7.5 Integrals Involving Inverse Trig Functions

Aside from the Museum Problem and its sporting variations introduced in the previous section, the primary use of the inverse trigonometric functions in calculus involves their role as antiderivatives of rational and algebraic functions. Each of the six differentiation patterns from the previous section provides us with an integral formula, but they give rise to only three essentially different patterns:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

Valid for: $-1 < x < 1$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

Valid for: $-\infty < x < \infty$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C$$

Valid for: $|x| > 1$

Most of the related antiderivative patterns you will need in practice arise from variations of these basic ones. Typically, you need to transform an integrand so that it exactly matches one of the basic patterns.

Why are the derivative patterns for arccos, arccot and arccsc of little use to us when finding antiderivatives of algebraic functions?

Example 1. Evaluate $\displaystyle\int \frac{1}{16+x^2}\,dx$.

Solution. We can transform this integrand into the arctangent pattern by factoring 16 from the denominator

$$\int \frac{1}{16+x^2}\,dx = \int \frac{1}{16\left(1+\frac{x^2}{16}\right)}\,dx = \frac{1}{16}\int \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$$

and then using the substitution $u = \dfrac{x}{4} \Rightarrow du = \dfrac{1}{4}\,dx \Rightarrow 4\,du = dx$:

$$\frac{1}{16}\int \frac{1}{1+u^2}\cdot 4\,du = \frac{4}{16}\int \frac{1}{1+u^2}\,du = \frac{1}{4}\arctan(u) + C$$

Replacing u with $\dfrac{x}{4}$ we get $\dfrac{1}{4}\arctan\left(\dfrac{x}{4}\right) + C$ as our final answer. ◀

Practice 1. Evaluate $\displaystyle\int \frac{1}{1+9x^2}\,dx$ and $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx$.

The integrands that arise most often contain patterns with the forms $a^2 - x^2$, $a^2 + x^2$ and $x^2 - a^2$, where a is some positive constant, so it is worthwhile to develop general integral patterns for these forms, list them in Appendix I, and refer to them when necessary:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

Valid for: $-a < x < a$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Valid for: $-\infty < x < \infty$

$$\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$$

Valid for: $|x| > a$

You can arrive at each of these general formulas by factoring the a^2 out of the denominator and making a suitable change of variable (as in Example 1, with a in place of 4). You can then check the result by differentiating. The arctan pattern is, by far, the most common. The arcsin pattern appears occasionally, and the arcsec pattern only rarely.

Example 2. Develop the general formula for $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$ from the known formula for $\int \frac{1}{\sqrt{1 - x^2}}\, dx$. (Assume that $a > 0$.)

Solution. Using Example 1 as a guide, factor a^2 out of the denominator:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \int \frac{1}{a\sqrt{1 - \frac{x^2}{a^2}}}\, dx = \frac{1}{a}\int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}}\, dx$$

Now substitute $u = \frac{x}{a} \Rightarrow du = \frac{1}{a}\, dx \Rightarrow a\, du = dx$ to get:

$$\frac{1}{a}\int \frac{1}{\sqrt{1 - u^2}} \cdot a\, du = \frac{a}{a}\int \frac{1}{\sqrt{1 - u^2}}\, du = \arcsin(u) + C$$

and replace u with $\frac{x}{a}$ to get $\arcsin\left(\frac{x}{a}\right) + C$, the desired result. ◀

Practice 2. Verify that the derivative of $\frac{1}{a} \cdot \arctan\left(\frac{x}{a}\right)$ is $\frac{1}{a^2 + x^2}$.

Example 3. Evaluate $\int \frac{1}{\sqrt{5 - x^2}}\, dx$ and $\int_1^3 \frac{1}{5 + x^2}\, dx$.

Solution. The constant a needn't be an integer, so take $a^2 = 5 \Rightarrow a = \sqrt{5}$:

$$\int \frac{1}{\sqrt{5 - x^2}}\, dx = \arcsin\left(\frac{x}{\sqrt{5}}\right) + C$$

using the pattern from Example 2, while the general arctan pattern yields:

$$\int_1^3 \frac{1}{5 + x^2}\, dx = \left[\frac{1}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right)\right]_1^3 = \frac{1}{\sqrt{5}}\left[\arctan\left(\frac{3}{\sqrt{5}}\right) - \arctan\left(\frac{1}{\sqrt{5}}\right)\right]$$

or about 0.228. ◀

The easiest way to integrate certain rational functions is to split the original integrand into two pieces.

Example 4. Evaluate $\int \frac{6x + 7}{25 + x^2}\, dx$.

Solution. The integrand splits nicely into the sum of two other functions that you can integrate more easily:

$$\int \frac{6x+7}{25+x^2}\,dx = \int \frac{6x}{25+x^2}\,dx + \int \frac{7}{25+x^2}\,dx$$

In the first integral, use the subsitution $u = 25 + x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$:

$$\int \frac{6x}{25+x^2}\,dx = \frac{6}{2}\int \frac{1}{u}\,du = 3\ln(|u|) + C_1 = 3\ln\left(25+x^2\right) + C_1$$

Why can we remove the absolute value signs in the last step?

Meanwhile, the second integral matches the general arctangent pattern with $a = 5$:

$$\int \frac{7}{25+x^2}\,dx = 7\int \frac{1}{5^2+x^2}\,dx = 7\cdot\frac{1}{5}\arctan\left(\frac{x}{5}\right) + C_2$$

Combining these two results yields:

$$\int \frac{6x+7}{25+x^2}\,dx = \ln\left(25+x^2\right)^3 + \frac{7}{5}\arctan\left(\frac{x}{5}\right) + C$$

The two constants C_1 and C_2 add up to another arbitrary constant, which we can simply call C.

as our final answer. ◀

The antiderivative of a linear function divided by an irreducible quadratic polynomial will typically result in the sum of a logarithm and an arctangent.

Practice 3. Evaluate $\displaystyle\int \frac{4x+3}{x^2+7}\,dx$.

7.5 Problems

In Problems 1–24, evaluate the integral.

1. $\displaystyle\int \frac{7}{\sqrt{9-x^2}}\,dx$
2. $\displaystyle\int \frac{9}{\sqrt{7-y^2}}\,dy$
3. $\displaystyle\int_0^1 \frac{3}{x^2+25}\,dx$
4. $\displaystyle\int_5^7 \frac{5}{x\sqrt{x^2-16}}\,dx$
5. $\displaystyle\int \frac{9}{\sqrt{49-x^2}}\,dx$
6. $\displaystyle\int_1^4 \frac{2}{7+x^2}\,dx$
7. $\displaystyle\int_6^{10} \frac{3}{x\sqrt{x^2-25}}\,dx$
8. $\displaystyle\int \frac{7}{(x-5)^2+9}\,dx$
9. $\displaystyle\int \frac{1}{(x-1)^2+1}\,dx$
10. $\displaystyle\int \frac{1}{x^2-2x+2}\,dx$
11. $\displaystyle\int_{-1}^1 \frac{e^x}{1+e^{2x}}\,dx$
12. $\displaystyle\int_1^e \frac{1}{x}\cdot\frac{3}{1+[\ln(x)]^2}\,dx$
13. $\displaystyle\int \frac{\cos(\theta)}{\sqrt{9-\sin^2(\theta)}}\,d\theta$
14. $\displaystyle\int \frac{8x}{16+x^2}\,dx$
15. $\displaystyle\int \frac{3x}{\sqrt{9+x^2}}\,dx$
16. $\displaystyle\int \frac{3x}{\sqrt{9-x^2}}\,dx$
17. $\displaystyle\int \frac{6x}{9+x^4}\,dx$
18. $\displaystyle\int \frac{6x}{\sqrt{9-x^4}}\,dx$
19. $\displaystyle\int \frac{1}{1+4x^2}\,dx$
20. $\displaystyle\int \frac{1}{\sqrt{1-9x^2}}\,dx$
21. $\displaystyle\int_0^\infty \frac{1}{3+x^2}\,dx$
22. $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$
23. $\displaystyle\int_0^{\sqrt{7}} \frac{1}{\sqrt{7-x^2}}\,dx$
24. $\displaystyle\int_0^\infty \frac{x}{1+x^4}\,dx$

In Problems 25–28, solve the initial value problem.

25. $\dfrac{dy}{dx} = \dfrac{y}{\sqrt{1-x^2}}, \ y(0) = e$

26. $\dfrac{dy}{dx} = \dfrac{1}{y(1+x^2)}, \ y(0) = 4$

27. $\dfrac{dy}{dx} = \dfrac{y^2}{9+x^2}, \ y(1) = 2$

28. $\dfrac{dy}{dx} \cdot \sqrt{16-x^2} = y, \ y(4) = 1$

In Problems 29–32, evaluate the integral by splitting the integrand into two simpler functions.

29. $\displaystyle\int \frac{8x+5}{x^2+9}\,dx$

30. $\displaystyle\int \frac{1-4x}{x^2+1}\,dx$

31. $\displaystyle\int \frac{7x+3}{x^2+10}\,dx$

32. $\displaystyle\int \frac{x+5}{x^2+16}\,dx$

Problems 33–40 illustrate how we can sometimes decompose a difficult integral into simpler ones. (Hints: For 33, complete the square in the denominator; for 34, let $u =$ denominator; for 35, write $4x+20 = (4x+12)+8$.)

33. $\displaystyle\int \frac{8}{x^2+6x+10}\,dx$

34. $\displaystyle\int \frac{4x+12}{x^2+6x+10}\,dx$

35. $\displaystyle\int \frac{4x+20}{x^2+6x+10}\,dx$

36. $\displaystyle\int \frac{7}{x^2+4x+5}\,dx$

37. $\displaystyle\int \frac{12x+24}{x^2+4x+5}\,dx$

38. $\displaystyle\int \frac{12x+31}{x^2+4x+5}\,dx$

39. $\displaystyle\int \frac{6x+15}{x^2+4x+20}\,dx$

40. $\displaystyle\int \frac{2x+5}{x^2-4x+13}\,dx$

7.5 Practice Answers

1. For the first integral, write:

$$\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx$$

and substitute $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$ to get:

$$\int \frac{1}{1+u^2} \cdot \frac{1}{3}\,du = \frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan(u) + C = \frac{1}{3}\arctan(3x) + C$$

For the second integral, factor out 25 to get:

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{5\sqrt{1-\frac{x^2}{25}}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\,dx$$

and then substitute $u = \dfrac{x}{5} \Rightarrow du = \dfrac{1}{5}\,dx \Rightarrow 5\,du = dx$:

$$\frac{1}{5}\int \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}} \cdot 5\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + K$$

Replacing u with $\dfrac{x}{5}$ yields:

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\left(\frac{x}{5}\right) + K$$

2. Using the Chain Rule:

$$\mathbf{D}\left[\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right] = \frac{1}{a} \cdot \frac{1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1}{a} = \frac{1}{a^2\left[1+\left(\frac{x}{a}\right)^2\right]} = \frac{1}{a^2+x^2}$$

3. Split the integrand into two pieces:

$$\int \frac{4x+3}{x^2+7}\,dx = \int \frac{4x}{x^2+7}\,dx + \int \frac{3}{x^2+7}\,dx$$

For the first integral, let $u = x^2+7 \;\Rightarrow\; du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$:

$$\int \frac{4x}{x^2+7}\,dx = \int \frac{2}{u}\,du = 2\ln\left(|u|\right) + C_1 = 2\ln\left(x^2+7\right) + C_1$$

The second integral matches the arctangent pattern with $a^2 = 7 \Rightarrow$ $a = \sqrt{7}$:

$$\int \frac{3}{x^2+7}\,dx = 3\int \frac{1}{\left(\sqrt{7}\right)^2 + x^2}\,dx = 3 \cdot \frac{1}{\sqrt{7}} \arctan\left(\frac{x}{\sqrt{7}}\right) + C_2$$

Combining these results yields:

$$\int \frac{4x+3}{x^2+7}\,dx = \ln\left(\left[x^2+7\right]^2\right) + \frac{3}{\sqrt{7}} \arctan\left(\frac{x}{\sqrt{7}}\right) + C$$