int_a^b k\,dx$ represents the area of a rectangle with base $b-a$ and height $k$, so:

$$\int_a^b k\,dx = (\text{height}) \cdot (\text{base}) = k \cdot (b-a)$$

Alternatively, for any $\mathcal{P} = \{x_0 = a, x_1, x_2, x_3, \ldots, x_{n-1}, x_n = b\}$ that partitions the interval $[a,b]$, and every choice of points $c_j$ from the subintervals of that partition, the Riemann sum is:

$$\sum_{j=1}^{n} f\left(c_j\right) \cdot \Delta x_j = \sum_{j=1}^{n} k \cdot \Delta x_j = k \sum_{j=1}^{n} \Delta x_j = k \cdot (b-a)$$

Because every Riemann sum equals $k \cdot (b-a)$, the limit of those sums, as $\|\mathcal{P}\| \to 0$, must also be $k \cdot (b-a)$.

$$\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx \qquad (k \text{ is any constant})$$

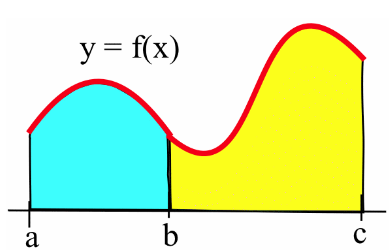
In words, this property says that multiplying an integrand by a constant $k$ has the same result as multiplying the value of the definite integral by that constant.

Geometrically, multiplying a function by a positive constant $k$ stretches the graph of $y = f(x)$ by a factor of $k$ in the vertical direction, which should multiply the area of the region between that graph and the $x$-axis by the same factor.

Thinking in terms of Riemann sums:

$$\sum_{j=1}^{n} k \cdot f\left(c_j\right) \cdot \Delta x_j = k \cdot \sum_{j=1}^{n} f\left(c_j\right) \cdot \Delta x_j$$

so the limit of the sum on the left over all possible partitions $\mathcal{P}$, as $\|\mathcal{P}\| \to 0$, is $\int_a^b k \cdot f(x)\,dx$, while the corresponding limit of the sums on the right yields $k \cdot \int_a^b f(x)\,dx$.




$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

This property is most easily understood (and believed) in terms of a picture (see margin). We can also justify this property using Riemann sums by restricting our partitions to include the point $x = b$ between $x = a$ and $x = c$ and then splitting that partition into two sub-partitions that partition $[a,b]$ and $[b,c]$, respectively.

This property remains true, however, even when $b \geq c$ or $b \leq a$.

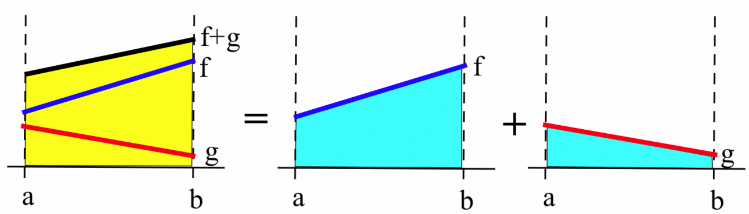
*Properties of Definite Integrals of Combinations of Functions*

The next two properties relate the values of integrals of sums and differences of functions to the sums and differences of integrals of the individual functions. You will find these properties very useful when computing integrals of functions that involve the sum or difference of several terms (such as a polynomial): you can integrate each term and then add or subtract the individual results to get the answer. Both properties have natural interpretations as statements about areas.

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

In words, this says "the integral of a sum is the sum of the integrals."

The following graph supplies a geometrical justification:




Using Riemann sums, we can write:

$$\sum_{j=1}^{n}\left[f\left(c_j\right)+g\left(c_j\right)\right]\cdot\Delta x_j = \sum_{j=1}^{n} f\left(c_j\right)\cdot\Delta x_j + \sum_{j=1}^{n} g\left(c_j\right)\cdot\Delta x_j$$

and then take the limit on each side as $\|\mathcal{P}\| \to 0$.

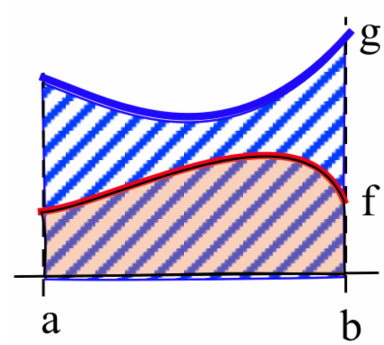
$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

In words, this says "the integral of a difference is the difference of the integrals."

The justification for this difference property is quite similar to the justification of the sum property. (Or we can combine the sum property with the constant-multiple property, setting $k=-1$.)

**Practice 1.** Given that $\int_1^4 f(x)\,dx = 7$ and that $\int_1^4 g(x)\,dx = 3$, evaluate the definite integral $\int_1^4 [f(x)-g(x)]\,dx$.

If $f(x) \le g(x)$ for all $x$ in $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$




Geometrically, the margin figure illustrates that if $f$ and $g$ are both positive and that $f(x) \le g(x)$ on the interval $[a,b]$, then the area of region between the graph of $f$ and the $x$-axis is smaller than the area of region between the graph of $g$ and the $x$-axis.

Similar sketches for the situations where $f$ or $g$ are sometimes or always negative illustrate that the property holds in other situations as well, but we can avoid all of those different cases using Riemann sums.

If we use the same partition $\mathcal{P}$ and chosen points $c_j$ for Riemann sums for $f$ and $g$, then $f(c_j) \le g(c_j)$ for each $j$, so:

$$\sum_{j=1}^{n} f(c_j) \cdot \Delta x_j \le \sum_{j=1}^{n} g(c_j) \cdot \Delta x_j$$

Taking the limit over all such partitions as the mesh of those partitions approaches 0, we get $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

| | |
|---|---|
| If | $m \le f(x) \le M$ for all $x$ in $[a,b]$ |
| then | $m \cdot (b-a) \le \int_a^b f(x)\,dx \le M \cdot (b-a)$ |

You may have noticed that we haven't called the justifications of these properties "proofs," in part because we haven't precisely defined what $\lim_{\|\mathcal{P}\|\to 0}$ means, but also because of some other technical details left to more advanced textbooks.

This property follows easily from the previous one. First let $g(x) = M$ so that $f(x) \le M = g(x)$ for all $x$ in $[a,b]$, hence

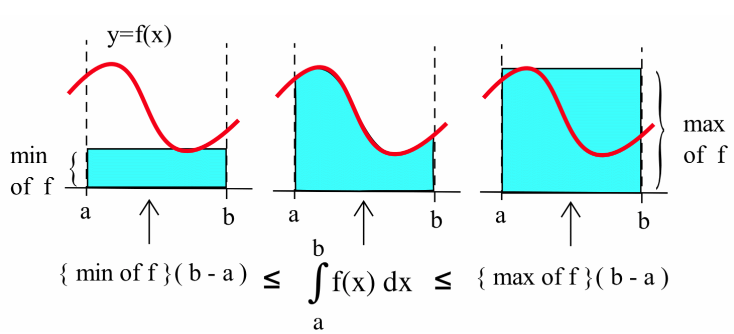
$$\int_a^b f(x)\,dx \le \int_a^b M\,dx = M \cdot (b-a)$$

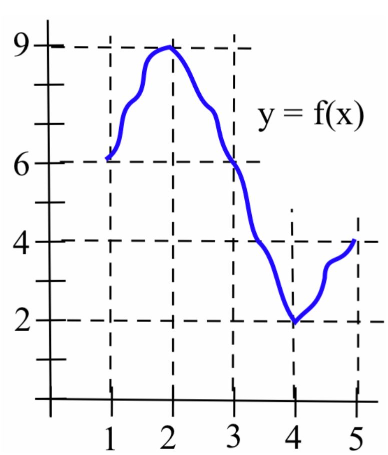
(using one of our previous properties). Likewise, taking $g(x) = m$ so that $f(x) \ge m = g(x)$ for all $x$ in $[a,b]$:

$$\int_a^b f(x)\,dx \ge \int_a^b m\,dx = m \cdot (b-a)$$

Geometrically, this says that if we can "trap" the output values of a function on the interval $[a,b]$ between two upper and lower bounds, $m$ and $M$, then the value of the definite integral must lie between the areas of the rectangles with heights $m$ and $M$.

If $f$ is continuous on the closed interval $[a,b]$, then we know that $f$ takes on a minimum value on that interval (call it $m$) and a maximum value (call it $M$), in which case this property just uses the lower and upper Riemann sums for the simplest possible partition of $[a,b]$:




**Example 1.** Determine lower and upper bounds for the value of $\int_1^5 f(x)\,dx$ with $f(x)$ given graphically in the margin.

**Solution.** If $1 \le x \le 5$, then we can estimate (from the graph) that $2 \le f(x) \le 9$ so a lower bound for $\int_1^5 f(x)\,dx$ is

$$(b-a)\cdot(\text{minimum of } f \text{ on } [a,b]) = (4)(2) = 8$$

and an upper bound is:

$$(b-a)\cdot(\text{maximum of } f \text{ on } [a,b]) = (4)(9) = 36$$

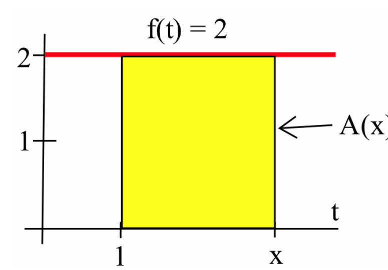
We can conclude that $8 \le \int_1^5 f(x)\,dx \le 36$. ◀

Knowing that the value of a definite integral is somewhere between 8 and 36 is not useful for finding its exact value, but the preceding estimation property is very easy to use and provides a "ballpark estimate" that will help you avoid reporting an unreasonable value.

**Practice 2.** Determine a lower bound and an upper bound for the value of $\int_3^5 f(x)\,dx$ with $f$ as in the previous Example.

## *Functions Defined by Integrals*

If one of the endpoints $a$ or $b$ of the interval $[a,b]$ changes, then the value of the integral $\int_a^b f(t)\,dt$ typically changes. A definite integral of the form $\int_a^x f(t)\,dt$ defines a function of $x$ that possesses interesting and useful properties. The next examples illustrate one such property: the derivative of a function defined by an integral is closely related to the integrand, the function "inside" the integral.




**Example 2.** For the function $f(t) = 2$, define $A(x)$ to be the area of the region bounded by $f$, the $t$-axis, and vertical lines at $t = 1$ and $t = x$.

(a) Evaluate $A(1)$, $A(2)$, $A(3)$ and $A(4)$.

(b) Find an algebraic formula for $A(x)$ valid for $x \ge 1$.

(c) Calculate $A'(x)$.

(d) Express $A(x)$ as a definite integral.

**Solution.** (a) Referring to the graph in the margin, we can see that $A(1) = 0$, $A(2) = 2$, $A(3) = 4$ and $A(4) = 6$. (b) Using the same area idea to compute a more general area:

$$A(x) = \text{area of a rectangle} = (\text{base})\,(\text{height}) = (x-1)(2) = 2x-2$$

(c) $A'(x) = \frac{d}{dx}(2x-2) = 2$ (d) $A(x) = \int_1^x 2\,dt$ ◀

**Practice 3.** Answer the questions in the previous Example for $f(x) = 3$.

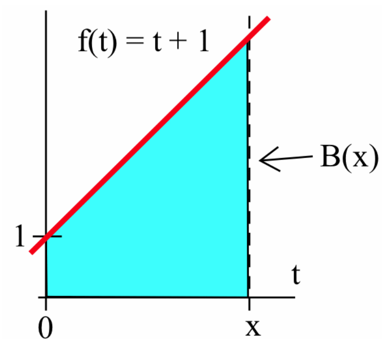
**Example 3.** For the function $f(t) = 1 + t$, define $B(x)$ to be the area of the region bounded by the graph of $f$, the $t$-axis, and vertical lines at $t = 0$ and $t = x$ (see margin).

(a) Evaluate $B(0)$, $B(1)$, $B(2)$ and $B(3)$.

(b) Find an algebraic formula for $B(x)$ valid for $x \geq 0$.

(c) Calculate $B'(x)$.

(d) Express $B(x)$ as a definite integral.




**Solution.** (a) From the graph, $B(0) = 0$, $B(1) = 1.5$, $B(2) = 4$ and $B(3) = 7.5$. (b) Using the same area concept:

$$\begin{aligned} B(x) &= \text{area of trapezoid} = (\text{base}) \cdot (\text{average height}) \\ &= (x) \cdot \left(\frac{1 + (1 + x)}{2}\right) = x + \frac{1}{2}x^2 \end{aligned}$$

(c) $B'(x) = \dfrac{d}{dx}\left(x + \dfrac{1}{2}x^2\right) = 1 + x$ (d) $B(x) = \displaystyle\int_0^x [1 + t]\, dt$ ◀

**Practice 4.** Answer the questions in the previous Example for $f(t) = 2t$.

A curious "coincidence" appeared in each of these Examples and Practice problems: the derivative of the function defined by the integral was the same as the integrand, the function "inside" the integral. Stated another way, the function defined by the integral was an "antiderivative" of the function "inside" the integral. In Section 4.4 we will see that this "coincidence" is actually a property shared by all functions defined by an integral in this way. And it is such an important property that it is part of a result called the Fundamental Theorem of Calculus. Before we study the Fundamental Theorem of Calculus, however, we need to consider an "existence" question: Which functions can be integrated?

## *Which Functions Are Integrable?*

This important question was finally answered in the 1850s by Bernhard Riemann, a name that should be familiar to you by now. Riemann proved that a function must be *badly* discontinuous in order to not be integrable.

Due to our inexact definition of the limit involved in the definition of the definite integral, we defer a proof of this theorem to more advanced textbooks.

| **Theorem:** Every continuous function is integrable. |
|---|

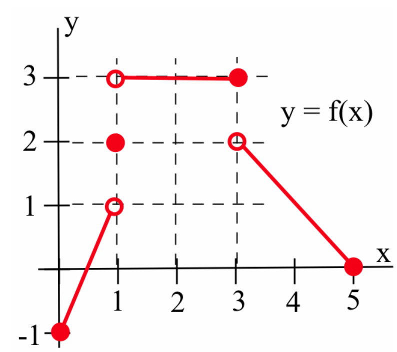
This result says that if $f$ is continuous on the interval $[a, b]$, then $\displaystyle\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ approaches the same finite number, $\displaystyle\int_a^b f(x)\, dx$, as $\|\mathcal{P}\| \to 0$, no matter how we choose the partitions $\mathcal{P}$.

In fact, we can generalize this result to functions that have a finite number of breaks or jumps, as long as the function is bounded:

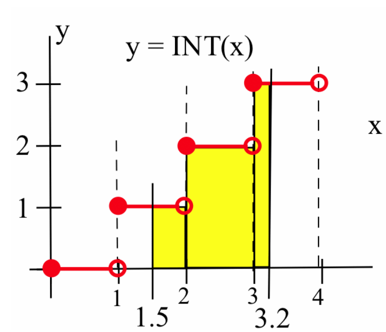
**Theorem:**
If $f$ is defined on an interval $[a, b]$ and bounded ($|f(x)| \leq M$ for some number $M$ for all $x$ in $[a, b]$) and continuous except at a finite number of points in $[a, b]$
then $f$ is integrable on $[a, b]$.

The function $f$ graphed in the margin is always between $-3$ and 3 (in fact, always between $-1$ and 3), so it is bounded, and it is continuous except at $x = 1$ and $x = 3$. As long as the values of $f(1)$ and $f(3)$ are finite numbers, their actual values will not affect the value of the definite integral, and we can compute the value of the integral by computing the areas of the (triangular and rectangular) regions between the graph of $f$ and the $x$-axis:

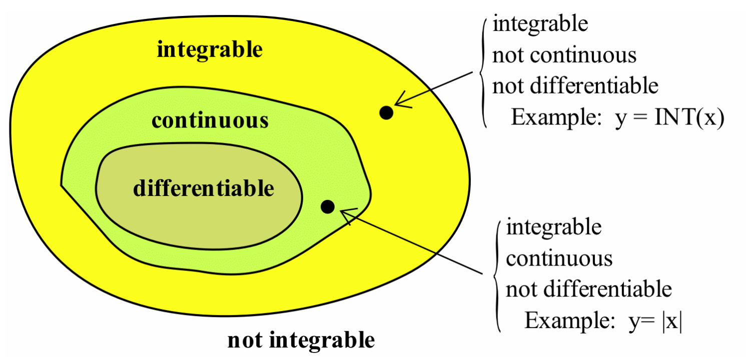
$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = 0 + 6 + 2 = 8$$




**Practice 5.** Evaluate $\int_{1.5}^{3.2} \lfloor x \rfloor \, dx$ (see margin).

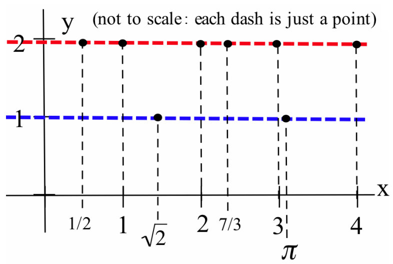



The figure below depicts graphically the relationships between differentiable, continuous and integrable functions:




This says:

- Every differentiable function is continuous, but there are continuous functions that are not differentiable: a simple example of the latter is $f(x) = |x|$, which is continuous but not differentiable at $x = 0$.
- Every continuous function is integrable, but there are integrable functions that are not continuous: a simple example of the latter situation is the function $f(x)$ graphed in the margin, which is integrable on $[0, 5]$ but discontinuous at $x = 2$ and $x = 3$.
- Finally, as demonstrated by the next example, there are functions that are not integrable.




### *A Non-integrable Function*

If $f$ is continuous or piecewise continuous on $[a,b]$, then $f$ is integrable on $[a,b]$. Fortunately, nearly all of the functions we will use throughout the rest of this book are integrable, as are the functions you are likely to need for common applications.

There are functions, however, for which the limit of the Riemann sums does not exist and hence, by definition, are not integrable. Recall the "holey" function from Section 0.4:

> The function
>
> $$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$
>
> is **not** integrable on $[0,3]$.

*Proof.* For any partition $\mathcal{P}$ of $[0,3]$, suppose that you, a very rational person, always choose values of $c_k$ that are rational numbers. (Any open interval on the real-number line contains rational numbers and irrational numbers, so for each subinterval of the partition $\mathcal{P}$ you can always choose $c_k$ to be a rational number.)

Then $h(c_k) = 2$, so for your Riemann sum:

$$\text{YS}_{\mathcal{P}} = \sum_{k=1}^{n} h(c_k)\cdot\Delta x_k = \sum_{k=1}^{n} 2\cdot\Delta x_k = 2\cdot\sum_{k=1}^{n}\Delta x_k = 2\cdot(3-0) = 6$$

Suppose your friend, however, always selects values of $c_k$ that are irrational numbers. Then $h(c_k) = 1$ for each $c_k$, so for your friend's Riemann sum:

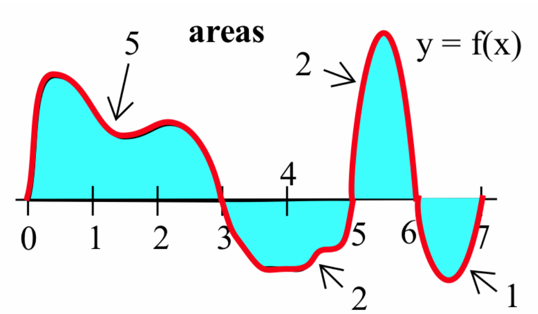
$$\text{FS}_{\mathcal{P}} = \sum_{k=1}^{n} h(c_k)\cdot\Delta x_k = \sum_{k=1}^{n} 1\cdot\Delta x_k = 1\cdot\sum_{k=1}^{n}\Delta x_k = 1\cdot(3-0) = 3$$

So the limit of your Riemann sums, as the mesh of $\mathcal{P}$ approaches 0, will be 6, while the limit of your friend's sums will be 3. This means that $\displaystyle\lim_{\|\mathcal{P}\|\to 0}\left(\sum_{k=1}^{n} h(c_k)\cdot\Delta x_k\right)$ does not exist (because there is no single limiting value of the Riemann sums as $\|\mathcal{P}\|\to 0$) so $h(x)$ is not integrable on $[0,3]$. □

A similar argument shows that $h(x)$ is not integrable on *any* interval of the form $[a,b]$ (where $a<b$).

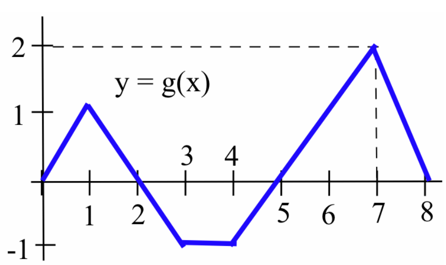
## 4.3 Problems

In Problems 1–20, refer to the graph of $f$ given below to determine the value of each definite integral.




1. $\int_0^3 f(x)\,dx$
2. $\int_3^5 f(x)\,dx$
3. $\int_2^2 f(x)\,dx$
4. $\int_6^7 f(w)\,dw$
5. $\int_0^5 f(x)\,dx$
6. $\int_0^7 f(x)\,dx$
7. $\int_3^6 f(t)\,dt$
8. $\int_5^7 f(x)\,dx$
9. $\int_3^0 f(x)\,dx$
10. $\int_5^3 f(x)\,dx$
11. $\int_6^0 f(x)\,dx$
12. $\int_0^3 2\cdot f(x)\,dx$
13. $\int_4^4 f^2(s)\,ds$
14. $\int_0^3 [1+f(x)]\,dx$
15. $\int_0^3 [x+f(x)]\,dx$
16. $\int_3^5 [3+f(x)]\,dx$
17. $\int_0^5 [2+f(x)]\,dx$
18. $\int_3^5 |f(x)|\,dx$
19. $\int_0^5 |f(x)|\,dx$
20. $\int_7^3 [1+|f(x)|]\,dx$

Problems 21–30 refer to the graph of $g$ given below. Use the graph to evaluate each integral.




21. $\int_0^2 g(x)\,dx$
22. $\int_1^3 g(t)\,dt$
23. $\int_0^5 g(x)\,dx$
24. $\int_4^2 g(x)\,dx$
25. $\int_0^8 g(s)\,ds$
26. $\int_1^4 |g(x)|\,dx$
27. $\int_0^3 2\cdot g(t)\,dt$
28. $\int_5^8 [1+g(x)]\,dx$
29. $\int_6^3 g(u)\,du$
30. $\int_0^8 [t+g(t)]\,dt$

For 31–34 , use the constant functions $f(x)=4$ and $g(x)=3$ on the interval $[0,2]$. Calculate the value of each integral and verify that the value obtained in part (a) is **not** equal to the value in part (b).

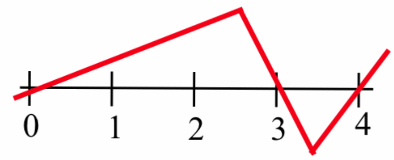
31. (a) $\int_0^2 f(x)\,dx\cdot\int_0^2 g(x)\,dx$ (b) $\int_0^2 f(x)\cdot g(x)\,dx$
32. (a) $\dfrac{\int_0^2 f(x)\,dx}{\int_0^2 g(x)\,dx}$ (b) $\int_0^2 \dfrac{f(x)}{g(x)}\,dx$
33. (a) $\int_0^2 [f(x)]^2\,dx$ (b) $\left(\int_0^2 f(x)\,dx\right)^2$
34. (a) $\int_0^2 \sqrt{f(x)}\,dx$ (b) $\sqrt{\int_0^2 f(x)\,dx}$

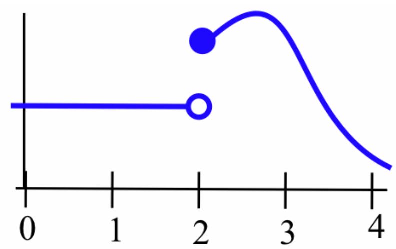
For 35–42, sketch a graph of the integrand function and use it to help evaluate the integral.

35. $\int_0^4 |x|\,dx$
36. $\int_0^4 [1+|t|]\,dt$
37. $\int_{-1}^2 |x|\,dx$
38. $\int_0^2 [|x|-1]\,dx$
39. $\int_1^3 \lfloor u\rfloor\,du$
40. $\int_1^{3.5} \lfloor x\rfloor\,dx$
41. $\int_1^3 [2+\lfloor t\rfloor]\,dt$
42. $\int_3^1 \lfloor x\rfloor\,dx$

For Problems 43–46, sketch (a) a graph of $y=A(x)=\int_0^x f(t)\,dt$ and (b) a graph of $y=A'(x)$.

43. $f(x)=x$
44. $f(x)=x-2$

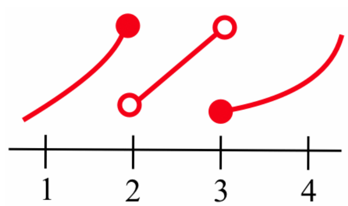
45. 

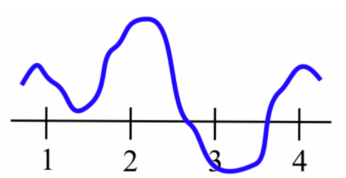

46. 


For 47–50, state whether or not each function is: (a) continuous on $[1,4]$ (b) differentiable on $[1,4]$ (c) integrable on $[1,4]$

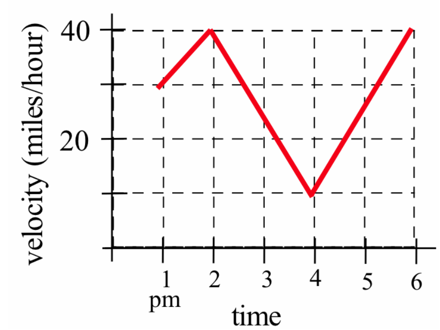
47. $f(x)$ from Problem 45.

48. $f(x)$ from Problem 46.

49. 


50. 


51. The figure below shows the velocity of a car. Write the total distance traveled by the car between 1:00 p.m. and 4:00 p.m. as a definite integral and estimate the value of that integral.




52. Write the total distance traveled by the car in the previous problem between 3:00 p.m. and 6:00 p.m. as a definite integral and estimate the value of that integral.

53. Define $g(x) = 7$ for $x \neq 2$ and $g(2) = 5$.
   (a) Show that the Riemann sum for $g(x)$ for any partition $\mathcal{P}$ of the interval $[1,4]$ is equal to $5w + 7(3 - w)$, where $w$ is the width of the subinterval that includes $x = 2$.
   (b) Compute the limit of these sums, as $\|\mathcal{P}\| \to 0$
   (c) Compare the values of $\int_1^4 g(x)\,dx$ and $\int_1^4 7\,dx$.
   (d) What can you conclude about how changing the value of an integrable function at a single point affects the value of its definite integral?

## *4.3 Practice Answers*

1. $\displaystyle\int_1^4 [f(x) - g(x)]\,dx = 7 - 3 = 4$

2. $m = 2$ and $M = 6$ so $(2)(5-3) = 4 \leq \displaystyle\int_3^5 f(x)\,dx \leq 12 = (6)(5-3)$

3. (a) $A(1) = 0$, $A(2) = 3$, $A(3) = 6$, $A(4) = 9$
   (b) $A(x) = (x-1)(3) = 3x - 3$ (c) $A'(x) = 3$ (d) $A(x) = \displaystyle\int_1^x 3\,dt$

4. (a) $B(0) = 0$, $B(1) = 1$, $B(2) = 4$, $B(3) = 9$
   (b) $B(x) = \frac{1}{2}(x)(2x) = x^2$ (c) $B'(x) = 2x$ (d) $A(x) = \displaystyle\int_1^x 2t\,dt$

5. $(0.5)(1) + (1)(2) + (0.2)(3) = 3.1$