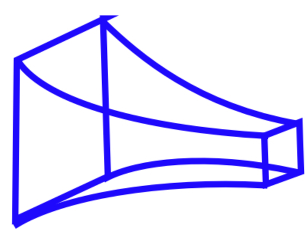
5
Applications of Definite Integrals

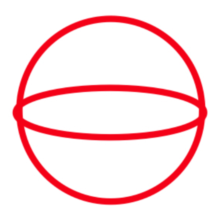
The previous chapter introduced the concepts of a definite integral as an "area" and as a limit of Riemann sums, demonstrated some of the properties of integrals, introduced some methods to compute values of definite integrals, and began to examine a few of their uses. This chapter focuses on several common applications of definite integrals.

An obvious goal of the chapter is to enable you to use integration when you encounter these particular applications later in mathematics or in other fields. A deeper goal is to illustrate the process of going from a problem to an integral, a process much broader than these particular applications. If you understand the process, then you can understand the use of integrals in many other fields and can even develop the integrals needed to solve problems in new areas. Another goal is to give you additional practice evaluating definite integrals.

Each section in this chapter follows the same basic format. First we describe a problem and present some background information. Then we approximate the solution to the basic problem using a Riemann sum. An exact answer comes from taking a limit of the Riemann sum, and we get a definite integral. After looking at several examples of the same basic application, we will examine some variations.



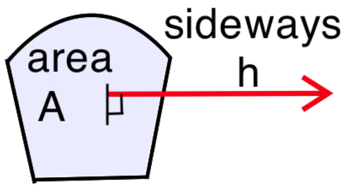
5.1 Volumes by Slicing



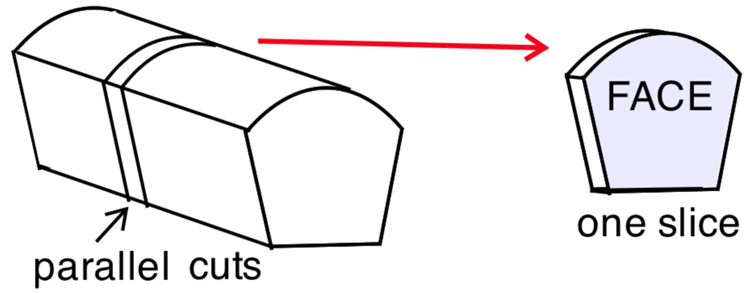
The previous chapter emphasized a geometric interpretation of definite integrals as "areas" in two dimensions. This section emphasizes another geometrical use of integration, computing volumes of solid three-dimensional objects such as those shown in the margin.

Our basic approach will involve cutting the whole solid into thin "slices" whose volumes we can approximate, adding the volumes of these "slices" together (to get a Riemann sum), and finally obtaining an exact answer by taking a limit of those sums to get a definite integral.

The Building Blocks: Right Solids



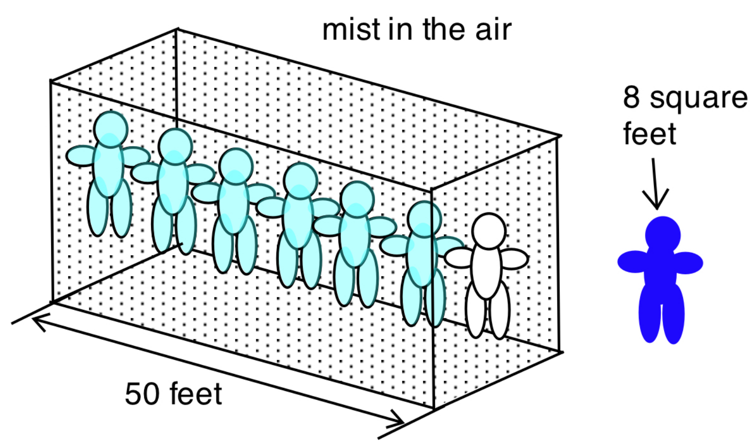
A **right solid** is a three-dimensional shape swept out by moving a planar region A some distance h along a line perpendicular to the plane of A (see margin). We call the region A a **face** of the solid and use the word "right" to indicate that the movement occurs along a line perpendicular—at a right angle—to the plane of A. Two parallel cuts produce one slice with two faces:



A slice has volume, and a face has area.

Example 1. Suppose a fine, uniform mist is suspended in the air and that every cubic foot of mist contains 0.02 ounces of water droplets. If you run 50 feet in a straight line through this mist, how wet do you get? Assume that the front (or a cross section) of your body has an area of 8 square feet.

Solution. As you run, the front of your body sweeps out a "tunnel" through the mist:

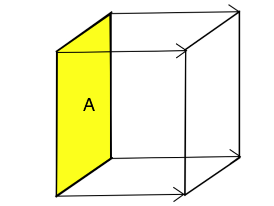


The volume of the "tunnel" is the area of the front of your body multiplied by the length of the tunnel:

$$\text{volume} = \left(8\ \text{ft}^2\right)(50\ \text{ft}) = 400\ \text{ft}^3$$

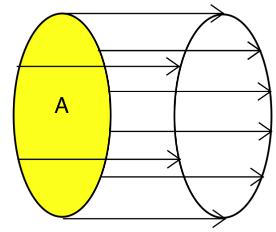
Because each cubic foot of mist held 0.02 ounces of water (which is now on you), you swept out a total of $\left(400\ \text{ft}^3\right)\left(0.02\ \frac{\text{oz}}{\text{ft}^3}\right) = 8$ ounces of water. If the water were truly suspended and not falling, would it matter how fast you ran? ◀

If A is a rectangle, then the "right solid" formed by moving A along a line (see margin) is a 3-dimensional solid box B. The volume of B is:



$$(\text{area of } A)\,(\text{distance along the line}) = (\text{base})\,(\text{height})\,(\text{width})$$

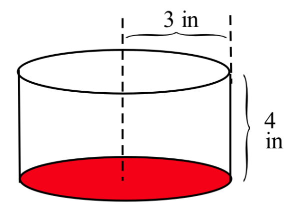
If A is a circle with radius r meters (see margin), then the "right solid" formed by moving A along a line a distance of h meters is a right circular cylinder with volume equal to:

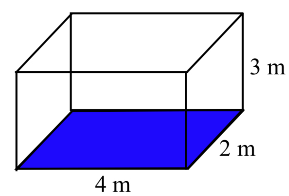


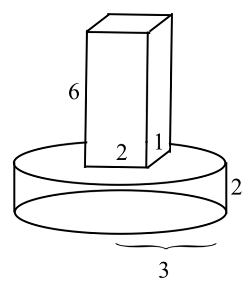
$$(\text{area of } A)\,(\text{distance along the line}) = \left[\pi\,(r\text{ ft})^2\right]\cdot[h\text{ ft}] = \pi r^2 h\text{ ft}^3$$

If we cut a right solid perpendicular to its axis (like slicing a block of cheese), then each face (cross-section) has the same two-dimensional shape and area. In general, if a 3-dimensional right solid B is formed by moving a 2-dimensional shape A along a line perpendicular to A, then the **volume** of B is defined to be:

$$(\text{area of } A)\cdot(\text{distance moved along the line perpendicular to } A)$$







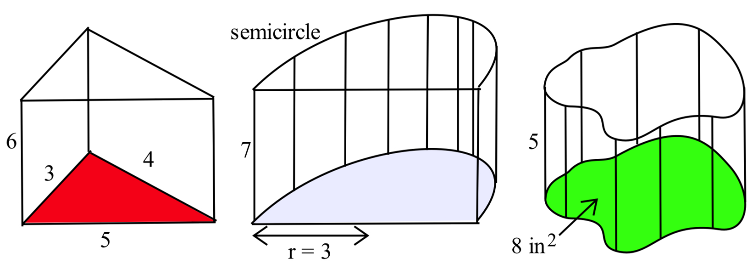
Example 2. Calculate the volumes of the right solids in the margin.

Solution. The cylinder is formed by moving the circular base with cross-sectional area $\pi r^2 = 9\pi\text{ in}^2$ a distance of 4 inches along a line perpendicular to the base, so the volume is $\left(9\pi\text{ in}^2\right)\cdot(4\text{ in}) = 36\pi\text{ in}^3$.

The volume of the box is $(\text{base area})\cdot(\text{distance base is moved}) = (8\text{ m}^2)\cdot(3\text{ m}) = 24\text{ m}^3$. We can also simply multiply "length times width times height" to get the same answer.

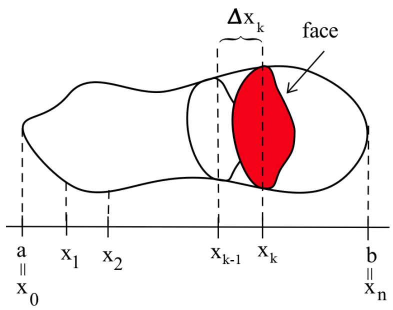
The last shape consists of two "easy" right solids with volumes $V_1 = \left(\pi\cdot 3^2\right)\cdot(2) = 18\pi\text{ cm}^3$ and $V_2 = (6)(1)(2) = 12\text{ cm}^3$, so the total volume is $(18\pi + 12)\text{ cm}^3 \approx 68.5\text{ cm}^3$. ◀

Practice 1. Calculate the volumes of the right solids shown below.



Volumes of General Solids

We can cut a general solid into "slices," each of which is "almost" a right solid if the cuts are close together. The volume of each slice will then be approximately equal to the volume of a right solid, so we can approximate the total volume of the entire solid by adding up the approximate volumes of the right-solid "slices."



First we position an x-axis below the solid shape (see margin) and let $A(t)$ be the area of the face formed when we cut the solid perpendicular to the x-axis where $x = t$. If $\mathcal{P} = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ is a partition of $[a, b]$ and we cut the solid at each x_k, then each slice of the solid is "almost" a right solid and the volume of each slice is approximately

$$\text{(area of a face of the slice) (thickness of the slice)} \approx A(x_k) \cdot \Delta x_k$$

The total volume V of the solid is approximately equal to the sum of the volumes of the slices:

$$V = \sum \text{(volume of each slice)} \approx \sum A(x_k) \cdot \Delta x_k$$

which is a Riemann sum.

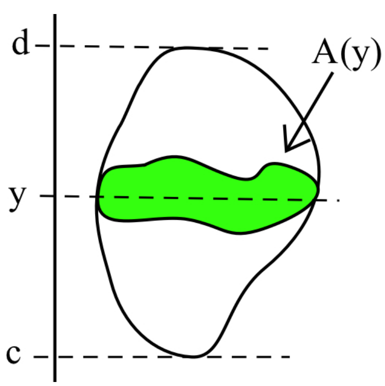
The limit, as the mesh of the partitions approaches 0 (taking thinner and thinner slices), of the Riemann sum is the definite integral of $A(x)$:

$$V \approx \sum A(x_k) \cdot \Delta x_k \longrightarrow \int_a^b A(x)\, dx$$

Volume by Slices Formula

If	S is a solid and $A(x)$ is the area of the face formed by a cut at x made perpendicular to the x-axis
then	the volume V of the part of S sitting above $[a, b]$ is:

$$V = \int_a^b A(x)\, dx$$

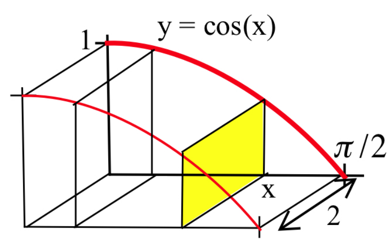


If S is a solid (see margin), and $A(y)$ is the area of a face formed by a cut at y perpendicular to the y-axis, then the volume of a slice with thickness Δy_k is approximately $A(y_k) \cdot \Delta y_k$. The volume of the part of S between cuts at $y = c$ and $y = d$ on the y-axis is therefore:

$$V = \int_c^d A(y)\, dy$$

Whether you slice a region with cuts perpendicular to the x-axis or cuts perpendicular to the y-axis depends on which slicing method results in slices with cross-sectional areas that are easiest to compute. Furthermore, slicing one way may result in a definite integral that is difficult to compute, while slicing the other way may result in a much easier definite integral (although you often can't tell which method will result in an easier integration process until you actually set up the integrals).

Example 3. For the solid shown in the margin, the cross-section formed by a cut at x is a rectangle with a base of 2 inches. (a) Find a formula for the approximate volume of the slice between x_{k-1} and x_k. (b) Compute the volume of the solid for x between 0 and $\frac{\pi}{2}$.



Solution. (a) The volume of a "slice" is approximately:

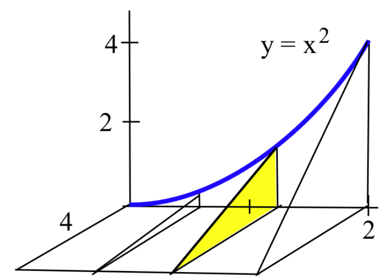
$$\begin{aligned}(\text{area of the face}) \cdot (\text{thickness}) &= (\text{base}) \cdot (\text{height}) \cdot (\text{thickness}) \\ &= (2 \text{ in})\,(\cos(x_k) \text{ in}) \cdot (\Delta x_k \text{ in}) \\ &= 2\cos(x_k)\Delta x_k \text{ in}^3\end{aligned}$$

(b) If we cut the solid into n slices of equal thickness Δx and add up the approximate volumes of the slices, we get a Riemann sum

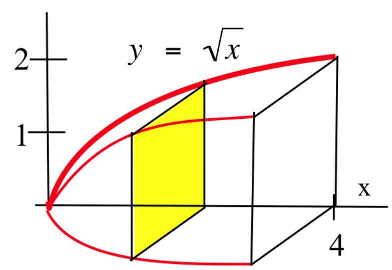
$$\sum_{k=1}^{n} 2\cos(x_k)\Delta x \longrightarrow \int_0^{\frac{\pi}{2}} 2\cos(x)\,dx = 2\sin(x)\Big|_0^{\frac{\pi}{2}} = 2$$

so the volume of the solid is 2 in^3. ◀

Practice 2. For the solid shown in the margin, the face formed by a cut at x is a triangle with a base of 4 inches. (a) Find a formula for the approximate volume of the slice between x_{k-1} and x_k. (b) Use a definite integral to compute the volume of the solid for x between 1 and 2.



Example 4. For the solid shown in margin, each face formed by a cut at x is a square. Compute the volume of the solid.



Solution. The volume of a "slice" is approximately:

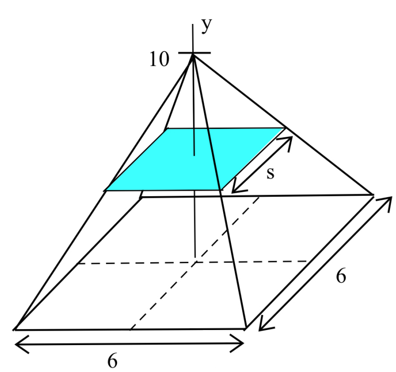
$$\begin{aligned}(\text{area of the face}) \cdot (\text{thickness}) &= (\text{base})^2 \cdot (\text{thickness}) \\ &= (\sqrt{x_k})^2 \cdot \Delta x_k = x_k \cdot \Delta x_k\end{aligned}$$

Adding up the approximate volumes of n slices, we get a Riemann sum that approximates the volume of the entire solid:

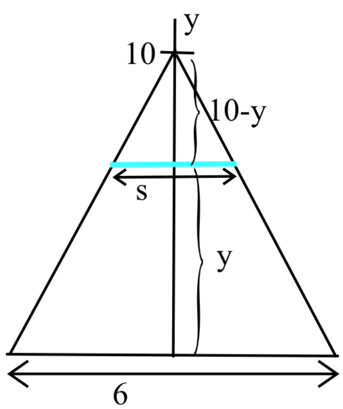
$$\sum_{k=1}^{n} x_k \cdot \Delta x_k \longrightarrow \int_0^4 x\,dx = \frac{1}{2}x^2\Big|_0^4 = 8$$

so the volume of the solid is 8. ◀

Example 5. Find the volume of the square-based pyramid shown in the margin.



Solution. Each cut perpendicular to the y-axis yields a square face, but in order to find the area of each square we need a formula for the



length of one side s of the square as a function of y, the location of the cut. Using similar triangles (see margin), we know that:

$$\frac{s}{10-y} = \frac{6}{10} \quad \Rightarrow \quad s = \frac{6}{10}(10-y) = 6 - \frac{3}{5}y$$

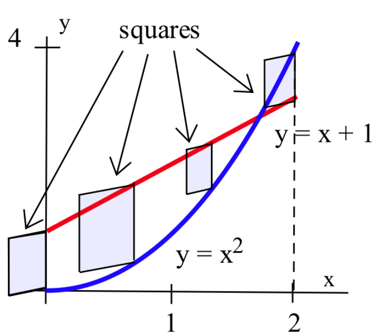
The rest of the solution is straightforward:

$$A(y) = (\text{side})^2 = \left[\frac{3}{5}(10-y)\right]^2 = \frac{9}{25}\left(100 - 20y + y^2\right)$$

so the volume of the solid is:

$$\begin{aligned} V &= \int_0^{10} A(y)\,dy = \int_0^{10} \frac{9}{25}\left(100-20y+y^2\right)\,dy \\ &= \frac{9}{25}\left[100y - 10y^2 + \frac{1}{3}y^3\right]_0^{10} \\ &= \frac{9}{25}\left[\left(1000 - 1000 + \frac{1000}{3}\right) - (0-0+0)\right] = 120 \end{aligned}$$

You may recall from geometry that the formula for the volume of a pyramid is $\frac{1}{3}Bh$ where B is the area of the base, which yields the same result as the definite integral: $\frac{1}{3}\left(6^2\right)(10) = 120$. ◀



Example 6. Form a solid with a base that is the region between the graphs of $f(x) = x+1$ and $g(x) = x^2$ for $0 \le x \le 2$ by building squares with heights (sides) equal to the vertical distance between the graphs of f and g (see margin). Find the volume of this solid.

Solution. The area of a square face is $A(x) = (\text{side})^2$ and the length of a side is either $f(x) - g(x)$ or $g(x) - f(x)$, depending on whether $f(x) \ge g(x)$ or $g(x) \ge f(x)$. We can express this side length as $|f(x) - g(x)|$ but the side length is squared in the area formula, so $A(x) = |f(x) - g(x)|^2 = (f(x) - g(x))^2$. Then:

$$\begin{aligned} V &= \int_a^b A(x)\,dx = \int_0^2 (f(x)-g(x))^2\,dx = \int_0^2 \left[(x+1) - x^2\right]^2\,dx \\ &= \int_0^2 \left[1 + 2x - x^2 - 2x^3 + x^4\right]\,dx \\ &= \left[x + x^2 - \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^2 \end{aligned}$$

which results in a volume of $\frac{26}{15}$. ◀

Wrap-Up

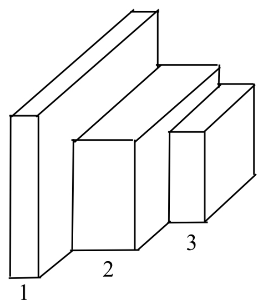
At first, all of these volumes may seem overwhelming — there are so many possible solids and formulas and different cases. If you concentrate on the differences, things can indeed seem very complicated.

Instead, focus on the pattern of cutting, finding areas of faces, volumes of slices, and adding those volumes. Then reason your way to a definite integral. Try to make cuts so the resulting faces have regular shapes (rectangles, triangles, circles) whose areas you can calculate easily. Try not to let the complexity of the whole solid confuse you. Sketch the shape of one face and label its dimensions. If you can find the area of one face in the middle of the solid, you can usually find the pattern for all of the faces — and then you can easily set up the integral.

5.1 Problems

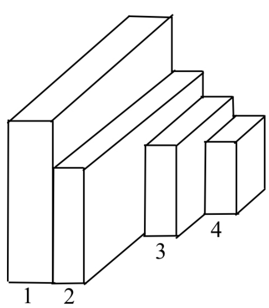
In Problems 1–5, compute the volume of the solid using the values provided in the table.

1.



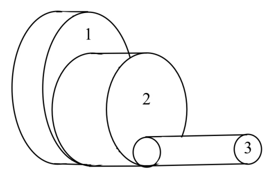
box	base	height	width
1	8	6	1
2	6	4	2
3	3	3	1

2.



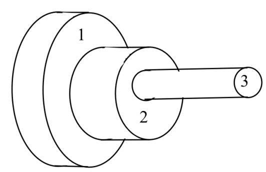
box	base	height	width
1	8	6	1
2	8	4	2
3	4	3	2
4	2	2	1

3.



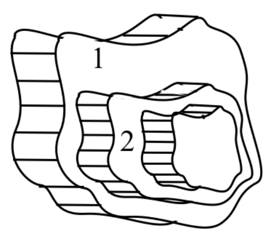
disk	radius	width
1	4	0.5
2	3	1.0
3	1	2.0

4.



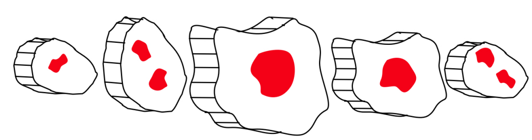
disk	diameter	width
1	8	0.5
2	6	1.0
3	2	2.0

5.



slice	face area	width
1	9	0.2
2	6	0.2
3	2	0.2

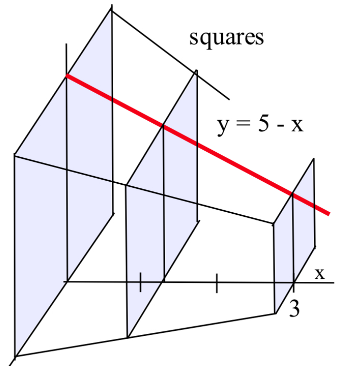
6. Five rock slices are embedded with mineral deposits. Use the information in the table to estimate the total rock volume.



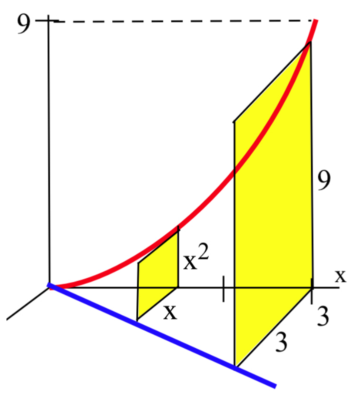
slice	face area	min. area	width
1	4	1	0.6
2	12	2	0.6
3	20	4	0.6
4	10	3	0.6
5	8	2	0.6

In Problems 7–12, represent the volume of each solid as a definite integral, then evaluate the integral.

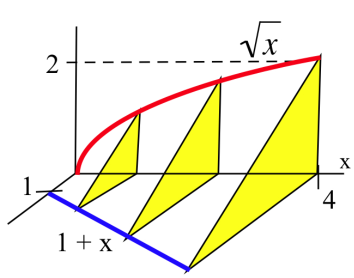
7. For $0 \le x \le 3$, each face is a square with height $5 - x$ inches.



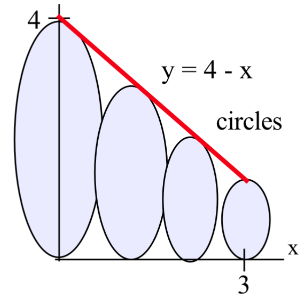
8. For $0 \le x \le 3$, each face is a rectangle with base x inches and height x^2 inches.



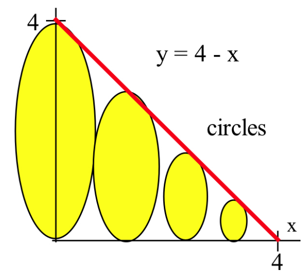
9. For $0 \le x \le 4$, each face is a triangle with base $x + 1$ m and height $\sqrt{x}$ m.



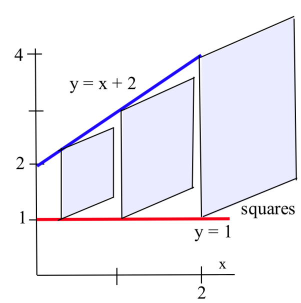
10. For $0 \le x \le 3$, each face is a circle with height (diameter) $4 - x$ m.



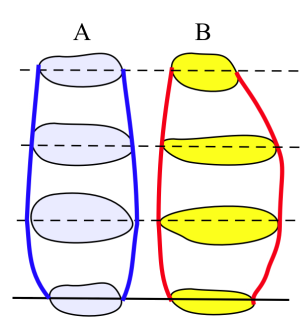
11. For $0 \le x \le 4$, each face is a circle with height (diameter) $4 - x$ m.



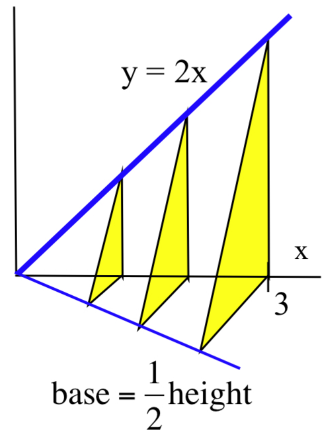
12. For $0 \le x \le 2$, each face is a square with a side extending from $y = 1$ to $y = x + 2$.

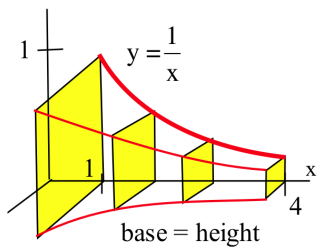


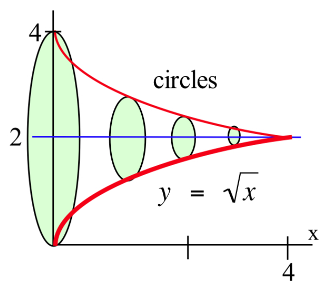
13. Suppose A and B are solids (see below) so that every horizontal cut produces faces of A and B that have equal areas. What can we conclude about the volumes of A and B? Justify your answer.

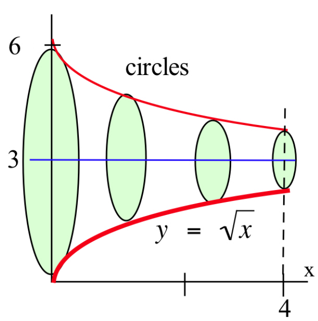


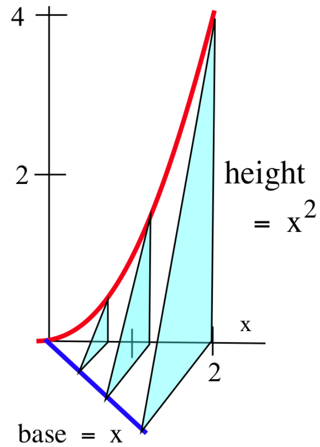
In 14–18, represent the volume of each solid as a definite integral, then evaluate the integral.

14. 

15. 

16. 

17. 

18. 

In 19–28, represent the volume of each solid as a definite integral, then evaluate the integral.

19. The base of a solid is the region between one arch of the curve $y = \sin(x)$ and the x-axis, and cross-sections ("slices") of the solid perpendicular to the base (and to the x-axis) are squares.

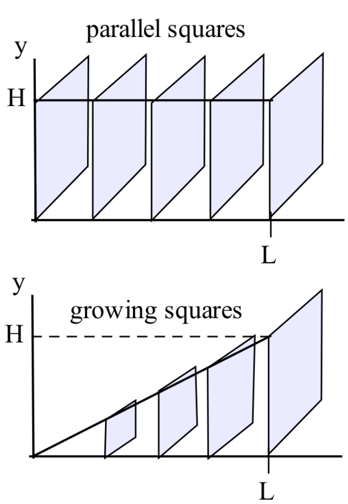
20. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis and the curve $y = \cos(x)$, and cross-sections ("slices") of the solid perpendicular to the base (and to the x-axis) are squares.

21. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis and the curve $y = \cos(x)$, and slices perpendicular to the base (and to the x-axis) are semicircles.

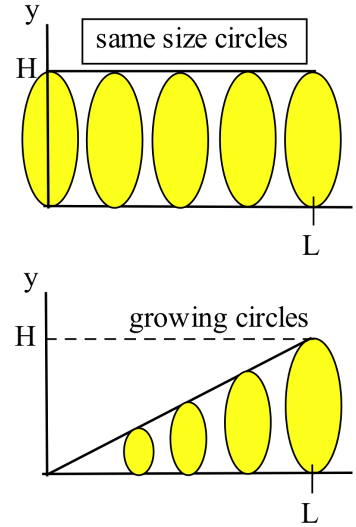
22. The base of a solid is the region between one arch of the curve $y = \sin(x)$ and the x-axis, and slices perpendicular to the base (and to the x-axis) are equilateral triangles.

23. The base of a solid is the region bounded by the parabolas $y = x^2$ and $y = 3 + x - x^2$, and slices perpendicular to the base (and to the x-axis) are:
   (a) squares.
   (b) semicircles.
   (c) rectangles twice as tall as they are wide.
   (d) isosceles right triangles with a hypotenuse in the base of the solid.

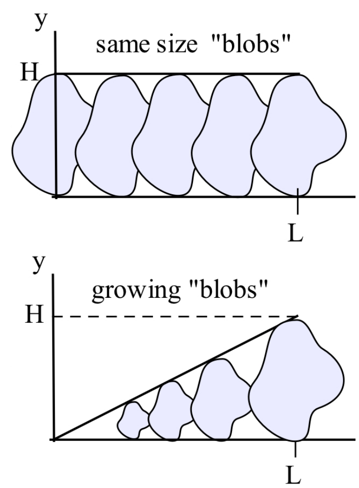
24. The base of a solid is the first-quadrant region bounded by the y-axis, the curve $y = \sin(x)$ and the curve $y = \cos(x)$, and slices perpendicular to the base (and to the x-axis) are:
    (a) squares.
    (b) semicircles.
    (c) rectangles twice as tall as they are wide.
    (d) isosceles right triangles with a hypotenuse in the base of the solid.
25. The base of a solid is the region bounded by the x-axis, the y-axis and the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are squares.
26. The base of a solid is the region bounded by the x-axis, the line $y = 3$ and the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are squares.
27. The base of a solid is the region bounded below by the line $y = 1$, on the left by the line $x = 2$ and above by the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are semicircles.
28. The base of a solid is the region bounded below by $y = 1$, on the left by $x = 2$ and above by $y = 8 - x^2$, and slices perpendicular to the base (and to the x-axis) are semicircles.
29. Calculate (a) the volume of the right solid in the top figure (b) the volume of the "right cone" in the bottom figure and (c) the ratio of the "right cone" volume to the right solid volume.



30. Calculate (a) the volume of the right solid in the top figure (b) the volume of the "right cone" in the bottom figure and (c) the ratio of the "right cone" volume to the right solid volume.



31. Calculate (a) the volume of the right solid in the top figure if each "blob" has area B (b) the volume of the "right cone" in the bottom figure, using "similar blobs" to conclude that the cross-section x units from the y-axis has area $A(x) = \frac{B}{L^2}x^2$ and (c) the ratio of the "right cone" volume to the right solid volume.



32. "Personal calculus": Describe a practical way to determine the volume of your hand and arm up to the elbow.

5.1 Practice Answers

1. triangular base: $V = (\text{base area}) \cdot (\text{height}) = \left(\frac{1}{2} \cdot 3 \cdot 4\right)(6) = 36$

   semicircular base: $V = (\text{base area}) \cdot (\text{height}) = \left(\frac{1}{2}\pi \cdot 3^2\right)(7) \approx 98.96$

   "blob"-shaped base: $V = (\text{base area}) \cdot (\text{height}) = (8)(5) = 40 \text{ in}^3$

2. (a) The base of each triangular slice is 4 and the height is approximately ${x_k}^2$ so $A(x_k) \approx \frac{1}{2}(4)\left({x_k}^2\right) = 2{x_k}^2$ and the volume of the k-th slice is this approximately $2{x_k}^2 \cdot \Delta x_k$.

   (b) Adding up the approximate volumes of all n slices yields $\sum_{n=1}^{\infty} 2{x_k}^2 \cdot \Delta x_k$, which is a Riemann sum with limit:

$$\int_0^2 2x^2\,dx = \frac{2}{3}x^3\Big|_1^2 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$