4.5 *The Fundamental Theorem of Calculus*

This section contains the most important and most frequently used theorem of calculus, **THE** Fundamental Theorem of Calculus. Discovered independently by Newton and Leibniz during the late 1600s, it establishes a connection between derivatives and integrals, provides a way to easily calculate many definite integrals, and was a key step in the development of modern mathematics to support the rise of science and technology. Calculus is one of the most significant intellectual structures in the history of human thought, and the Fundamental Theorem of Calculus is the most important brick in that beautiful structure.

Prior sections have emphasized the meaning of the definite integral, defined it, and began to explore some of its applications and properties. In this section, the emphasis shifts to the Fundamental Theorem of Calculus. You will use this theorem often in later sections.

The Fundamental Theorem has two parts. They resemble results in the previous section but apply to more general situations. The first part (FTC^1) says that every continuous function has an antiderivative and shows how to differentiate a function defined as an integral. The second part (FTC^2) shows how to evaluate the definite integral of any function if we know a formula for an antiderivative of that function.

Part 1: Antiderivatives

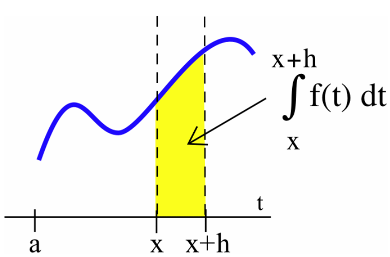
Every continuous function has an antiderivative, even functions with "corners," such as the absolute value function $f(x) = |x|$, that fail to be differentiable at one or more points.

The Fundamental Theorem of Calculus Part 1 (FTC^1)

If f is continuous and $A(x) = \int_a^x f(t)\,dt$

then $A'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

so $A(x)$ is an antiderivative of $f(x)$.



Proof. For a continuous function f, let $A(x) = \int_a^x f(t)\,dt$. By the definition of derivative,

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

Using one of the integral properties from Section 4.3, we know that:

$$\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt$$

$$\Rightarrow \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

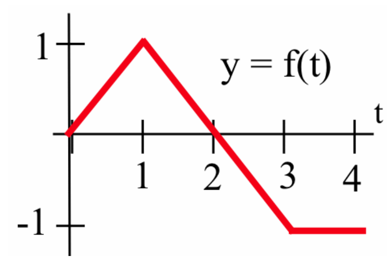
Assume for the moment that $h > 0$. Because f is continuous on $[x, x+h]$ we know that f attains a maximum and minimum on that interval, so there are values m_h and M_h with $x < m_h < x+h$ and $x < M_h < x+h$ so that $f(m_h) \le f(t) \le f(M_h)$ when $x \le t \le x+h$. Hence:

$$\begin{aligned} &\int_x^{x+h} f(m_h)\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} f(M_h)\,dt \\ \Rightarrow\quad & f(m_h)\cdot h \le \int_x^{x+h} f(t)\,dt \le f(M_h)\cdot h \\ \Rightarrow\quad & f(m_h) \le \frac{\int_x^{x+h} f(t)\,dt}{h} \le f(M_h) \end{aligned}$$

Because $x < m_h < x+h$, we know $\lim_{h\to 0^+} m_h = x$; consequently—because $f(t)$ is continuous—we also know that $\lim_{h\to 0^+} f(m_h) = f(x)$. Likewise, $\lim_{h\to 0^+} f(M_h) = f(x)$, so the Squeezing Theorem tells us that:

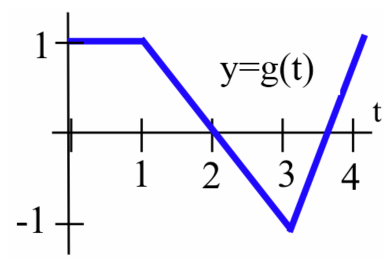
$$\lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)\,dt}{h} = f(x)$$

Repeating this argument for $h < 0$ is relatively straightforward. □

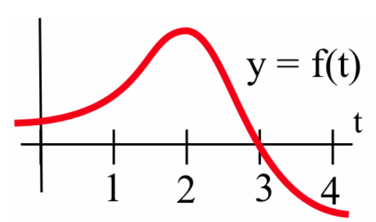


Example 1. Define $A(x) = \int_0^x f(t)\,dt$ for f in the margin figure. Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$ and 4.

Solution. $A(1) = \int_0^1 f(t)\,dt = \frac{1}{2}$, $A(2) = \int_0^2 f(t)\,dt = 1$, $A(3) = \int_0^3 f(t)\,dt = \frac{1}{2}$ and $A(4) = \int_0^4 f(t)\,dt = -\frac{1}{2}$. Because f is continuous, FTC[1] tells us that $A'(x) = f(x)$, so $A'(1) = f(1) = 1$, $A'(2) = f(2) = 0$, $A'(3) = f(3) = -1$ and $A'(4) = f(4) = -1$. ◀



Practice 1. Define $A(x) = \int_0^x g(t)\,dt$ for g in the margin figure. Evaluate $A(x)$ and $A'(x)$ for $x = 1, 2, 3$ and 4.



Example 2. Define $A(x) = \int_0^x f(t)\,dt$ for f in the margin figure. For which value of x is $A(x)$ maximum? For which x is the rate of change of A maximum?

Solution. Because A is differentiable, the only critical points are where $A'(x) = 0$ or at endpoints. $A'(x) = f(x) = 0$ at $x = 3$, and A has a maximum at $x = 3$. Notice that the values of $A(x)$ increase as x goes from 0 to 3 and then the values of A decrease. The rate of change of $A(x)$ is $A'(x) = f(x)$, and $f(x)$ appears to have a maximum at $x = 2$, so the rate of change of $A(x)$ is maximum when $x = 2$. Near $x = 2$, a slight increase in the value of x yields the maximum increase in the value of $A(x)$. ◀

Part 2: Evaluating Definite Integrals

If we know a formula for an antiderivative of a function, then we can compute any definite integral of that function.

The Fundamental Theorem of Calculus Part 2 (FTC²)

If $f(x)$ is continuous
and $F(x)$ is any antiderivative of f (so that $F'(x) = f(x)$)

then $\displaystyle\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$.

Proof. Define $A(x) = \displaystyle\int_a^x f(t)\,dt$. If F is an antiderivative of f, then $F'(x) = f(x)$ and by FTC¹ we know that $A'(x) = f(x)$ so $F'(x) = A'(x)$, hence $F(x)$ and $A(x)$ differ by a constant: $A(x) - F(x) = C$ for all x and some constant C. At $x = a$, we have $C = A(a) - F(a) = 0 - F(a) = -F(a)$ so $C = -F(a)$ and the equation $A(x) - F(x) = C$ becomes $A(x) - F(x) = -F(a)$. Then $A(x) = F(x) - F(a)$ for all x, so setting $x = b$ yields $A(b) = F(b) - F(a)$, hence $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, the formula we wanted. □

We can evaluate the definite integral of a continuous function f by finding an antiderivative of f (*any* antiderivative of f will work) and then doing some arithmetic with this antiderivative. FTC² does not tell us *how* to find an antiderivative of f, and it does not tell us how to find the definite integral of a discontinuous function. It is possible to evaluate definite integrals of some discontinuous functions (as we saw in Section 4.3) but not by using FTC² directly.

Example 3. Evaluate $\displaystyle\int_0^2 \left(x^2 - 1\right)\,dx$.

Solution. $F(x) = \frac{1}{3}x^3 - x$ is an antiderivative of $f(x) = x^2 - 1$ (you should check that $\mathbf{D}\left(\frac{1}{3}x^3 - x\right) = x^2 - 1$), so:

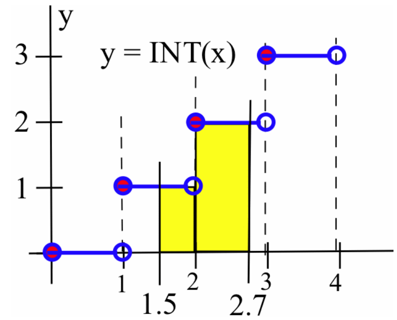
$$\int_0^2 \left(x^2 - 1\right)\,dx = \left[\frac{1}{3}x^3 - x\right]_0^2 = \left[\frac{1}{3}\cdot 2^3 - 2\right] - \left[\frac{1}{3}\cdot 0^3 - 0\right] = \frac{2}{3}$$

If your friend had picked a different antiderivative of $x^2 - 1$, say $G(x) = \frac{1}{3}x^3 - x + 4$, then her calculations would be slightly different :

$$\begin{aligned}\int_0^2 \left(x^2 - 1\right)\,dx &= \left[\frac{1}{3}x^3 - x + 4\right]_0^2 \\ &= \left[\frac{1}{3}\cdot 2^3 - 2 + 4\right] - \left[\frac{1}{3}\cdot 0^3 - 0 + 4\right] = \frac{2}{3} + 4 - 4 = \frac{2}{3}\end{aligned}$$

but the result would be the same. ◀

Practice 2. Evaluate $\int_1^3 \left(3x^2 - 1\right)\,dx$.



Example 4. Evaluate $\int_{1.5}^{2.7} \lfloor x \rfloor\,dx$ (where $\lfloor x \rfloor = \text{INT}(x)$ is the largest integer less than or equal to x, as in the margin figure).

Solution. $f(x) = \lfloor x \rfloor$ is not continuous at $x = 2$ in the interval $[1.5, 2.7]$, so we cannot employ the Fundamental Theorem of Calculus directly. We can, however, use our understanding of the geometric meaning of a definite integral to compute:

$$\begin{aligned}\int_{1.5}^{2.7} \lfloor x \rfloor\,dx &= (\text{area below } y = \lfloor x \rfloor \text{ for } 1.5 \le x \le 2) + (\text{area below } y = \lfloor x \rfloor \text{ for } 2 \le x \le 2.7)\\ &= (\text{first base})\,(\text{first height}) + (\text{second base})\,(\text{second height})\\ &= (0.5)(1) + (0.7)(2) = 1.9\end{aligned}$$

We could also split the integral into two pieces:

$$\begin{aligned}\int_{1.5}^{2.7} \lfloor x \rfloor\,dx &= \int_{1.5}^{2.0} \lfloor x \rfloor\,dx + \int_{2.0}^{2.7} \lfloor x \rfloor\,dx\\ &= \int_{1.5}^{2.0} 1\,dx + \int_{2.0}^{2.7} 2\,dx = \Big[x\Big]_{1.5}^{2.0} + \Big[x\Big]_{2.0}^{2.7}\\ &= [2.0 - 1.5] + [2(2.7) - 2(2.0)] = 0.5 + 1.4 = 1.9\end{aligned}$$

using the fact that $\lfloor x \rfloor = 1$ for $1.5 \le x < 2.0$ and the fact that $\lfloor x \rfloor = 2$ for $2.0 \le x \le 2.7$. (We also need to redefine the first integrand to equal 1 at its right endpoint and the second integrand to equal 2 at its right endpoint so that each integrand is continuous on a closed interval). ◀

Problem 53 in Section 4.3 indicates that this redefinition is perfectly legal.

Practice 3. Evaluate $\int_{1.3}^{3.4} \lfloor x \rfloor\,dx$.

Calculus is the study of derivatives and integrals, their meanings and their applications. The Fundamental Theorem of Calculus demonstrates how differentiation and integration are closely related processes: integration is really anti-differentiation, the inverse of differentiation.

Applications: The Future

Calculus is important for many reasons, but students are usually required to study calculus because they will need to *apply* calculus concepts in a variety of fields. Most applied problems in integral calculus require the following steps to get from a real-life problem to a numerical answer:

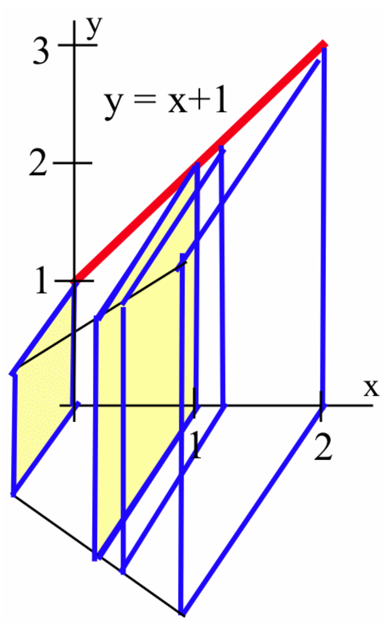
$$\boxed{\text{applied problem}} \xrightarrow{\ 1\ } \boxed{\text{Riemann sum}} \xrightarrow{\ 2\ } \boxed{\text{definite integral}} \xrightarrow{\ 3\ } \boxed{\text{number}}$$

Step 1 is absolutely vital. If we can not translate the ideas of an applied problem into an area or a Riemann sum or a definite integral, then we can not use integral calculus to solve the problem. For a few special types of applied problems, we will be able to move directly from the problem to an integral, but usually it will be easier to first break the problem into smaller pieces and to build a Riemann sum. Section 4.7 and all of Chapter 5 focus on translating different types of applied problems into Riemann sums and definite integrals. **Computers and calculators are seldom of any help with Step 1.**

Step 2 is usually easy. If we have a Riemann sum $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ on an interval $[a,b]$, then the limit of the sum (as $n \to \infty$) is simply the definite integral $\int_a^b f(x)\,dx$.

Step 3 can be handled in several ways.

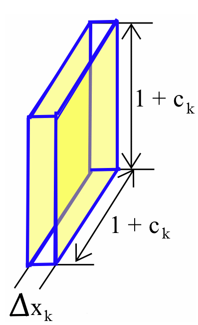
- If the function f is relatively simple, we may be able to find an antiderivative for f (using techniques from Section 4.6 and Chapter 8) and then apply FTC^2 to get a numerical answer.
- If the function f is more complicated, then integral tables or computers (Section 4.8) may help us find an antiderivative for f, in which case we can apply FTC^2 to get a numerical answer.
- If we cannot find an antiderivative for f, we can compute approximate numerical answers for the definite integral using various approximation methods (Sections 4.9 and 8.7); we typically employ computers to carry out the heavy-duty arithmetic.



Usually any difficulties in solving an applied problem arise in the first and third steps. There are techniques and details to master and understand, but it is also important to keep in mind where these techniques and details fit into the bigger picture.

The next Example illustrates these steps for the problem of finding a volume of a solid. We will explore techniques for finding volumes of solids in greater detail in Chapter 5.

Example 5. Find the volume of the solid shown in the margin for $0 \le x \le 2$. (Each "slice" perpendicular to the xy-plane is a square.)



Solution. Step 1: Going from the figure to a Riemann sum.
If we break the solid into n "slices" with cuts perpendicular to the x-axis (and the xy-plane) using a partition $\mathcal{P}$ with cuts at x_1, x_2, x_3, ..., x_{n-1} (like slicing a block of cheese or a loaf of bread), then the volume of the original solid is equal to the sum of the volumes of the "slices."

The volume of the k-th slice is *approximately* equal to the volume of a thin, rectangular box:

$$(\text{height}) \cdot (\text{base}) \cdot (\text{thickness}) \approx (c_k + 1)\,(c_k + 1) \cdot \Delta x_k$$

where c_k is any chosen value between x_{k-1} and x_k. Therefore:

$$\text{total volume} = \sum_{k=1}^{n} (\text{volume of the } k\text{-th slice}) = \sum_{k=1}^{n} (c_k + 1)^2 \, \Delta x_k$$

which is a Riemann sum.

Step 2: Going from the Riemann sum to a definite integral.
We can improve the Riemann sum approximation of the total volume from Step 1 by taking thinner slices (making all of the Δx_k smaller and smaller) so that the mesh of the partition $\mathcal{P}$ approaches 0:

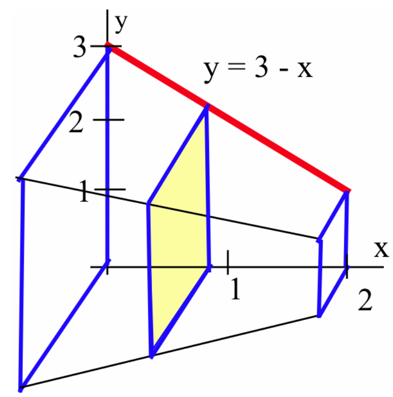
$$\lim_{\|\mathcal{P}\| \to 0} \sum_{k=1}^{n} (c_k + 1)^2 \, \Delta x_k = \int_0^2 (x+1)^2 \, dx = \int_0^2 \left[x^2 + 2x + 1 \right] \, dx$$

Step 3: Going from the definite integral to a numerical answer.
We can now use FTC^2 to evaluate the integral: $F(x) = \frac{1}{3}x^3 + x^2 + x$ is an antiderivative of $x^2 + 2x + 1$ (check this by differentiating $F(x)$), so:

$$\int_0^2 \left[x^2 + 2x + 1 \right] \, dx = \left[\frac{1}{3}x^3 + x^2 + x \right]_0^2$$

$$= \left[\frac{1}{3} \cdot 2^3 + 2^2 + 2 \right] - \left[\frac{1}{3} \cdot 0^3 + 0^2 + 0 \right] = \frac{26}{3}$$

The volume of the solid shape is exactly $\frac{26}{3}$ cubic inches. ◀



Practice 4. Find the volume of the solid shape in the margin figure for $0 \le x \le 2$. (Each "slice" perpendicular to the xy-plane is a square.)

Leibniz's Rule For Differentiating Integrals

If the endpoint of an integral is a function of x rather than simply x, then we need to use the Chain Rule together with FTC^1 to calculate the derivative of the integral. For example:

$$A'(x) = f(x) \qquad \Rightarrow \qquad \frac{d}{dx}\left[A\left(x^2\right) \right] = A'(x) \cdot 2x = f\left(x^2\right) \cdot 2x$$

We can generalize this result by applying the Chain Rule to the derivative of the integral:

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t) \, dt \right] = \frac{d}{dx}\left[A\left(g(x)\right) \right] = f\left(g(x)\right) \cdot g'(x)$$

and combine this with some integral properties to further extend FTC^1.

Leibniz's Rule

If f is a continuous function, $A(x) = \int_a^x f(t) \, dt$
and $g_1(x)$ and $g_2(x)$ are both differentiable functions

then $\frac{d}{dx}\left[\int_{g_1(x)}^{g_2(x)} f(t) \, dt \right] = f\left(g_2(x)\right) \cdot g_2'(x) - f\left(g_1(x)\right) \cdot g_1'(x)$

Proof. Assume for simplicity that f, g_1 and g_2 are continuous on $(-\infty, \infty)$ and let c be any number. Then:

$$\int_{g_1(x)}^{g_2(x)} f(t)\,dt = \int_c^{g_2(x)} f(t)\,dt + \int_{g_1(x)}^{c} f(t)\,dt$$
$$= \int_c^{g_2(x)} f(t)\,dt - \int_c^{g_1(x)} f(t)\,dt$$

Now apply the preceding result. □

Example 6. If a is any constant, compute the derivatives $\frac{d}{dx}\left[\int_a^{5x} t^2\,dt\right]$, $\frac{d}{dx}\left[\int_a^{x^2} \cos(u)\,du\right]$ and $\frac{d}{dw}\left[\int_{\pi w}^{\sin w} z^3\,dz\right]$.

Solution. Applying Leibniz's Rule:

$$\frac{d}{dx}\left[\int_a^{5x} t^2\,dt\right] = (5x)^2 \cdot 5 = 125x^2$$
$$\frac{d}{dx}\left[\int_a^{x^2} \cos(u)\,du\right] = \cos(x^2) \cdot 2x = 2x\cos(x^2)$$
$$\frac{d}{dw}\left[\int_{\pi w}^{\sin(w)} z^3\,dz\right] = (\sin(w))^3 \cdot \cos(w) - (\pi w)^3 \cdot \pi$$

The last quantity simplifies to $\sin^3(w)\cos(w) - \pi^4 w^3$. ◀

Practice 5. Compute $\frac{d}{dx}\left[\int_0^{x^3} \sin(t)dt\right]$.

4.5 Problems

In Problems 1–2, (a) Use FTC[2] to find a formula for $A(x)$, differentiate $A(x)$ to obtain a formula for $A'(x)$, and evaluate $A'(x)$ at $x = 1, 2$ and 3. (b) Use FTC[1] to evaluate $A'(x)$ at $x = 1, 2$ and 3.

1. $A(x) = \int_0^x 3t^2\,dt$
2. $A(x) = \int_1^x (1 + 2t)\,dt$

In Problems 3–8, compute $A'(1)$, $A'(2)$ and $A'(3)$.

3. $A(x) = \int_0^x 2t\,dt$
4. $A(x) = \int_1^x 2t\,dt$
5. $A(x) = \int_{-3}^x 2t\,dt$
6. $A(x) = \int_0^x \left(3 - t^2\right)\,dt$
7. $A(x) = \int_0^x \sin(t)\,dt$
8. $A(x) = \int_1^x |t - 2|\,dt$

In 9–12, $A(x) = \int_0^x f(t)\,dt$, with $f(t)$ given graphically. Evaluate $A'(1)$, $A'(2)$ and $A'(3)$.

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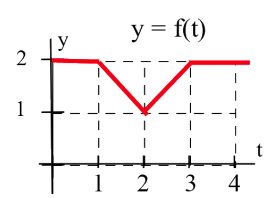

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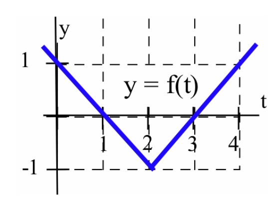

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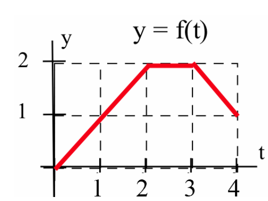

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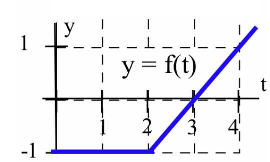


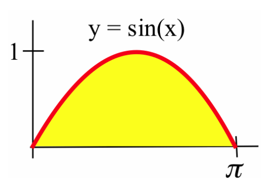
In 13–33, verify that $F(x)$ is an antiderivative of the integrand and use FTC[2] to evaluate the integral.

13. $\int_0^1 2x\,dx, \quad F(x) = x^2 + 5$
14. $\int_1^4 3x^2\,dx, \quad F(x) = x^3 + 2$
15. $\int_1^3 x^2\,dx, \quad F(x) = \frac{1}{3}x^3$
16. $\int_0^3 \left[x^2 + 4x - 3\right]\,dx,\ F(x) = \frac{1}{3}x^3 + 2x^2 - 3x$
17. $\int_1^5 \frac{1}{x}\,dx, \quad F(x) = \ln(x)$
18. $\int_2^5 \frac{1}{x}\,dx, \quad F(x) = \ln(x) + 4$
19. $\int_{\frac{1}{2}}^3 \frac{1}{x}\,dx, \quad F(x) = \ln(x)$
20. $\int_1^3 \frac{1}{x}\,dx, \quad F(x) = \ln(x) + 2$
21. $\int_0^{\frac{\pi}{2}} \cos(x)\,dx, \quad F(x) = \sin(x)$
22. $\int_0^{\pi} \sin(x)\,dx, \quad F(x) = -\cos(x)$
23. $\int_0^1 \sqrt{x}\,dx, \quad F(x) = \frac{2}{3}x^{\frac{3}{2}}$
24. $\int_1^4 \sqrt{x}\,dx, \quad F(x) = \frac{2}{3}x^{\frac{3}{2}}$
25. $\int_1^7 \sqrt{x}\,dx, \quad F(x) = \frac{2}{3}x^{\frac{3}{2}}$
26. $\int_1^4 \frac{1}{2\sqrt{x}}\,dx, \quad F(x) = \sqrt{x}$
27. $\int_1^9 \frac{1}{2\sqrt{x}}\,dx, \quad F(x) = \sqrt{x}$
28. $\int_2^5 \frac{1}{x^2}\,dx, \quad F(x) = -\frac{1}{x}$
29. $\int_{-2}^3 e^x\,dx, \quad F(x) = e^x$
30. $\int_0^3 \frac{2x}{1+x^2}\,dx, \quad F(x) = \ln(1+x^2)$
31. $\int_0^{\frac{\pi}{4}} \sec^2(x)\,dx, \quad F(x) = \tan(x)$
32. $\int_1^e \ln(x)\,dx, \quad F(x) = x \cdot \ln(x) - x$
33. $\int_0^3 2x\sqrt{1+x^2}\,dx, \quad F(x) = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}$

For 34–48, find an antiderivative of the integrand and use FTC[2] to evaluate the definite integral.

34. $\int_2^5 3x^2\,dx$
35. $\int_{-1}^2 x^2\,dx$
36. $\int_1^3 \left[x^2 + 4x - 3\right]\,dx$
37. $\int_1^e \frac{1}{x}\,dx$
38. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x)\,dx$
39. $\int_{25}^{100} \sqrt{x}\,dx$
40. $\int_3^5 \sqrt{x}\,dx$
41. $\int_1^{10} \frac{1}{x^2}\,dx$
42. $\int_1^{1000} \frac{1}{x^2}\,dx$
43. $\int_0^1 e^x\,dx$
44. $\int_{-2}^2 \frac{2x}{1+x^2}\,dx$
45. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2(x)\,dx$
46. $\int_0^1 e^{2x}\,dx$
47. $\int_3^3 \sin(x) \cdot \ln(x)\,dx$
48. $\int_2^4 (x-2)^3\,dx$

In 49–54, find the area of the shaded region.

49.


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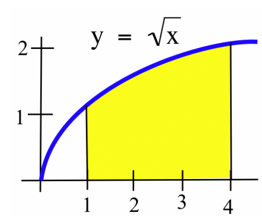
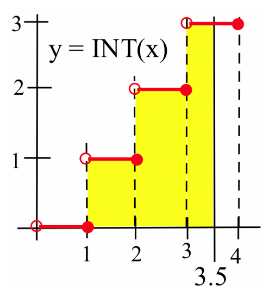


51.


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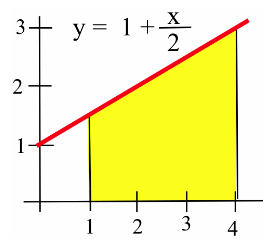

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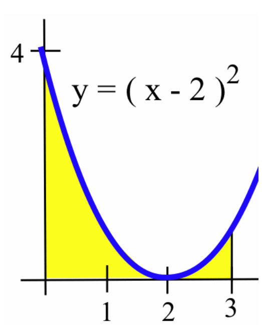

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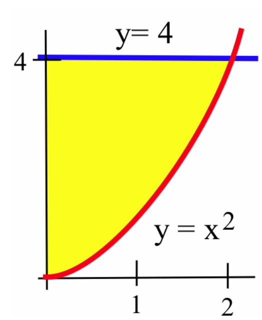


55. Given that $A'(x) = \tan(x)$, find $\mathbf{D}(A(3x))$, $\mathbf{D}\left(A(x^2)\right)$ and $\mathbf{D}(A(\sin(x)))$.

56. Given that $B'(x) = \sec(x)$, find $\mathbf{D}(B(3x))$, $\mathbf{D}\left(B(x^2)\right)$ and $\mathbf{D}(B(\sin(x)))$.

In 57–68, apply Leibniz's Rule.

57. $\frac{d}{dx}\left[\int_1^{5x} \sqrt{1+t}\,dt\right]$

58. $\frac{d}{dx}\left[\int_2^{x^2} \sqrt{1+t}\,dt\right]$

59. $\frac{d}{dx}\left[\int_0^{\sin(x)} \sqrt{1+t}\,dt\right]$

60. $\frac{d}{dx}\left[\int_1^{2+3x} \left(t^2+5\right)\,dt\right]$

61. $\frac{d}{dx}\left[\int_0^{1-2x} \left(3t^2+2\right)\,dt\right]$

62. $\frac{d}{dx}\left[\int_x^{9} \left(3t^2+2\right)\,dt\right]$

63. $\frac{d}{dx}\left[\int_x^{\pi} \cos(3t)\,dt\right]$

64. $\frac{d}{dx}\left[\int_{7x}^{\pi} \cos(2t)\,dt\right]$

65. $\frac{d}{dx}\left[\int_x^{x^2} \tan(t)\,dt\right]$

66. $\frac{d}{dx}\left[\int_0^{\pi} \cos(3t)\,dt\right]$

67. $\frac{d}{dx}\left[\int_2^{\ln(x)} 5t\cdot\cos(3t)\,dt\right]$

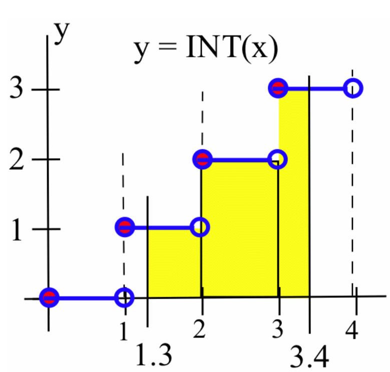
68. $\frac{d}{dx}\left[\int_0^{\pi} \tan(7t)\,dt\right]$

69. $\frac{d}{dy}\left[\int_0^{y^2} \tan(\theta)\,d\theta\right]$

4.5 Practice Answers

1. $A(1) = 1$, $A(2) = 1.5$, $A(3) = 1$, $A(4) = 0.5$; $A'(x) = fgx)$ so $A'(1) = g(1) = 1$, $A'(2) = g(2) = 0$, $A'(3) = -1$, $A'(4) = 0$.

2. $F(x) = x^3 - x$ is an antiderivative of $f(x) = 3x^2 - 1$ so:
$$\int_1^3 \left[3x^2-1\right]\,dx = \left[x^3-x\right]_1^3 = \left[3^3-3\right] - \left[1^3-1\right] = 24$$
$F(x) = x^3 - x + 7$ is another antiderivative of $f(x) = 3x^2 - 1$ so:
$$\int_1^3 \left[3x^2-1\right]\,dx = \left[x^3-x+7\right]_1^3 = \left[3^3-3+7\right] - \left[1^3-1+7\right] = 24$$
No matter which antiderivative of $f(x) = 3x^2 - 1$ you use, the value of the definite integral $\int_1^3 \left[3x^2-1\right]\,dx$ is 24.



3. Because $f(x) = \lfloor x \rfloor$ is not continuous on $[1.3, 3.4]$ we cannot use the Fundamental Theorem of Calculus. Instead, we can think of the definite integral as an area (see margin figure) and compute:
$$\int_{1.3}^{3.4} \lfloor x \rfloor\,dx = 3.9$$

4. First break the solid into "slices" and approximate the volume of the k-th slice by $(3-c_k)^2 \cdot \Delta x_k$ where c_k is any point in the k-th subinterval. Next add up these approximate volumes to get a Riemann Sum:
$$\sum_{k=1}^{n} (3-c_k)^2 \cdot \Delta x_k$$

and then take the limit of these Riemann sums as the mesh of the partitions approaches 0 (and $n \to \infty$, where n is the number of subintervals in the partition):

$$\begin{aligned}
\lim_{\|\mathcal{P}\| \to 0} \left[\sum_{k=1}^{n} (3 - c_k)^2 \cdot \Delta x_k \right] &= \int_0^2 (3-x)^2 \, dx \\
&= \int_0^2 \left(9 - 6x + x^2\right) dx \\
&= \left[9x - 3x^2 + \frac{1}{3}x^3 \right]_0^2 \\
&= \left[18 - 12 + \frac{8}{3} \right] - [0 - 0 + 0] = \frac{26}{3}
\end{aligned}$$

5. $\dfrac{d}{dx}\left[\displaystyle\int_0^{x^3} \sin(t)\, dt\right] = \sin(x^3) \cdot \dfrac{d}{dx}\left[x^3\right] = 3x^2 \sin(x^3)$