4.6 Finding Antiderivatives

In order to use the second part of the Fundamental Theorem of Calculus, we need an antiderivative of the integrand, but sometimes it is not easy to find one. This section collects some of the information we already know about general properties of antiderivatives and about antiderivatives of particular functions. It shows how to use this information to find antiderivatives of more complicated functions and introduces a "change of variable" technique to make that job easier.

Indefinite Integrals and Antiderivatives

Antiderivatives arise so often that there is a special notation to indicate the antiderivative of a function:

> $\int f(x)\,dx$, read as "the **indefinite integral** of f" or as "the antiderivatives of f," represents the collection (or family) of all functions whose derivatives are f.

If you've been wondering why we called $\int_a^b f(t)\,dt$ a **definite** integral, now you know. A definite integral has specific upper and lower limits, while an indefinite integral does not.

If F is an antiderivative of f, then any member of the family $\int f(x)\,dx$ has the form $F(x)+C$ for some constant C. We write $\int f(x)\,dx = F(x)+C$, where C represents an arbitrary constant. There are no small families in the world of antiderivatives: if f has one antiderivative F, then f has an *infinite* number of antiderivatives and each has the form $F(x)+C$, which means there are many ways to write a particular indefinite integral and some of them may look very different. You can check that $F(x)=\sin^2(x)$, $G(x)=-\cos^2(x)$ and $H(x)=2\sin^2(x)+\cos^2(x)$ all have the same derivative, $f(x)=2\sin(x)\cos(x)$, so the indefinite integral of $2\sin(x)\cos(x)$, $\int 2\sin(x)\cos(x)\,dx$, can be written in several ways: $\sin^2(x)+C$ or $-\cos^2(x)+K$ or $2\sin^2(x)+\cos^2(x)+C$.

Practice 1. Verify that $\int 2\tan(x)\cdot\sec^2(x)\,dx = \tan^2(x)+C$ and that $\int 2\tan(x)\cdot\sec^2(x)\,dx = \sec^2(x)+K$.

Properties of Antiderivatives (Indefinite Integrals)

These sum, difference and constant-multiple properties follow directly from corresponding properties for derivatives.

> If f and g are integrable functions, then
>
> - $\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
> - $\int k\cdot f(x)\,dx = k\cdot\int f(x)\,dx$

Although we know general rules for *derivatives* of products and quotients, unfortunately there are no easy general patterns for *antiderivatives* of products and quotients—we will only be able to add one more general property to this list (in Section 8.2).

We already know antiderivatives for several important functions.

Constant Functions: $\int k\,dx = kx + C$

Powers of x: $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$ if $p \neq -1$, $\int \frac{1}{x}\,dx = \ln|x| + C$

Exponential Functions: $\int e^x\,dx = e^x + C$

Trig Functions: $\int \cos(x)\,dx = \sin(x) + C$, $\int \sin(x)\,dx = -\cos(x) + C$

$\int \sec^2(x)\,dx = \tan(x) + C$, $\int \csc^2(x)\,dx = -\cot(x) + C$

$\int \sec(x)\cdot\tan(x)\,dx = \sec(x) + C$, $\int \csc(x)\cdot\cot(x)\,dx = -\csc(x) + C$

Our list of antiderivatives of particular functions will grow in coming chapters and will eventually include antiderivatives of additional trigonometric functions, the inverse trigonometric functions, logarithms, rational functions and more. (See Appendix I.)

All of these antiderivatives can be verified by differentiating. For $\int \frac{1}{x}\,dx$ you may be wondering about the presence of the absolute value signs in the antiderivative. If $x > 0$, you can check that:

$$\mathbf{D}(\ln(|x|)) = \mathbf{D}(\ln(x)) = \frac{1}{x}$$

If $x < 0$, then you can check that:

$$\mathbf{D}(\ln(|x|)) = \mathbf{D}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}$$

When computing a definite integral of the form $\int_a^b \frac{1}{x}\,dx$, either a and b will both be positive or both be negative, because the integrand is not defined at $x = 0$, so $x = 0$ cannot be included in the interval of integration.

Antiderivatives of More Complicated Functions

Antiderivatives are very sensitive to small changes in the integrand, so we should be very careful.

Fortunately, an antiderivative can always be checked by differentiating, so even though we may not find the correct antiderivative, we should be able to determine whether or not an antiderivative candidate is actually an antiderivative.

Example 1. We know $\mathbf{D}(\sin(x)) = \cos(x)$, so $\int \cos(x)\,dx = \sin(x) + C$.
Find: (a) $\int \cos(2x+3)\,dx$ (b) $\int \cos(5x-7)\,dx$ (c) $\int \cos(x^2)\,dx$

Solution. (a) Because $\sin(x)$ is an antiderivative of $\cos(x)$, it is reasonable to hope that $\sin(2x+3)$ will be an antiderivative of $\cos(2x+3)$. Unfortunately, we see that $\mathbf{D}(\sin(2x+3)) = \cos(2x+3)\cdot 2$, exactly twice the result we want. Let's try again by modifying our "guess" to be half the original guess:

$$\mathbf{D}\left(\frac{1}{2}\sin(2x+3)\right) = \frac{1}{2}\cos(2x+3)\cdot 2 = \cos(2x+3)$$

which is what we want, so $\int \cos(2x+3)\,dx = \frac{1}{2}\sin(2x+3) + C$.

(b) $\mathbf{D}(\sin(5x-7)) = \cos(5x-7)\cdot 5$, so dividing the original guess by 5 we get $\mathbf{D}\left(\frac{1}{5}\sin(5x-7)\right) = \frac{1}{5}\cos(5x-7)\cdot 5 = \cos(5x-7)$ and conclude that $\int \cos(5x-7)\,dx = \frac{1}{5}\sin(5x-7) + C$.

(c) $\mathbf{D}\left(\sin(x^2)\right) = \cos(x^2)\cdot 2x$. It was easy enough in parts (a) and (b) to modify our "guesses" to eliminate the constants 2 and 5, but here the x is much harder to eliminate:

$$\mathbf{D}\left(\frac{1}{2x}\sin\left(x^2\right)\right) = \mathbf{D}\left(\frac{\sin(x^2)}{2x}\right)$$
$$= \frac{2x\cdot\mathbf{D}\left(\sin(x^2)\right) - \sin(x^2)\cdot\mathbf{D}(2x)}{(2x)^2)}$$
$$= \frac{(2x)^2\cos(x^2) - 2\sin(x^2)}{(2x)^2}$$
$$= \cos(x^2) - \frac{\sin(x^2)}{2x^2} \neq \cos(x^2)$$

Our guess did not check out — we're stuck. ◀

Advanced mathematical techniques beyond the scope of this text can show that $\cos(x^2)$ does not have an "elementary" antiderivative composed of polynomials, roots, trigonometric functions, exponential functions or their inverses.

The value of a definite integral of $\cos(x^2)$ could still be approximated as accurately as needed by using Riemann sums or one of the numerical techniques in Sections 4.9 and 8.7, but no matter how hard we try, we cannot find a concise formula for an antiderivative of $\cos(x^2)$ in order to use the Fundamental Theorem of Calculus. Even a simple-looking integrand can be very difficult. At this point, there is no quick way to tell the difference between an "easy" indefinite integral and a "difficult" or "impossible" one.

Getting the Constants Right

The previous example illustrated one technique for finding antiderivatives: "guess" the form of the answer, differentiate your "guess" and then modify your original "guess" so its derivative is exactly what you want it to be.

Example 2. Knowing that $\int \sec^2(x)\,dx = \tan(x) + C$ and $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$, find (a) $\int \sec^2(3x+7)\,dx$ (b) $\int \frac{1}{\sqrt{5x+3}}\,dx$.

Solution. (a) If we "guess" an answer of $\tan(3x+7)$ and then differentiate it, we get $\mathbf{D}\left(\tan(3x+7)\right) = \sec^2(3x+7)\cdot\mathbf{D}(3x+7) = 3\sec^2(3x+7)$, which is three times what we want. If we divide our original guess by 3 and try again, we have:

$$\mathbf{D}\left(\frac{1}{3}\tan(3x+7)\right) = \frac{1}{3}\mathbf{D}\left(\tan(3x+7)\right) = \frac{1}{3}\sec^2(3x+7)\cdot 3$$
$$= \sec^2(3x+7)$$

so $\int \sec^2(3x+7)\,dx = \frac{1}{3}\tan(3x+7) + C$.

(b) If we "guess" $2\sqrt{5x+3}$ and then differentiate it, we get:

$$\mathbf{D}\left(2\,(5x+3)^{\frac{1}{2}}\right) = 2\cdot\frac{1}{2}\,(5x+3)^{-\frac{1}{2}}\,\mathbf{D}(5x+3) = 5\cdot(5x+3)^{-\frac{1}{2}}$$

which is five times what we want. Dividing our guess by 5 and differentiating, we have:

$$\mathbf{D}\left(\frac{2}{5}(5x+3)^{\frac{1}{2}}\right) = \frac{2}{5}\cdot\frac{1}{2}(5x+3)^{-\frac{1}{2}}\cdot 5 = \frac{1}{\sqrt{5x+3}}$$

so $\displaystyle\int \frac{1}{\sqrt{5x+3}}\,dx = \frac{2}{5}\sqrt{5x+3}+C$. ◀

Practice 2. Find $\displaystyle\int \sec^2(7x)\,dx$ and $\displaystyle\int \frac{1}{\sqrt{3x+8}}\,dx$.

The "guess and check" method is a very effective technique if you can make a good first guess, one that misses the desired result only by a constant multiple. In that situation, just divide the first guess by the unwanted constant multiple. If the derivative of your guess misses by something other than a constant multiple, then more drastic modifications are needed. Sometimes the next technique can help.

Making Patterns More Obvious: Changing the Variable

Successful integration is mostly a matter of recognizing patterns. The "change of variable" technique can make some underlying patterns of an integral easier to recognize. Essentially, the technique involves rewriting an integral that is originally in terms of one variable, say x, in terms of another variable, say u, with the hope that it will be easier to find an antiderivative of the new integrand.

For example, we can rewrite $\displaystyle\int \cos(5x+1)\,dx$ by setting $u = 5x+1$. Then $\cos(5x+1)$ becomes $\cos(u)$ but we must also convert the dx in the original integral. We know that $\dfrac{du}{dx} = 5$, so rewriting this last expression in differential notation, we get $du = 5\,dx$; isolating dx yields $dx = \frac{1}{5}\,du$ so:

We first discussed differential notation in Section 2.8; although you may not have used them much in differential calculus, you will now use them extensively.

$$\int \cos(5x+1)\,dx = \int \cos(u)\cdot\frac{1}{5}\,du = \frac{1}{5}\int \cos(u)\,du$$

This new integral is easier:

$$\frac{1}{5}\int \cos(u)\,du = \frac{1}{5}\sin(u)+C$$

but our original problem was in terms of x and our answer is in terms of u, so we must "resubstitute" using the relationship $u = 5x+1$:

$$\frac{1}{5}\sin(u)+C = \frac{1}{5}\sin(5x+1)+C$$

We can now conclude that:

$$\int \cos(5x+1)\,dx = \frac{1}{5}\sin(5x+1)+C$$

As always, you can check this result by differentiating.

We can summarize the steps of this "change of variable" (or "u-substitution") method as:

Often u is set equal to some "interior" part of the original integrand function.

- set a new variable, say u, equal to some function of the original variable x
- calculate the differential du in terms of x and dx
- rewrite the original integral in terms of u and du
- integrate the new integral to get an answer in terms of u
- resubstitute for u to get a result in terms of the original variable x

Example 3. Make the suggested change of variable, rewrite each integral in terms of u and du, and evaluate the integral.

(a) $\int \cos(x) \cdot e^{\sin(x)}\, dx \quad \text{with} \quad u = \sin(x)$

(b) $\int \frac{2x}{5+x^2}\, dx \quad \text{with} \quad u = 5 + x^2$

Solution. (a) $u = \sin(x) \Rightarrow du = \cos(x)\, dx$ and $e^{\sin(x)} = e^u$:

$$\int \cos(x) e^{\sin(x)}\, dx = \int e^u\, du = e^u + C = e^{\sin(x)} + C$$

(b) $u = 5 + x^2 \Rightarrow du = 2x\, dx$, so:

$$\int \frac{2x}{5+x^2}\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln\left|5+x^2\right| + C$$

Because $5 + x^2 > 0$, we can also write the answer as $\ln\left(5 + x^2\right)$. ◀

In each example, the change of variable did not find the antiderivative, but it did make the pattern of the integrand more obvious, which in turn made it easier to determine an antiderivative.

Practice 3. Make the suggested change of variable, rewrite each integral in terms of u and du and evaluate the integral.

(a) $\int (7x+5)^3\, dx \quad \text{with} \quad u = 7x + 5$

(b) $\int 3x^2 \cdot \sin\left(x^3 - 1\right)\, dx \quad \text{with} \quad u = x^3 - 1$

The previous examples have supplied a suggested substitution, but in the future *you* will need to decide what u should equal. Unfortunately there are no rules that guarantee your choice will lead to an easier integral—sometimes you will need to resort to trial and error until you find a particular u-substitution that works for your integrand. There is, however, a "rule of thumb" that frequently results in easier integrals. Even though the following suggestion comes with no guarantees, it is often worth trying.

A "Rule of Thumb" for Changing the Variable

If part of the integrand consists of a composition of functions, $f(g(x))$, try setting $u = g(x)$, the "inner" function.

The key to becoming skilled at selecting a good u and correctly making the substitution is **practice**.

If part of the integrand is being raised to a power, try setting u equal to the part being raised to the power. For example, if the integrand includes $(3+\sin(x))^5$, try $u = 3+\sin(x)$. If part of the integrand involves a trigonometric (or exponential or logarithmic) function of another function, try setting u equal to the "inside" function: if the integrand includes the function $\sin\left(3+x^2\right)$, try $u = 3+x^2$.

Example 4. Select a u for each integrand and rewrite the associated integral in terms of u and du.

(a) $\int \cos(3x)\sqrt{2+\sin(3x)}\,dx$ (b) $\int \frac{5e^x}{2+e^x}\,dx$ (c) $\int e^x \cdot \sin\left(e^x\right)\,dx$

Solution. (a) If $u = 2+\sin(3x)$, $du = 3\cos(3x)\,dx \Rightarrow \frac{1}{3}\,du = \cos(3x)\,dx$ so the integral becomes $\int \frac{1}{3}\sqrt{u}\,du$. (b) With $u = 2+e^x \Rightarrow du = e^x\,dx$, the integral becomes $\int \frac{5}{u}\,du$. (c) With $u = e^x \Rightarrow du = e^x\,dx$, the integral becomes $\int \sin(u)\,du$. ◀

Changing Variables with Definite Integrals

If we need to change variables in a *definite* integral, we have two choices:

- First work out the corresponding *indefinite* integral and then use that antiderivative and FTC2 to evaluate the definite integral.
- Change variables in the definite integral, which requires changing the limits of integration from x limits to u limits.

For the second option, if the original integral had endpoints $x = a$ and $x = b$, and we make the substitution $u = g(x) \Rightarrow du = g'(x)\,dx$, then the new integral will have endpoints $u = g(a)$ and $u = g(b)$:

$$\int_{x=a}^{x=b} f(g(x)) \cdot g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

Example 5. Evaluate $\int_0^1 (3x-1)^4\,dx$.

Solution. Using the first option with $u = 3x-1 \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$, the corresponding indefinite integral becomes:

$$\int (3x-1)^4\,dx = \int u^4 \cdot \frac{1}{3}\,du = \frac{1}{3} \cdot \frac{1}{5}u^5 + C = \frac{1}{15}(3x-1)^5 + C$$

We now use this result to evaluate the original definite integral:

$$\int_0^1 (3x-1)^4\,dx = \left[\frac{1}{15}(3x-1)^5\right]_0^1 = \left[\frac{1}{15}\cdot 2^5\right] - \left[\frac{1}{15}\cdot(-1)^5\right]$$
$$= \frac{32}{15} - \frac{-1}{15} = \frac{33}{15} = \frac{11}{5}$$

For the second option, we make the same substitution $u = 3x - 1 \Rightarrow \frac{1}{3}\,du = dx$ while also computing $x = 0 \Rightarrow u = 3\cdot 0 - 1 = -1$ and $x = 1 \Rightarrow u = 3\cdot 1 - 1 = 2$:

Both options require roughly the same amount of work and computation. In practice you should choose the option that seems easiest for you and poses the least risk of error.

$$\int_{x=0}^{x=1} (3x-1)^4\,dx = \int_{u=-1}^{u=2} \frac{1}{3}u^4\,du = \frac{1}{3}\cdot\frac{1}{5}u^5\Big|_{-1}^{2} = \frac{2^5}{15} - \frac{(-1)^5}{15} = \frac{33}{15}$$

We arrive at the same answer either way. ◀

Practice 4. If the original integrals in Example 4 had endpoints (a) $x = 0$ to $x = \pi$ (b) $x = 0$ to $x = 2$ or (c) $x = 0$ to $x = \ln(3)$, then the new integrals should have what endpoints?

Special Transformations for $\int \sin^2(x)\,dx$ *and* $\int \cos^2(x)\,dx$

The integrals of $\sin^2(x)$ and $\cos^2(x)$ arise often, and we can find their antiderivatives with the help of some trigonometric identities. Solving the first identity in the margin for $\sin^2(x)$, we get:

$$\cos(2x) = 1 - 2\sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$
$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

and solving the second identity for $\cos^2(x)$, we get:

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

Integrating the first of these new identities yields:

$$\int \sin^2(x)\,dx = \int \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

Using the identity $\sin(2x) = 2\sin(x)\cos(x)$, we can also write:

$$\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin(x)\cos(x) + C$$

In practice, it's easier to remember the new trig identities and use them to work out these antiderivatives, rather than memorizing the antiderivatives directly.

Similarly, using $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ yields:

$$\int \cos^2(x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C = \frac{1}{2}x + \frac{1}{2}\sin(x)\cos(x) + C$$

4.6 Problems

For Problems 1–4, put $f(x) = x^2$ and $g(x) = x$ to verify the inequality.

1. $\int_1^2 f(x) \cdot g(x)\, dx \neq \left(\int_1^2 f(x)\, dx\right)\left(\int_1^2 g(x)\, dx\right)$
2. $\int_1^2 \frac{f(x)}{g(x)}\, dx \neq \frac{\int_1^2 f(x)\, dx}{\int_1^2 g(x)\, dx}$
3. $\int_0^1 f(x) \cdot g(x)\, dx \neq \left(\int_0^1 f(x)\, dx\right)\left(\int_0^1 g(x)\, dx\right)$
4. $\int_1^4 \frac{f(x)}{g(x)}\, dx \neq \frac{\int_1^4 f(x)\, dx}{\int_1^4 g(x)\, dx}$

For 5–14, use the suggested u to find du and rewrite the integral in terms of u and du. Then find an antiderivative in terms of u and, finally, rewrite your answer in terms of x.

5. $\int \cos(3x)\, dx, \quad u = 3x$
6. $\int \sin(7x)\, dx, \quad u = 7x$
7. $\int e^x \sin(2+e^x)\, dx, \quad u = 2 + e^x$
8. $\int e^{5x}\, dx, \quad u = 5x$
9. $\int \cos(x) \sec^2(\sin(x))\, dx, \quad u = \sin(x)$
10. $\int \frac{\cos(x)}{\sin(x)}\, dx, \quad u = \sin(x)$
11. $\int \frac{5}{3+2x}\, dx, \quad u = 3 + 2x$
12. $\int x^2 \left(5 + x^3\right)^7 dx, \quad u = 5 + x^3$
13. $\int x^2 \sin\left(1 + x^3\right) dx, \quad u = 1 + x^3$
14. $\int \frac{e^x}{1+e^x}\, dx, \quad u = 1 + e^x$

For 15–26, use the change-of-variable technique to evaluate the indefinite integral.

15. $\int \cos(4x)\, dx$
16. $\int e^{3x}\, dx$
17. $\int x^3 \left(5 + x^4\right)^{11} dx$
18. $\int x \cdot \sin\left(x^2\right) dx$
19. $\int \frac{3x^2}{2+x^3}\, dx$
20. $\int \frac{\sin(x)}{\cos(x)}\, dx$
21. $\int \frac{\ln(x)}{x}\, dx$
22. $\int x\sqrt{1+x^2}\, dx$
23. $\int (1+3x)^7\, dx$
24. $\int \frac{1}{x} \cdot \sin(\ln(x))\, dx$
25. $\int e^x \cdot \sec\left(e^x\right) \cdot \tan\left(e^x\right) dx$
26. $\int \frac{1}{\sqrt{x}} \cos\left(\sqrt{x}\right) dx$

In 27–42, evaluate the integral.

27. $\int_0^{\frac{\pi}{2}} \cos(3x)\, dx$
28. $\int_0^{\pi} \cos(4x)\, dx$
29. $\int_0^1 e^x \cdot \sin\left(2 + e^x\right) dx$
30. $\int_0^1 e^{5x}\, dx$
31. $\int_{-1}^1 x^2 \left(1 + x^3\right)^5 dx$
32. $\int_0^1 x^4 \left(x^5 - 1\right)^{10} dx$
33. $\int_0^2 \frac{5}{3+2x}\, dx$
34. $\int_0^{\ln(3)} \frac{e^x}{1+e^x}\, dx$
35. $\int_0^1 x\sqrt{1-x^2}\, dx$
36. $\int_2^5 \frac{2}{1+x}\, dx$
37. $\int_0^1 \sqrt{1+3x}\, dx$
38. $\int_0^1 \frac{1}{\sqrt{1+3x}}\, dx$
39. $\int \sin^2(5x)\, dx$
40. $\int \cos^2(3x)\, dx$
41. $\int \left[\frac{1}{2} - \sin^2(x)\right] dx$
42. $\int \left[e^x + \sin^2(x)\right] dx$
43. Find the area under one arch of $y = \sin^2(x)$.
44. Evaluate $\int_0^{2\pi} \sin^2(x)\, dx$.

In 45–53, expand the integrand first.

45. $\int \left(x^2 + 1\right)^3 dx$
46. $\int \left(x^3 + 5\right)^2 dx$
47. $\int \left(e^x + 1\right)^2 dx$
48. $\int \left(x^2 + 3x - 2\right)^2 dx$
49. $\int (x^2+1)(x^3+5)\, dx$
50. $\int (7 + \sin(x))^2\, dx$
51. $\int e^x \left(e^x + e^{3x}\right) dx$
52. $\int (2 + \sin(x)) \sin(x)\, dx$
53. $\int \sqrt{x} \left(x^2 + 3x - 2\right) dx$

In 54–64, divide, then find an antiderivative.

54. $\int \frac{x+1}{x}\, dx$

55. $\int \frac{3x}{x+1}\, dx$

56. $\int \frac{x-1}{x+2}\, dx$

57. $\int \frac{x^2-1}{x+1}\, dx$

58. $\int \frac{2x^2-13x+15}{x-1}\, dx$

59. $\int \frac{2x^2-13x+18}{x-1}\, dx$

60. $\int \frac{2x^2-13x+11}{x-1}\, dx$

61. $\int \frac{x+2}{x-1}\, dx$

62. $\int \frac{e^x+e^{3x}}{e^x}\, dx$

63. $\int \frac{x+4}{\sqrt{x}}\, dx$

64. $\int \frac{\sqrt{x}+3}{x}\, dx$

The definite integrals in 65–70 involve areas associated with parts of circles; use your knowledge of circles and their areas to evaluate them. (Suggestion: Sketch a graph of the integrand function.)

65. $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

66. $\int_{0}^{1} \sqrt{1-x^2}\, dx$

67. $\int_{-3}^{3} \sqrt{9-x^2}\, dx$

68. $\int_{-4}^{0} \sqrt{16-x^2}\, dx$

69. $\int_{-1}^{1} \left[2+\sqrt{1-x^2}\right]\, dx$

70. $\int_{0}^{2} \left[3-\sqrt{1-x^2}\right]\, dx$

4.6 Practice Answers

1. $\mathbf{D}\left(\tan^2(x)+C\right) = 2\tan^1(x)\cdot\mathbf{D}\left(\tan(x)\right) = 2\tan(x)\sec^2(x)$
$\mathbf{D}\left(\sec^2(x)+C\right) = 2\sec^1(x)\cdot\mathbf{D}\left(\sec(x)\right) = 2\sec(x)\cdot\sec(x)\tan(x)$

2. We know $\mathbf{D}\left(\tan(x)\right) = \sec^2(x)$, so it is reasonable to try $\tan(7x)$: $\mathbf{D}\left(\tan(7x)\right) = \sec^2(7x)\cdot\mathbf{D}(7x) = 7\sec^2(7x)$, a result seven times the result we want, so divide the original "guess" by 7 and try again:

$$\mathbf{D}\left(\frac{1}{7}\tan(7x)\right) = \frac{1}{7}\sec^2(7x)\cdot 7 = \sec^2(7x)$$

so $\int \sec^2(7x)\, dx = \frac{1}{7}\tan(7x)+C$.
$\mathbf{D}\left((3x+8)^{\frac{1}{2}}\right) = \frac{1}{2}(3x+8)^{-\frac{1}{2}}\,\mathbf{D}(3x+8) = \frac{3}{2}(3x+8)^{-\frac{1}{2}}$ so multiply our original "guess" by $\frac{2}{3}$:

$$\mathbf{D}\left(\frac{2}{3}(3x+8)^{\frac{1}{2}}\right) = \frac{2}{3}\cdot\frac{1}{2}\cdot(3x+8)^{-\frac{1}{2}}\cdot\mathbf{D}(3x+8) = \frac{2}{3}\cdot\frac{3}{2}\cdot\frac{1}{\sqrt{3x+8}}$$

hence $\int \frac{1}{\sqrt{3x+8}}\, dx = \frac{2}{3}\sqrt{3x+8}+C$.

3. (a) $u = 7x+5 \Rightarrow du = 7\, dx \Rightarrow dx = \frac{1}{7}\, du$ so:

$$\int (7x+5)^3\, dx = \int u^3\cdot\frac{1}{7}\, du = \frac{1}{7}\cdot\frac{1}{4}u^4+C = \frac{1}{28}(7x+5)^4+C$$

(b) $u = x^3-1 \Rightarrow du = 3x^2\, dx$ so $\int \sin(x^3-1)\cdot 3x^2\, dx$ becomes:

$$\int \sin(u)\, du = -\cos(u)+C = -\cos\left(x^3-1\right)+C$$

4. (a) $u = 2+\sin(3x)$ so $x=0 \Rightarrow u = 2+\sin(3\cdot 0) = 2$ and $x=\pi \Rightarrow u = 2+\sin(3\pi) = 2$. (This integral is now easy; why?)
(b) $u = 2+e^x$ so $x = 0 \Rightarrow u = 2+e^0 = 3$ and $x=2 \Rightarrow u = 2+e^2$
(c) $u = e^x$ so $x = 0 \Rightarrow u = e^0 = 1$ and $x = \ln(3) \Rightarrow u = e^{\ln(3)} = 3$