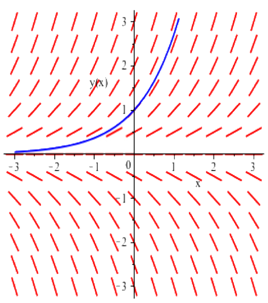
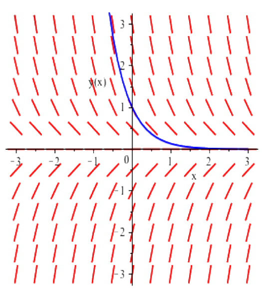
6.4 *Exponential Growth and Decay*

The separable differential equation $y' = ky$ is relatively simple to solve, but it can model a wealth of important situations, including population growth, radioactive decay and drug absorption in the bloodstream. In this section we will solve this ODE and explore some related applications.

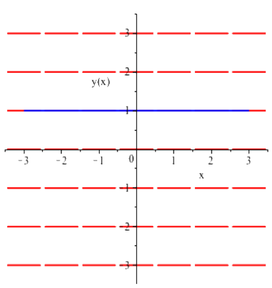
The Differential Equation $y' = ky$



y ' = 1y, y(0)=1



y ' =-2y, y(0)=1



y ' =0y, y(0)=1

Direction fields for y ' = ky

The differential equation $y' = ky$ says that the rate of change of a quantity y is proportional to the value of y. The margin figures show direction fields for $y' = 1y$ (growth) and $y' = -2y$ (decay). The ODE $y' = ky$ can model the behavior of populations (the rate at which babies are born is proportional to the number of people currently in the population), radioactive decay (the rate at which atoms decay is proportional to the number of atoms present), the absorption of some medicines by our bodies, and many other situations. The solutions of $y' = ky$ will help us determine how long it takes a population to double in size, the age of some prehistoric artifacts, and even how often some medicines should be taken in order to maintain a safe and effective concentration of that medicine in a patient's body.

If $y' = ky$ (for $y > 0$), then $y(t) = y(0) \cdot e^{kt}$.

Proof. The ODE $y' = ky$ is separable, so we can employ the method of Section 6.3 to solve it:

$$\frac{dy}{dt} = k \cdot y \;\Rightarrow\; \frac{1}{y} \cdot \frac{dy}{dt} = k \;\Rightarrow\; \frac{1}{y}\,dy = k\,dt \;\Rightarrow\; \int \frac{1}{y}\,dy = \int k\,dt$$
$$\Rightarrow\; \ln(|y|) = kt + C \;\Rightarrow\; e^{\ln(|y|)} = e^{kt+C}$$
$$\Rightarrow\; |y| = e^{kt} \cdot e^{C} \;\Rightarrow\; y = \pm e^{C} e^{kt}$$

Because we assumed that $y > 0$, we didn't need to worry about dividing by y, and we didn't really need the absolute values (or the $\pm$) in the solution above. But $y = 0$ is also a solution to $y' = ky$, so $y = Ae^{kt}$ solves $y' = ky$ for any value of A.

We've found an infinite family of solutions for $y' = ky$, but how do we know that we've found *all* solutions to that ODE? Let $f(t)$ be any solution to $y' = ky$ so that $f'(t) = k \cdot f(t)$. Then define another function $g(t) = f(t) \cdot e^{-kt}$ so that:

$$g(t) = \frac{f(t)}{e^{kt}} \;\Rightarrow\; g'(t) = \frac{e^{kt} \cdot f'(t) - f(t) \cdot e^{kt}}{\left[e^{kt}\right]^2} = \frac{e^{kt}\left[f'(t) - k \cdot f(t)\right]}{e^{2kt}}$$

The last expression in brackets is 0 (because $f'(t) = k \cdot f(t)$), so:

$$g'(t) = 0 \;\Rightarrow\; g(t) = C \;\Rightarrow\; \frac{f(t)}{e^{kt}} = C \;\Rightarrow\; f(t) = Ce^{kt}$$

We now know that any function of the form $y = Ae^{kt}$ solves $y' = ky$ *and* that any solution of $y' = ky$ must have the form $y = Ae^{kt}$.

Finally, putting $t = 0$ into the general solution:

$$y(0) = Ae^{k \cdot 0} = A \ \Rightarrow\ A = y(0) \ \Rightarrow y(t) = y(0) \cdot e^{kt}$$

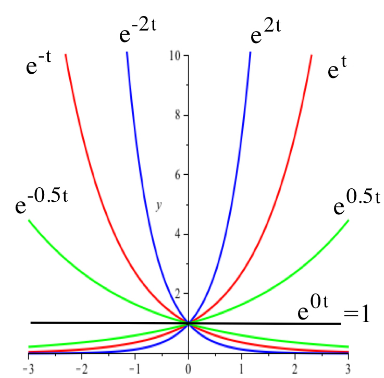
which holds for any value of $y(0)$, but in particular for any $y(0) > 0$. □

Exponential Growth

A population of people, a chunk of radioactive material and the amount of money in a bank account can all share a common trait. In each situation, the rate at which an amount changes at a particular time is often proportional to the value of that amount at that time. For example:

- the number of births per year is proportional to the number of people in the population
- the number of atoms per hour that release a particle is proportional to the number of atoms present
- the number of dollars of interest per year added to a bank account is proportional to the amount of money in that bank account

These situations can all be modeled with the separable ODE solved above. Our focus in this section will be on using those equations and their solutions to answer questions about applied problems. The applications here all involve the rate of change of some quantity with respect to time, so the input variable will generally be time t (instead of x). We might also write the output quantity as $f(t)$ (instead of y). The ODE $y' = ky$ then becomes $f'(t) = k \cdot f(t)$ and the solution $y = y_0 \cdot e^{kx}$ becomes $f(t) = f(0) \cdot e^{kt}$.



When $k > 0$, $f(t) = f(0) \cdot e^{kt}$ represents **exponential growth** and we call k the **growth constant**. When $k < 0$, $f(t) = f(0) \cdot e^{kt}$ represents **exponential decay** and we call k the **decay constant**. The margin figure shows the graphs of $f(t) = e^{kt}$ for several values of k.

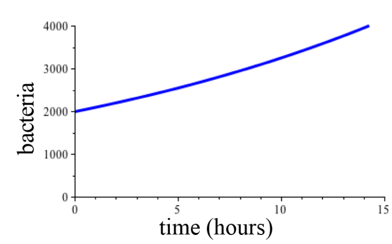
Example 1. The number of bacteria on a Petri plate t hours after an experiment starts is $2000 \cdot e^{0.0488t}$.

(a) How many bacteria are on the plate after one hour? Two hours?

(b) What is the percentage growth of the population from $t = 0$ to $t = 1$? From $t = 1$ to $t = 2$?

(c) How long does it take for the population to reach 3000? To double?

When you know the initial population $f(0)$ and the growth constant k, you can write an equation for $f(t)$, the population at any time t, and use it to answer questions about the population.



Solution. See margin figure for a graph of $f(t) = 2000 \cdot e^{0.0488t}$.

(a) $f(1) = 2000 \cdot e^{0.0488} \approx 2100$; $f(2) = 2000 \cdot e^{0.0976} \approx 2205$

(b) The percentage growth from $t = 0$ to $t = 1$ is:

$$\frac{f(1) - f(0)}{f(0)} = \frac{2100 - 2000}{2000} = \frac{100}{2000} = 0.05 = 5\%$$

The percentage growth from $t = 1$ to $t = 2$ is:

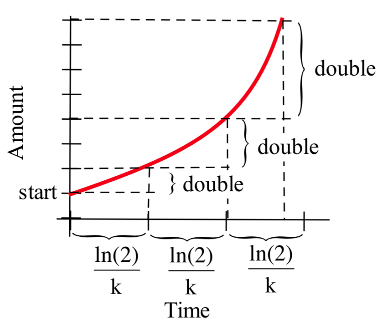
$$\frac{f(2) - f(1)}{f(1)} = \frac{2205 - 2100}{2100} = \frac{105}{2100} = 0.05 = 5\%$$

During the first hour, the population grows by 100 and during the second hour it grows by 105, but the percentage growth during each hour remains constant at 5%.

(c) We need the value of T so that $3000 = f(T) = 2000 \cdot e^{0.0488T}$:

$$1.5 = e^{0.0488T} \Rightarrow \ln(1.5) = \ln\left(e^{0.0488T}\right) = 0.0488T$$
$$\Rightarrow T = \frac{\ln(1.5)}{0.0488} \approx 8.31 \text{ hours}$$

The original population is 2000, so the doubled population is 4000 and the doubling time is $\frac{\ln(2)}{0.0488} \approx 14.2$ hours. ◀



Constant doubling times

When we know the growth constant k, the doubling time is simple to find (as in the preceding Example). If $f(t) = f(0) \cdot e^{kt}$ then the doubling time is the time t_d so that:

$$2f(0) = f(0) \cdot e^{kt_d} \Rightarrow 2 = e^{kt_d} \Rightarrow \ln(2) = kt_d \Rightarrow t_d = \frac{\ln(2)}{k}$$

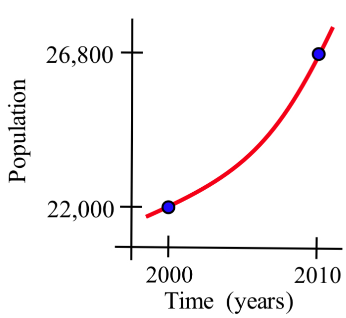
An important aspect of exponential growth is that the doubling time depends only on the growth constant k.

Practice 1. Use the information from the previous Example to:

(a) determine the population at $t = 5$.

(b) find how long it takes for the population to reach 5,000. To triple.

If you do not know the value of the growth constant k, your first step will typically be to use other information to find it.



Example 2. The population of a community was 22,000 in 2000 and 26,800 in 2010. Assuming that the community maintains the same rate of exponential growth (see margin figure):

(a) Find a formula for the population t years after 2000.

(b) Find the annual percentage rate of growth of the community.

Solution. Let t represent the number of years since 2000, so the year 2000 corresponds to $t = 0$ and the year 2010 corresponds to $t = 10$. Then $f(0) = 22000$, $f(10) = 26800$ and $f(t) = 22000 \cdot e^{kt}$.

(a) To find the value for k:

$$26800 = f(10) = 22000 \cdot e^{k(10)} \Rightarrow 1.218 = e^{10k} \Rightarrow \ln(1.218) = 10k$$
$$\Rightarrow k = 0.1 \ln(1.218) \approx 0.0197$$

so $f(t) \approx 22000 \cdot e^{0.0197t}$.

(b) $f(0) = 22000$ and $f(1) \approx 22000 \cdot e^{(0.0197)1} \approx 22438$ so the annual percentage increase was

$$\frac{f(1) - f(0)}{f(0)} = \frac{438}{22000} \approx 0.0199 = 1.99\%$$

during the first year. ◀

Practice 2. A scientist released 12,000 free neutrons into a material. Two seconds later, the material contained 18,000 free neutrons. If the number of free neutrons grows exponentially:

(a) Find a formula for the number of neutrons present t seconds after the beginning of the experiment.

(b) Find the doubling time for the number of free neutrons.

Compound interest provides another example of exponential growth.

Example 3. How long does it take $1,000 to double when invested in a savings account with interest compounded continuously at an annual rate of 5%? At an effective annual rate of return of 5% (compounded continuously)?

Solution. If the interest is compounded continuously at an annual rate of 5% and $A(t)$ is the amount of money in the account t years after the initial deposit of $1,000, then:

$$A'(t) = 0.05A(t),\ A(0) = 1000 \quad \Rightarrow \quad A(t) = 1000e^{0.05t}$$

and the doubling time is $t_d = \frac{\ln(2)}{0.05} \approx 13.86$ years.

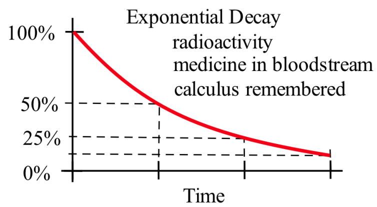
If the effective annual rate of return is 5%, then $A(1) = 1000 + 0.05(1000) = 1.05(1000) = 1050$ so:

$$1050 = 1000e^{k \cdot 1} \quad \Rightarrow \quad 1.05 = e^k \quad \Rightarrow \quad k = \ln(1.05) \approx 0.0488$$

hence the doubling time is $t_d = \frac{\ln(2)}{\ln(1.05)} \approx 14.21$ years. (An effective annual return of 5% corresponds to a continuously compounded annual interest rate of 4.88%.) ◀

Practice 3. How long does it take an investment to double if the effective annual rate of return is 2%?

Exponential Decay



Exponential decay occurs when the rate of loss of something is proportional to the amount present. One example of exponential decay is radioactive decay: the number of atoms of a radioactive substance that "decay" (split into nonradioactive atoms and release particles) during a short time interval is proportional to the number of radioactive atoms present at that time. Exponential decay (see margin) also models how quickly some medicines are absorbed from the bloodstream — and even how quickly you forget calculus concepts.

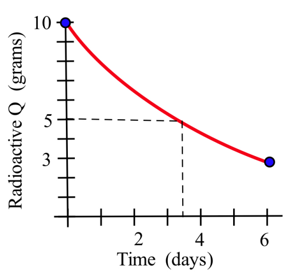
Exponential decay calculations are similar to those for growth, but the value of k is negative and we talk about "**half-life**," the time for half of the material to decay or be absorbed, instead of the doubling time. The margin table shows the half-lives of some isotopes.

element	half-life
iodine-131	8.07 days
strontium-90	29 years
argon-39	265 years
carbon-14	5700 years
plutonium-239	24400 years
uranium-238	4.51×10^5 years
uranium-234	2.47×10^9 years

Example 4. You started an experiment with 10 g of a radioactive substance, but after 6 days of decay only 3 g remained.

(a) Find a formula for the amount of radioactive material present t days after beginning the experiment.

(b) Find the half-life for the radioactive substance.



Solution. Let $f(t)$ represent the amount of the radioactive substance present after t days. Then $f(t) = 10e^{kt}$ (see margin figure).

(a) $3 = f(6) = 10e^{6k} \Rightarrow 0.3 = e^{6k} \Rightarrow \ln(0.3) = 6k$ so $k = \frac{1}{6}\ln(0.3) \approx -0.2007$ and $f(t) = 10e^{-0.2007t}$.

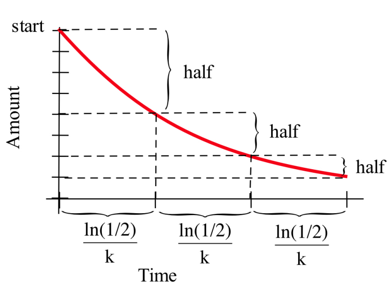
(b) The half-life t_h is the time required for half of the material to decay, so we need to solve $5 = 10e^{-0.2007t_h}$ for t_h:

$$0.5 = e^{-0.2007t_h} \Rightarrow \ln(0.5) = -0.2007t_h \Rightarrow t_h = \frac{\ln(0.5)}{-0.2007} \approx 3.5 \text{ days}$$

Note that $t_h = \dfrac{\ln(0.5)}{k}$, which will hold true generally. ◀

When you know the decay constant k, the half-life t_h is simple to find, as in the preceding Example:

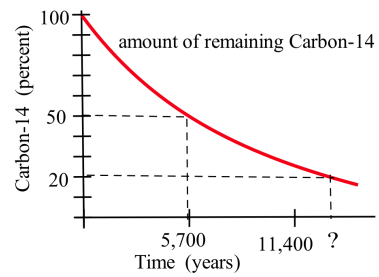
$$t_h = \frac{\ln(0.5)}{k} = \frac{\ln\left(\frac{1}{2}\right)}{k} = \frac{-\ln(2)}{k}$$



The half-life depends only on the decay constant k (see margin figure).

If you know the half-life of a substance and you know how much of the substance is present in a sample now, you can determine how much was present at some past time or determine how long ago the sample contained a particular amount of the substance.

Scientists use radioactive **carbon-14**, with a half-life of about 5,700 years, in this way to estimate how long ago plants and animals lived. A living plant continually exchanges carbon-14 and ordinary carbon with the atmosphere so that the ratio of carbon-14 to non-radioactive carbon remains relatively constant. But once the plant dies, this exchange stops. The ordinary carbon remains in the material, but the carbon-14 decays, so the ratio of carbon-14 to ordinary carbon decreases at a known rate. By measuring the ratio of carbon-14 to ordinary carbon in a sample of plant tissue, scientists can determine how long ago the plant died and obtain an estimate for the age of the sample.



Example 5. The amount of carbon-14 in plant fiber of a woven basket is 20% of the amount present in a living plant (see margin figure). Estimate the age of the basket.

Solution. Let $f(t)$ represent the relative amount of carbon-14 in a sample with age t years. Because the half-life of carbon-14 is 5,700 years:

$$t_h = \frac{-\ln(2)}{k} \Rightarrow 5700 = \frac{-\ln(2)}{k} \Rightarrow k = \frac{-\ln(2)}{5700} \approx -0.0001216$$

so that $f(t) = f(0) \cdot e^{-0.0001216t}$. Because 20% of the carbon-14 remains in our sample, we want the value of T so that:

$$\begin{aligned} 0.20f(0) = f(T) &\Rightarrow 0.20f(0) = f(0) \cdot e^{-0.0001216T} \\ &\Rightarrow 0.20 = e^{-0.0001216T} \Rightarrow \ln(0.20) = -0.0001216T \\ &\Rightarrow T = \frac{\ln(0.2)}{-0.0001216} \approx 13235 \end{aligned}$$

We can conclude that the basket was made from a plant that died about 13,200 years ago. (Does that mean the basket was made then?) ◀

This dating method is very sensitive to small changes in the measured amount of carbon-14.

Practice 4. The half-life of an isotope is 8 days. Write a formula for the amount of the isotope present t days after you begin an experiment with 10 mg of the isotope.

The rate at which many medicines are absorbed from the blood is proportional to the concentration of the medicine in the blood: the higher the concentration in the blood, the faster it is absorbed.

Example 6. Suppose a certain medicine has an absorption (decay) constant of -0.17 (determined experimentally) and that the lowest "effective"concentration of the medicine is 0.3 mg/l (milligrams of medicine per liter of blood). If a patient who has 5 liters of blood is injected with 20 mg of the medicine, how long will the medicine be effective?

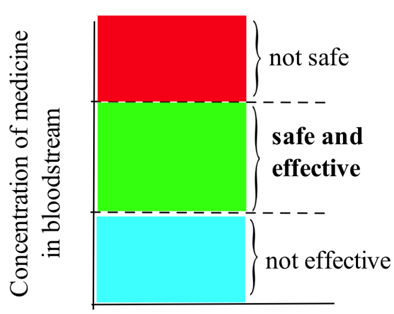
Solution. because the patient is starting with 20 mg of the medicine in 5 liters of blood, the initial concentration is $\frac{20 \text{ mg}}{5 \text{ L}} = 4$ mg/L. The amount of medicine in the blood t hours later is thus $f(t) = 4e^{-0.17t}$ and we want to find T so that $f(T) = 0.3$ mg/L

$$0.3 = 4e^{-0.17T} \Rightarrow \frac{0.3}{4} = e^{-0.17T} \Rightarrow \ln\left(\frac{0.3}{4}\right) = -0.17T \Rightarrow T \approx 15.2$$

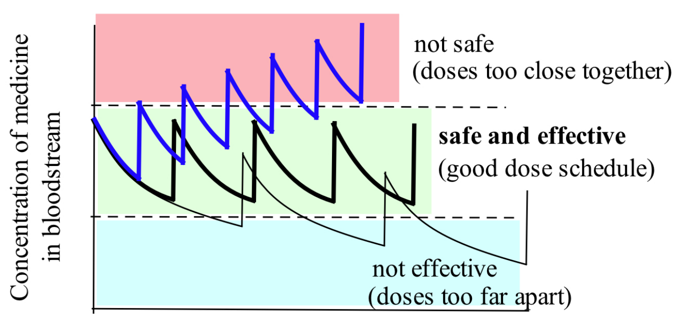
so a patient should receive a new does of the medicine about 15 hours after the first dose. ◀

Practice 5. Should the amount of the second dose in the preceding Example be the same as the initial dose?

Many medicines have a "safe and effective" interval of concentrations (see figure below), so the goal of a schedule for taking the medicine is to keep the concentration near the middle of that range. Taking doses too close together in time can result in an overdose, while taking them too far apart is eventually ineffective.



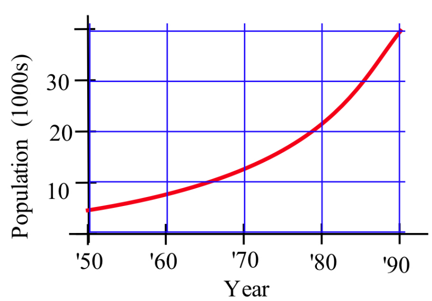
Safe and effective region



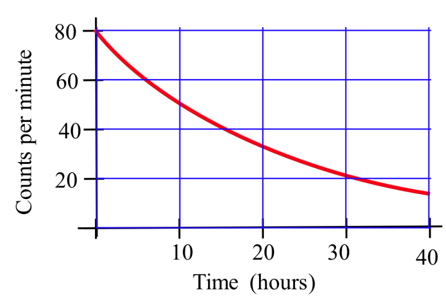
Safe and effective dosing schedule

6.4 Problems

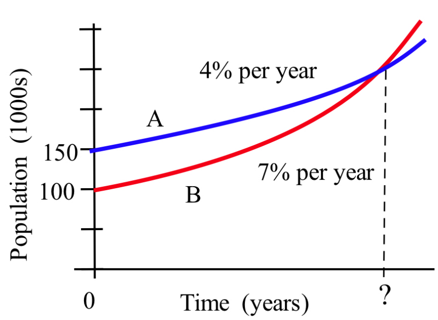
1. How long did it take the population of a city (see below) to double from 10,000 to 20,000? From 15,000 to 30,000? Approximate the doubling time.



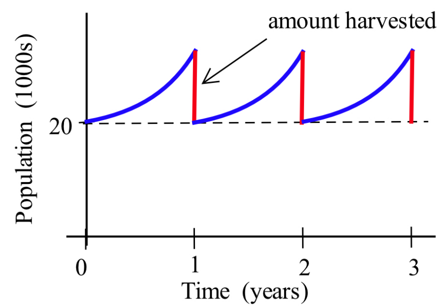
2. How long did it take the counts for a radioactive material to decay from 80 per minute to 40? From 60 to 30? From 40 to 20? What is the half-life?



3. The population of a community in 1990 was 48,000 people and in 2010 it was 64,000 people.
   (a) Write a formula for the population of the community t years after 1990.
   (b) Estimate the population in the year 2020.
   (c) When will the population be 100,000?
   (d) What is the doubling time of the population?
4. Repeat Problem 3 if the population of another community was was 40,000 people in 1990 and 60,000 people in 2010.
5. A terrific investment pays interest at an effective annual rate of 15%. How long will it take for a \$5,000 investment to double? To triple?
6. You have invested \$3,000 for 10 years at an effective annual rate of 7.5% and a friend has invested the same amount invested at an effective annual interest rate of 7.75%. Your friend will get back how much more money than you at the end of 10 years? At the end of 20 years?
7. Find a formula for the population in Problem 1.
8. Each bacterium of a certain species splits into two bacteria at the end of each minute. If we start with a few bacteria in a bowl at 3:00 p.m. and the bowl is full of bacteria at 4:30 p.m., when was the bowl half full? (Calculus is not required.)
9. A newscaster reports that the world's population is now doubling every 50 years. What annual rate of growth results in a 50-year doubling time?
10. Group A has a population of 150,000 and an annual growth rate of 4%; group B has a population of 100,000 and an annual growth rate of 7% (see figure below). After how many years will the two groups be the same size?



11. Group A has a population of 600,000 and an annual growth rate of 3%; group B has a population of 400,000 and an annual growth rate of 6%. After how many years will the two groups be the same size?
12. The unregulated population of fish in a certain lake grows by 30% per year under optimum conditions. A fish census reveals there are approximately 20,000 fish in the lake. How many fish can be harvested (see figure below) at the end of each year in order to maintain a stable population? (This is an example of calculating the yield for a "renewable resource." In practice, more sophisticated calculations also take into account the distribution of species, ages and genders.)



13. The annual exponential growth constant for a population of snails is $k = 0.14$. Currently you have 8,000 snails.
   (a) Determine the size of the population over the next 20 months if you harvest 2,000 snails every 2 months.
   (b) What happens if you harvest 3,000 snails every 2 months?
   (c) How many can we harvest every 2 months in order to maintain a stable population?
14. An exponential function $f(t) = Ae^{kt}$ has constant doubling time, but some non-exponential functions also have constant doubling times.
   (a) Show that the exponential function $f(t) = 2^t = e^{\ln(2)t}$ has a constant doubling time of 1. (Show that $f(t+1) = 2f(t)$.)
   (b) Graph $g(t) = 2^t\left[1 + A\sin(2\pi t)\right]$ for $A = 0.5$ and $A = 1.5$. Show that g has a constant doubling time 1 for any choice of A.

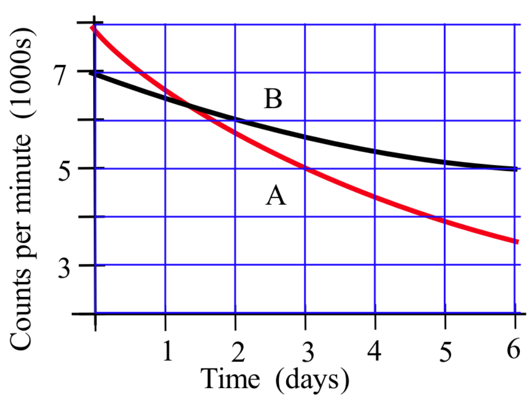
15. An experiment starts with 10 grams of a radioactive material and 14 days later 2 grams remain.
    (a) Find a formula for the amount of material remaining t days after the experiment starts.
    (b) Find the half-life of the material.
    (c) When will 0.7 grams of the material remain?

16. You start with 8 mg of a radioactive substance and 10 days later determine that 6.3 mg remains.
    (a) Find a formula for the amount left t days later.
    (b) Find the half-life of the substance.
    (c) When will 1 mg of the substance remain?

17. A Geiger counter initially recorded 187 counts per minute from a radioactive material, but 2 days later the count was down to 143 counts per minute. (The count per minute is proportional to the amount of radioactive material present.)
    (a) What is the half-life of the material?
    (b) When will the count be down to 20 counts per minute?

18. The initial Geiger counter measurement from a radioactive substance was 540 counts per minute, and a week later it was 500 counts per minute.
    (a) What is the half-life of the substance?
    (b) When will the count be down to 100 counts per minute?

19. Find a formula for the counts per minute for the radioactive material A in the figure below.



20. Find a formula for the counts per minute for the radioactive material B in the figure above.

21. Your friend plans to purchase a letter reputedly written by Isaac Newton (1642–1727), but an analysis of the paper shows that it contains 97.5% of the proportion of carbon-14 present in new paper of the same type. Can you be certain the letter is a forgery? If the age of the paper is consistent with time frame during which Newton lived, can you be certain the letter is genuine?

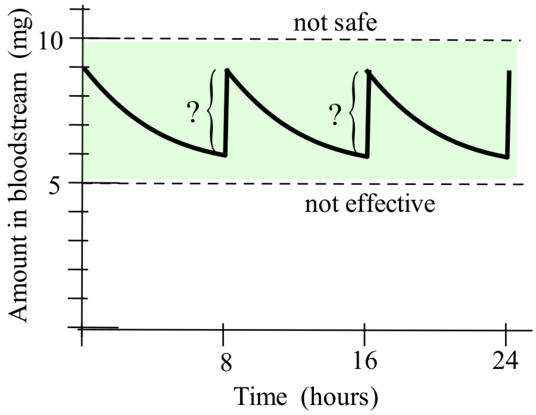
22. For several centuries, the Shroud of Turin was widely believed to be the shroud of Jesus. Three independent laboratories in England, Switzerland and the United States used carbon-14 dating on a few square centimeters of the cloth, and in 1988 they reported (*Science*, October 21, 1988) that the Shroud of Turin was probably made during the early 1300s and certainly after 1200 A.D.
    (a) If the Shroud was made in 1300 A.D., what percentage of the original carbon-14 was still present in 1988?
    (b) If the Shroud was made in 30 A.D., what percentage of the original carbon-14 was still present in 1988?

23. Half of a particular medicine is used up by the body every 6 hours, and the medicine is not effective if the concentration in the blood is less than 10 mg/l. If an ill person is given an initial dose of medicine to raise the concentration to 30 mg/l, for how long will the medicine be effective?

24. A particular controlled substance has a half-life of 12 hours, and it can be detected in concentrations as low as 0.002 mg/l in the blood.
    (a) If a person has an initial concentration of the substance of 15 mg/l in the blood, for how long can it be detected?
    (b) If the detection test is improved by a factor of 100, so it can detect a concentration of 0.00002 mg/l, for how long can an initial concentration of 15 mg/l be detected?

25. A doctor gave a patient 9 mg of a medicine that has half-life of 15 hours in the body. How much of the medicine does the patient need to take every 8 hours in order to maintain between 6 and 9 mg

of the medicine in the body all of the time? (See figure below.)



26. Each layer of a dark film transmits 40% of the light that strikes it.
    (a) How many layers are needed for an eye shield to transmit only 10% of the light?
    (b) How many layers are needed to transmit only 2% of the light?

27. A region has been contaminated with radioactive iodine-131 to a level five times the safe level. How long will it be until the area is safe?

28. A region has been contaminated with radioactive strontium-90 to a level five times the safe level. How long will it take until the area is safe?

29. The population of a country is 4 million and is growing at 5% per year. Currently the country has 10 million acres of forests that are being cut down (and not replanted) at a rate of 300,000 acres per year.
    (a) Find a formula for the number of acres of forest per person.
    (b) At what rate is the number of acres of forest per person changing?
    (c) If the population and harvest rates remain constant, in approximately how many years will there be one acre of forest per person?

6.4 Practice Answers

1. (a) $f(5) = 2000e^{0.0488(5)} \approx 2552$

   (b) $f(T) = 5000 \Rightarrow 5000 = 2000e^{0.0488T} \Rightarrow 2.5 = e^{0.0488T} \Rightarrow \ln(2.5) = 0.0488T$
   $\Rightarrow T = \dfrac{\ln(2.5)}{0.0488} \approx 18.78$ hours; tripling time is $\dfrac{\ln(3)}{0.0488} \approx 22.51$ hours.

2. (a) $f(0) = 12000$ so $f(t) = 12000e^{kt}$ and $f(2) = 18000$ so:

   $$18000 = 12000e^{2k} \Rightarrow 1.5 = e^{2k} \Rightarrow \ln(1.5) = 2k \Rightarrow k = \frac{1}{2}\ln(1.5) \approx 0.2027$$

   and thus $f(t) \approx 12000e^{0.2027t}$.

   (b) Doubling time is $t_d = \dfrac{\ln(2)}{k} \approx \dfrac{\ln(2)}{0.2027} \approx 3.42$ seconds.

3. After 1 year, each \$1 invested will become $\$1 + (0.02)(\$1) = \$1.02$ so:

   $$f(1) = 1.02 = 1 \cdot e^{k \cdot 1} \Rightarrow \ln(1.02) = k \Rightarrow k \approx 0.0198$$

   The doubling time is therefore: $t_d = \dfrac{\ln(2)}{0.0198} \approx 35$ years.

4. $8 = t_h = -\dfrac{\ln(2)}{k} \Rightarrow k = -\dfrac{\ln(2)}{8} \approx -0.0866$ so $f(t) = 10e^{-0.0866t}$.

5. No. After 15.2 hours, the patient still has 0.3 mg/l of medicine in his blood, or $5(0.3) = 1.5$ mg. A dose of $20 - 1.5 = 18.5$ mg would return the medicine in his blood to the original level of 20 mg.