4.8 Using Tables (and Technology) to Find Antiderivatives

Appendix I shows patterns for many antiderivatives — some of which you should already know based on your work in this chapter. Many reference books and Web sites contain far more than the ones listed in the appendix. A table of integrals helps you while you are learning calculus and serves as a reference later when you are using calculus.

Think of an integral table as a dictionary: something to use when you need to spell a challenging word or need the meaning of a new word. It would be difficult to write a report if you had to look up the spelling of *every* word, and it will be difficult to learn and use calculus if you have to look up every antiderivative. Tables of antiderivatives are limited by necessity and often take longer to use than finding an antiderivative from scratch, but they can also be very valuable and useful.

This section shows how to transform some integrals into forms found in Appendix I and how to use "recursion" formulas found in integral tables. The first Examples and Practice problems illustrate some of the techniques used to change an integral into a standard form.

These techniques are useful whether that standard form resides in a table or in your head.

Example 1. Use Appendix I to find $\int \frac{1}{9+x^2}\,dx$.

Solution. The integrand is a rational function, and the first entry you see listed in the "Rational Functions" section of Appendix I should be:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

This resembles the pattern we need, so replacing the a with 3 we have:

$$\int \frac{1}{9+x^2}\,dx = \int \frac{1}{3^2+x^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

You can (and should) check this answer by differentiating. ◀

Appendix I (like some other integral tables) omits the "$+C$" arbitrary constant for conciseness, but you need to remember to include it when using the results of the table to find an indefinite integral.

Practice 1. Use Appendix I to find $\int \frac{1}{25+x^2}\,dx$ and $\int \frac{1}{25-x^2}\,dx$.

Notice that a small change in the form of the integrand (from $+$ to $-$ here) can lead to a very different result.

Example 2. Use Appendix I to find $\int \frac{1}{5+x^2}\,dx$.

Solution. The integrand is again a rational function, and the general form is the same as in the previous Example:

$$\int \frac{1}{5+x^2}\,dx = \int \frac{1}{(\sqrt{5})^2+x^2}\,dx = \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

but here we needed to put $a = \sqrt{5}$. ◀

The constant in the denominator of this integrand was not a perfect square, but the process is exactly the same — even if the result looks a bit "messier" due to the presence of the radical.

Practice 2. Use Appendix I to find $\int \frac{1}{7+x^2}\,dx$ and $\int \frac{1}{7-x^2}\,dx$.

We often need to perform some algebraic manipulations to change an integrand into one that exactly matches a pattern in the table.

Example 3. Use Appendix I to find $\int \frac{1}{9+4x^2}\,dx$.

Solution. The integrand is again a rational function, and the general form resembles the one used in the previous Examples, but here we have a $4x^2$ where we only see x^2 in the table pattern. To get the integrand in the form we want, we can factor a 4 out of the denominator:

$$\begin{aligned}\int \frac{1}{9+4x^2}\,dx &= \int \frac{1}{4\left(\frac{9}{4}+x^2\right)}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{3}{2}\right)^2+x^2}\,dx \\ &= \frac{1}{4}\cdot\frac{1}{\frac{3}{2}}\cdot\arctan\left(\frac{x}{\frac{3}{2}}\right)+C = \frac{1}{6}\arctan\left(\frac{2x}{3}\right)+C\end{aligned}$$

Another approach involves a change of variable. First write:

$$\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{3^2+(2x)^2}\,dx$$

We have $2x$ where we would like to see a standalone variable. To get that pattern, put $u = 2x \Rightarrow du = 2\,dx \Rightarrow dx = \frac{1}{2}\,du$:

$$\begin{aligned}\int \frac{1}{3^2+(2x)^2}\,dx &= \int \frac{1}{3^2+u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right)+C \\ &= \frac{1}{6}\arctan\left(\frac{2x}{3}\right)+C\end{aligned}$$

which yields the same result as our previous method. ◀

Practice 3. Use Appendix I to find $\int \frac{1}{25+9x^2}\,dx$ and $\int \frac{1}{25-9x^2}\,dx$.

Sometimes a change of variable is absolutely necessary.

Example 4. Use Appendix I to find $\int \frac{e^x}{9+e^{2x}}\,dx$.

Solution. Here the integrand is *not* a rational function, but we can transform it into one by using the substitution $u = e^x \Rightarrow du = e^x\,dx$ so that $u^2 = (e^x)^2 = e^{2x}$:

$$\begin{aligned}\int \frac{e^x}{9+e^{2x}}\,dx &= \int \frac{1}{3^2+(e^x)^2}\cdot e^x\,dx = \int \frac{1}{3^2+u^2}\,du \\ &= \frac{1}{3}\arctan\left(\frac{u}{3}\right)+C = \frac{1}{3}\arctan\left(\frac{e^x}{3}\right)+C\end{aligned}$$

If you don't see the exact pattern you need in an integral table, try a substitution first. ◀

Practice 4. Evaluate $\int \frac{\cos(x)}{25+\sin^2(x)}\,dx$ and $\int \frac{\cos(x)}{25-\sin^2(x)}\,dx$.

How should you recognize whether algebra or a change of variable is needed? Experience and practice, practice, practice.

Using "Recursion" Formulas

A **recursion formula** gives one antiderivative in terms of another antiderivative. Usually the new antiderivative is somehow simpler than the original one. For example, the first recursion formula for a trigonometric function listed in Appendix I states:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

We will develop this formula from scratch in Problem 25 of Section 8.2. For now, you can check that it works by comparing the derivative of your answer to the original integrand for an integration problem that uses this — or any other — recursion formula.

This formula would allow us to write $\int \sin^8(x)\,dx$, for instance, in terms of $\int \sin^6(x)\,dx$, which should (theoretically, at least) be easier to compute than the original integral.

Example 5. Use a recursion formula to evaluate $\int \sin^4(x)\,dx$.

Solution. Applying the formula given in the discussion above:

$$\int \sin^4(x)\,dx = -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx$$

This new integral is one we already know how to evaluate:

$$\int \sin^2(x)\,dx = \int \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + K$$

Putting this together with the result of the recursion formula, we get:

$$\begin{aligned}\int \sin^4(x)\,dx &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right] + C \\ &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{8}x - \frac{3}{16}\sin(2x) + C\end{aligned}$$

We could have included the "$+K$" here but then the result at the next stage would have included the constant terms

$$\cdots + \frac{3}{4}K + C$$

which is also an arbitrary constant.

We could have used Appendix I to find $\int \sin^2(x)\,dx$ instead — or even applied the recursion formula a second time to rewrite $\int \sin^2(x)\,dx$ in terms of $\int \sin^0(x)\,dx = \int 1\,dx$. ◀

Practice 5. Use Appendix I to evaluate $\int \cos^4(x)\,dx$ and $\int \cos^4(7x)\,dx$.

Using Technology

Many Web sites (such as Wolfram|Alpha, `www.wolframalpha.com`), computer programs (wxMaxima is a good free one) and calculators (such as the TI-89 or TI-Nspire CAS) feature computer algebra systems that can find antiderivatives of a wide variety of functions. For example, typing `integral sin^4(x)` into Wolfram|Alpha yields:

$$\int \sin^4(x)\,dx = \frac{1}{32}\,(12\,x - 8\sin(2\,x) + \sin(4\,x)) + \text{constant}$$

which (applying some trig idenities) agrees with our result above.

Although technology can help us find an antiderivative and evaluate a definite integral, in an application problem **you** still need to set up the Riemann sum that leads to the definite integral.

4.8 Problems

In Problems 1–48, use patterns and recursion formulas from the integral table in Appendix I as necessary (along with any other antiderivatives and integration techniques you already know) to evaluate each integral.

1. $\displaystyle\int \frac{1}{4+x^2}\,dx$
2. $\displaystyle\int \frac{5}{4+x^2}\,dx$
3. $\displaystyle\int \left[2x+\frac{2}{25+x^2}\right]dx$
4. $\displaystyle\int \frac{1}{4-x^2}\,dx$
5. $\displaystyle\int \frac{2}{9-x^2}\,dx$
6. $\displaystyle\int \left[\cos(x)+\frac{3}{25-x^2}\right]dx$
7. $\displaystyle\int \frac{1}{3+x^2}\,dx$
8. $\displaystyle\int \frac{5}{7+x^2}\,dx$
9. $\displaystyle\int \left[e^x+\frac{7}{2+x^2}\right]dx$
10. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$
11. $\displaystyle\int \frac{3}{\sqrt{5-x^2}}\,dx$
12. $\displaystyle\int \frac{3}{\sqrt{4-x^2}}\,dx$
13. $\displaystyle\int \frac{1}{4+25x^2}\,dx$
14. $\displaystyle\int \frac{2}{\sqrt{9-16x^2}}\,dx$
15. $\displaystyle\int \frac{5}{\sqrt{1-4x^2}}\,dx$
16. $\displaystyle\int \sec(x+5)\,dx$
17. $\displaystyle\int \frac{2}{\sqrt{1+9x^2}}\,dx$
18. $\displaystyle\int x\cdot\sec(2x^2+7)\,dx$
19. $\displaystyle\int \ln(x+1)\,dx$
20. $\displaystyle\int \ln(3x-1)\,dx$
21. $\displaystyle\int 3x\cdot\ln(5x^2+7)\,dx$
22. $\displaystyle\int e^x\ln\left(e^x-3\right)\,dx$
23. $\displaystyle\int \cos(x)\cdot\ln\left(\sin(x)\right)\,dx$
24. $\displaystyle\int \frac{2}{\sqrt{x^2-9}}\,dx$
25. $\displaystyle\int \sqrt{4+x^2}\,dx$
26. $\displaystyle\int \sqrt{9+x^2}\,dx$
27. $\displaystyle\int \sqrt{16+x^2}\,dx$
28. $\displaystyle\int_0^1 \frac{1}{4+x^2}\,dx$
29. $\displaystyle\int_1^3 \left[2x+\frac{2}{25+x^2}\right]dx$
30. $\displaystyle\int_0^2 \frac{2}{9-x^2}\,dx$
31. $\displaystyle\int_{-1}^1 \frac{1}{3+x^2}\,dx$
32. $\displaystyle\int_0^1 \left[e^x+\frac{7}{2+x^2}\right]dx$
33. $\displaystyle\int_1^2 \frac{3}{\sqrt{5-x^2}}\,dx$
34. $\displaystyle\int_0^1 \frac{1}{4+25x^2}\,dx$
35. $\displaystyle\int_0^{0.1} \frac{5}{\sqrt{1-4x^2}}\,dx$
36. $\displaystyle\int_0^1 \frac{1}{\sqrt{9-4x^2}}\,dx$
37. $\displaystyle\int_0^6 \ln(x+1)\,dx$
38. $\displaystyle\int_0^3 3x\cdot\ln(5x^2+7)\,dx$
39. $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\cdot\ln\left(2+\sin(x)\right)\,dx$
40. $\displaystyle\int_0^2 \sqrt{4+x^2}\,dx$
41. $\displaystyle\int_{-3}^3 \sqrt{9+x^2}\,dx$
42. $\displaystyle\int_0^1 \sqrt{16+x^2}\,dx$
43. $\displaystyle\int \sin^3(x)\,dx$
44. $\displaystyle\int \cos^3(x)\,dx$
45. $\displaystyle\int \cos^5(x)\,dx$
46. $\displaystyle\int \sec^5(x)\,dx$
47. $\displaystyle\int x^2\cos(x)\,dx$
48. $\displaystyle\int x^2\sin^5(x)\,dx$

49. Before doing any calculations, predict which you expect to be larger:
    - the average value of $\sin(x)$ on $[0, \pi]$
    - the average value of $\sin^2(x)$ on $[0, \pi]$

    Then calculate each average to see if your prediction was correct.
50. Find the area of the region bounded by the graph of $f(x) = \ln(x)$, the x-axis and the lines $x = 1$ and $x = C$ when $C = e$, 10, 100 and 200.
51. Find the average value of $f(x) = \ln(x)$ on the interval $1 \leq x \leq C$ when $C = e$, 10, 100, 200.
52. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

    (a) $\displaystyle\int_0^1 e^x\,dx$ (b) $\displaystyle\int_0^1 xe^x\,dx$ (c) $\displaystyle\int_0^1 x^2e^x\,dx$
53. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

    (a) $\displaystyle\int_1^2 e^x\,dx$ (b) $\displaystyle\int_1^2 xe^x\,dx$ (c) $\displaystyle\int_1^2 x^2e^x\,dx$
54. Before doing any calculations, predict which of the following integrals you expect to be the largest, then evaluate each integral.

    (a) $\displaystyle\int_0^\pi \sin(x)\,dx$

    (b) $\displaystyle\int_0^\pi x\sin(x)\,dx$

    (c) $\displaystyle\int_0^\pi x^2\sin(x)\,dx$
55. Evaluate $\displaystyle\int_0^C \frac{2}{1+x^2}\,dx$ for $C = 1$, 10, 20 and 30. Before doing the calculation, estimate the value of the integral when $C = 40$.

4.8 Practice Answers

1. The integral $\displaystyle\int \frac{1}{25+x^2}\,dx$ resembles the pattern from Example 1:

$$\int \frac{1}{25+x^2}\,dx = \int \frac{1}{5^2+x^2}\,dx = \frac{1}{5}\arctan\left(\frac{x}{5}\right) + C$$

The integrand in $\displaystyle\int \frac{1}{25-x^2}\,dx$ is also a rational function, but we need a different pattern from Appendix I (see margin) with $a = 5$:

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|$$

$$\int \frac{1}{25-x^2}\,dx = \int \frac{1}{5^2-x^2}\,dx = \frac{1}{10}\ln\left|\frac{x+5}{x-5}\right| + C$$

2. The integral $\displaystyle\int \frac{1}{7+x^2}\,dx$ matches the pattern in Example 2:

$$\int \frac{1}{7+x^2}\,dx = \int \frac{1}{(\sqrt{7})^2+x^2}\,dx = \frac{1}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right) + C$$

For $\displaystyle\int \frac{1}{7-x^2}\,dx$ we need the pattern in the margin with $a = \sqrt{7}$:

$$\int \frac{1}{7-x^2}\,dx = \int \frac{1}{(\sqrt{7})^2-x^2}\,dx = \frac{1}{2\sqrt{7}}\ln\left|\frac{x+\sqrt{7}}{x-\sqrt{7}}\right| + C$$

3. For the integral $\int \frac{1}{25+9x^2}\,dx$ we can factor 9 from the denominator:

$$\int \frac{1}{25+9x^2}\,dx = \int \frac{1}{9\left(\frac{25}{9}+x^2\right)}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{5}{3}\right)^2+x^2}\,dx$$
$$= \frac{1}{9}\cdot\frac{1}{\frac{5}{3}}\arctan\left(\frac{x}{\frac{5}{3}}\right)+C = \frac{1}{15}\arctan\left(\frac{3x}{5}\right)+C$$

and proceed as before. We could proceed similarly for $\int \frac{1}{25-9x^2}\,dx$ or we could substitute $u = 3x$ (see margin):

$u = 3x \Rightarrow du = 3\,dx \Rightarrow dx = \frac{1}{3}\,du$

$$\int \frac{1}{25-9x^2}\,dx = \int \frac{1}{5^2-(3x)^2}\,dx = \int \frac{1}{5^2-u^2}\cdot\frac{1}{3}\,du$$
$$= \frac{1}{3}\cdot\frac{1}{2\cdot 5}\ln\left|\frac{u+5}{u-5}\right|+C = \frac{1}{30}\cdot\ln\left|\frac{3x+5}{3x-5}\right|+C$$

4. For $\int \frac{\cos(x)}{25+\sin^2(x)}\,dx$, first use the substitution in the margin:

$u = \sin(x) \Rightarrow du = \cos(x)\,dx$

$$\int \frac{\cos(x)}{25+\sin^2(x)}\,dx = \int \frac{1}{25+u^2}\,du$$

followed by the result of the first part of Practice 1:

$$\int \frac{1}{25+u^2}\,du = \frac{1}{5}\arctan\left(\frac{u}{5}\right)+C = \frac{1}{5}\arctan\left(\frac{\sin(x)}{5}\right)+C$$

For $\int \frac{\cos(x)}{25-\sin^2(x)}\,dx$ use the same substitution, followed by the result from the second part of Practice 1:

$$\int \frac{1}{25-u^2}\,du = \frac{1}{10}\ln\left|\frac{u+5}{u-5}\right|+C = \frac{1}{10}\ln\left|\frac{\sin(x)+5}{\sin(x)-5}\right|+C$$

5. For $\int \cos^4(x)\,dx$ we need the recursion formula:

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x)+\frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

with $n = 4$:

$$\int \cos^4(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x)+\frac{3}{4}\int \cos^2(x)\,dx$$
$$= \frac{1}{4}\cos^3(x)\sin(x)+\frac{3}{4}\int\left[\frac{1}{2}+\frac{1}{2}\cos(2x)\right]dx$$
$$= \frac{1}{4}\cos^3(x)\sin(x)+\frac{3}{4}\left[\frac{1}{2}x+\frac{1}{4}\sin(2x)\right]+C$$

$u = 7x \Rightarrow du = 7\,dx \Rightarrow dx = \frac{1}{7}\,du$

$\int \cos^4(7x)\,dx = \frac{1}{7}\int \cos^4(u)\,du$

For $\int \cos^4(7x)\,dx$, first use a substitution (see margin) and then the result of the previous integration:

$$\int \cos^4(7x)\,dx = \frac{1}{28}\cos^3(7x)\sin(7x)+\frac{3}{28}\left[\frac{1}{2}(7x)+\frac{1}{4}\sin(14x)\right]+C$$