8
Integration Techniques

In our journey through integral calculus, we have: developed the concept of a Riemann sum that converges to a definite integral; learned how to use the Fundamental Theorem of Calculus to evaluate a definite integral — as long as we can find an antiderivative for the integrand; examined numerical methods to approximate values of definite integrals; applied the concept of a Riemann sum to a variety of geometric and physical situations to compute lengths, areas, volumes, work and more; and employed integration to solve differential equations.

Finding an approximate value for a definite integral is often "good enough," but exact values are sometimes necessary — and this requires us to find antiderivatives of integrand functions. We have already learned how to find antiderivatives of many basic functions, and repeatedly employed substitution to turn complicated integrands into ones that are easier to integrate. This chapter begins with a review of these integration techniques you already know, then develops several new techniques that will allow you to integrate even more functions. It concludes by presenting a way to find "approximate antiderivatives" that will allow you to compute approximate numerical values for certain definite integrals much more efficiently than the techniques introduced in Section 4.9.

8.1 Finding Antiderivatives: A Review

Success at integration is primarily a matter of recognizing standard patterns and being able to manipulate functions into a form that corresponds to one of these patterns. Integral tables — such as the brief one on the next page and the longer one in Appendix I — list antiderivatives for many basic patterns of functions. Often a change of variable (employing the u-substitution method introduced in Section 4.6) will allow you to see a pattern more easily. For most people, developing the skill of recognizing these patterns comes with practice, and this section provides a variety of problems to review and hone your skills.

This is an extended version of the table listed in Section 4.6.

Constant Functions: $\int k\,du = ku + C$

Powers of u: $\int u^p\,du = \dfrac{u^{p+1}}{p+1} + C$ if $p \neq -1$, $\int \dfrac{1}{u}\,du = \ln|u| + C$

Exponential Functions: $\int e^u\,du = e^u + C$, $\int a^u\,du = \dfrac{a^u}{\ln(a)} + C$

Trig Functions: $\int \cos(u)\,du = \sin(u) + C$, $\int \sin(u)\,du = -\cos(u) + C$

$\int \tan(u)\,du = \ln(|\sec(u)|) + C$, $\int \cot(u)\,du = \ln(|\sin(u)|) + C$

These two patterns may be new to you. See the discussion below.

$\int \sec(u)\,du = \ln(|\sec(u) + \tan(u)|) + C$

$\int \csc(u)\,du = -\ln(|\csc(u) + \cot(u)|) + C$

$\int \sec^2(u)\,du = \tan(u) + C$, $\int \csc^2(u)\,du = -\cot(u) + C$

$\int \sec(u) \cdot \tan(u)\,du = \sec(u) + C$, $\int \csc(u) \cdot \cot(u)\,du = -\csc(u) + C$

Inverse-Trig–Related Functions: $\int \dfrac{1}{1+u^2}\,du = \arctan(u) + C$

$\int \dfrac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + C$, $\int \dfrac{1}{|u| \cdot \sqrt{u^2-1}}\,du = \text{arcsec}(u) + C$

The most generally useful and powerful integration technique remains Changing the Variable. The first Problems in this section provide additional practice changing variables to calculate integrals. As we develop more complicated and more specialized techniques for finding antiderivatives, your first thought should still be whether the integral can be simplified by changing the variable. Sometimes the appropriate change of variable is not obvious, and we may need to manipulate the integrand using algebra, trigonometric identities or some clever "tricks" before employing a u-substitution.

Antiderivatives of $\sec(\theta)$ *and* $\csc(\theta)$

In the list of basic antiderivatives at the top of this page, you may have noticed two unfamiliar patterns: those for $\int \sec(\theta)\,d\theta$ and $\int \csc(\theta)\,d\theta$. Antiderivatves for $\cos(\theta)$ and $\sin(\theta)$ essentially came "free" from the derivative patterns we discovered in Chapter 2. Antiderivatives for $\tan(\theta)$ and $\cot(\theta)$ were among the first applications of u-substitution: for example, we can write $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ and put $u = \cos(\theta)$ so that $du = -\sin(\theta)\,d\theta$ and the integral in question becomes:

$$\begin{aligned}\int \tan(\theta)\,d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\,d\theta = \int \frac{-1}{u}\,du &= -\ln(|u|) + C \\ = -\ln(|\cos(\theta)|) + C &= \ln(|\sec(\theta)|) + C\end{aligned}$$

Finding an antiderivative of $\sec(\theta)$, however, requires a special "trick."

Before attempting a substitution, write:

$$\sec(\theta) = \frac{\sec(\theta)}{1} \cdot \frac{(\sec(\theta) + \tan(\theta))}{(\sec(\theta) + \tan(\theta))} = \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}$$

This is an example of "multiplying by 1," a tactic often employed in mathematics where we multiply the top and bottom of a fraction by the same expression.

Why would we want to take the nice, simple function $\sec(\theta)$ and rewrite it as this monstrosity? Look at the denominator and notice that the derivative of $\sec(\theta)$ is $\sec(\theta) \cdot \tan(\theta)$, while the derivative of $\tan(\theta)$ is $\sec^2(\theta)$. Both of these derivatives appear in the numerator. So if we use the substitution:

$$u = \sec(\theta) + \tan(\theta) \;\Rightarrow\; du = \left[\sec(\theta)\tan(\theta) + \sec^2(\theta)\right] d\theta$$

the integral of $\sec(\theta)$ becomes:

$$\begin{aligned}\int \sec(\theta)\, d\theta &= \int \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}\, d\theta \\ &= \int \frac{1}{u}\, du = \ln(|u|) + C = \ln\left(|\sec(\theta) + \tan(\theta)|\right) + C\end{aligned}$$

proving the result listed on the previous page.

Practice 1. Prove that $\int \csc(\theta)\, d\theta = -\ln\left(|\csc(\theta) + \cot(\theta)|\right) + C$.

The trick used to integrate $\sec(\theta)$ and $\csc(\theta)$ only applies in these special situations, so rather than remembering the trick, you might want to simply memorize the result if you find yourself needing to integrate $\sec(\theta)$ on a regular basis.

See Problems 41–42 in Section 8.3 (and the related discussion on page 575) for a more intuitive — albeit more tedious — method to obtain antiderivatives for $\sec(\theta)$ and $\csc(\theta)$.

An Irreducible Quadratic Denominator

The following examples review and extend techniques (introduced in Section 7.5) involving variations on the arctangent derivative pattern.

Example 1. Evaluate $\int \frac{18}{1 + (x-3)^2}\, dx$ and $\int \frac{18}{x^2 - 6x + 10}\, dx$.

Solution. The form of the first integrand reminds us of the derivative of the arctangent function:

$$\mathbf{D}\left(\arctan(u)\right) = \frac{1}{1 + u^2}$$

If we make the substitution $u = x - 3 \Rightarrow du = dx$ the integral becomes:

$$\int \frac{18}{1 + (x-3)^2}\, dx = 18 \int \frac{1}{1 + u^2}\, du = 18 \arctan(u) + C$$

Replacing u with $x - 3$, we get $18 \arctan(x - 3) + C$. The second integrand appears much more complicated, until you notice that:

$$1 + (x-3)^2 = 1 + x^2 - 6x + 9 = x^2 - 6x + 10$$

These integrands are in fact equal, so the second integral also equals $18 \arctan(x - 3) + C$. ◀

In the preceding Example, we showed that the two integrands were equal by expanding the denominator of the first integrand to get the denominator of the second integrand. If we had started with the second integral in Example 1, we could have rewritten the second denominator employing the method of **completing the square**:

$$x^2 - 6x + 10 = \left(x^2 - 6x + 9\right) + (10 - 9) = (x-3)^2 + 1$$

so that we would make the substitution $u = x - 3$. For any irreducible quadratic polynomial of the form $ax^2 + bx + c$, we can write:

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}\right] + c = a\left[x^2 + \frac{b}{a} + \frac{b^2}{4a^2}\right] + c - a \cdot \frac{b^2}{4a^2}$$
$$= a\left(x - \frac{b}{2a}\right)^2 + \left[c - \frac{b^2}{4a}\right]$$

Rather than memorizing this formula, you should remember the *process* of completing the square.

Recall from algebra that to complete the square with a polynomial, you need to take half of the x coefficient:

$$\frac{-6}{2} = -3 \quad \text{or} \quad \frac{1}{2} \cdot \frac{b}{a}$$

and then square it:

$$(-3)^2 = 9 \quad \text{or} \quad \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

to get the number that must be added and subtracted to create a perfect square (plus a leftover constant term). A quadratic polynomial is **irreducible** if it can't be factored into two linear terms using real coefficients; for $ax^2 + bx + c$ this occurs when $b^2 - 4ac < 0$.

Practice 2. Evaluate $\displaystyle\int \frac{5}{x^2 + 8x + 25}\, dx$ by completing the square and making a substitution.

Example 2. Evaluate $\displaystyle\int \frac{2x}{x^2 - 6x + 10}\, dx$.

Solution. This would be an easy problem if the numerator were $2x - 6$: the numerator would then be the derivative of the denominator and the pattern of the integral would be $\int \frac{1}{u}\, du$ with $u = x^2 - 6x + 10$. Using a bit of cleverness, we can rewrite the numerator as $2x - 6 + 6$. Then the integral becomes:

$$\int \frac{2x - 6 + 6}{x^2 - 6x + 10}\, dx = \int \frac{2x - 6}{x^2 - 6x + 10}\, dx + \int \frac{6}{x^2 - 6x + 10}\, dx$$

This is an example of "adding 0," another tactic often employed in mathematics in which we add and subtract the same quantity from an expression.

For the first integral, substitute $u = x^2 - 6x + 10 \Rightarrow du = (2x - 6)\, dx$:

$$\int \frac{2x - 6}{x^2 - 6x + 10}\, dx = \int \frac{1}{u}\, du = \ln(|u|) + C_1 = \ln\left(x^2 - 6x + 10\right) + C_1$$

Why can we omit the absolute values in the last step?

For the second integral, complete the square in the denominator:

$$x^2 - 6x + 10 = x^2 - 6x + 9 + 1 = (x-3)^2 + 1$$

and use the substitution $w = x - 3 \Rightarrow dw = dx$ to get:

$$\int \frac{6}{x^2 - 6x + 10}\, dx = \int \frac{6}{(x-3)^2 + 1}\, dx = 6 \int \frac{1}{w^2 + 1}\, dw$$
$$= 6 \arctan(w) + C_2 = 6 \arctan(x - 3) + C_2$$

so that the final answer is $\ln\left(x^2 - 6x + 10\right) + 6 \arctan(x - 3) + C$. ◀

This "logarithm plus an arctangent" pattern that arose in the previous Example turns up quite often with integrals of linear functions divided by irreducible quadratic polynomials. If the quadratic denominator can be factored into a product of two linear factors, we will instead use a technique discussed in Section 8.3 (Partial Fraction Decomposition).

Practice 3. Evaluate $\int \frac{4x+21}{x^2+8x+25}\,dx$.

8.1 Problems

In Problems 1–54, evaluate the integral. A well-chosen substitution will often turn a complicated-looking integral into a much simpler one.

1. $\int 6x\left(x^2+7\right)^2\,dx$

2. $\int 6x\left(x^2-1\right)^3\,dx$

3. $\int_2^4 \frac{6t}{\sqrt{t^2-3}}\,dt$

4. $\int_0^{\pi} 12\cos(\theta)\,[2+\sin(\theta)]^2\,d\theta$

5. $\int \frac{12x}{x^2+3}\,dx$

6. $\int \frac{\cos(\varphi)}{2+\sin(\varphi)}\,d\varphi$

7. $\int \sin(3y+2)\,dy$

8. $\int \cos\left(\frac{x}{5}\right)\,dx$

9. $\int_{-1}^{0} e^x\cdot\sec^2(e^x+3)\,dx$

10. $\int_0^{\frac{\pi}{2}} \cos(\theta)\sqrt{1+\sin(\theta)}\,d\theta$

11. $\int \frac{\ln(x)}{x}\,dx$

12. $\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

13. $\int \cos(\theta)\cdot e^{\sin(\theta)}\,d\theta$

14. $\int e^z\sin(e^z)\,dz$

15. $\int_1^3 \frac{5}{1+9x^2}\,dx$

16. $\int_0^1 \frac{7}{1+(x+5)^2}\,dx$

17. $\int_1^2 \frac{1}{x^2}\cdot\cos\left(\frac{1}{x}\right)\,dx$

18. $\int_1^e \frac{\sec(2+\ln(x))}{x}\,dx$

19. $\int \frac{6\sin(\theta)\cos(\theta)}{5+\sin^2(\theta)}\,d\theta$

20. $\int \frac{6\cos(\alpha)}{5+\sin^2(\alpha)}\,d\alpha$

21. $\int \frac{10}{2x+5}\,dx$

22. $\int \frac{3}{8y+1}\,dy$

23. $\int_1^3 \frac{20x}{5x^2+3}\,dx$

24. $\int_1^5 \frac{4x}{x^2+9}\,dx$

25. $\int_0^1 \frac{7}{(x+3)^2+4}\,dx$

26. $\int_{-2.3}^{-2} \frac{1}{\sqrt{1-(x+2)^2}}\,dx$

27. $\int \frac{e^t}{1+e^{2t}}\,dt$

28. $\int \frac{4x+10}{x^2+5x+9}\,dx$

29. $\int_1^3 \frac{3}{x\,[1+\ln(x)]}\,dx$

30. $\int_0^1 \frac{e^t}{1+e^t}\,dt$

31. $\int_0^1 2x\sqrt{1-x^2}\,dx$

32. $\int_0^3 \frac{2x}{\sqrt{5+x^2}}\,dx$

33. $\int \cos(\theta)\,[1+\sin(\theta)]^3\,d\theta$

34. $\int \cos(\varphi)\sin^4(\varphi)\,d\varphi$

35. $\int_1^e \frac{\sqrt{\ln(x)}}{x}\,dx$

36. $\int_1^2 e^x\sqrt{2+e^x}\,dx$

37. $\int \frac{\sec^2(\theta)}{5+\tan(\theta)}\,d\theta$

38. $\int \frac{6x}{(x^2-1)^3}\,dx$

39. $\int \tan(y-5)\,dy$

40. $\int \left(x^3+3\right)^2\,dx$

41. $\int_0^1 e^{5u}\,du$

42. $\int \sec(2+3t)\,dt$

43. $\int t\cdot\sec(2+3t^2)\,dt$

44. $\int_1^1 \arctan\left(\sqrt{5-x^3}\right)\,dx$

45. $\int_e^e \ln\left(\sqrt{5+x^3}\right)\,dx$

46. $\int_1^\infty \frac{1}{1+9x^2}\,dx$

47. $\int_1^\infty \frac{x}{1+9x^4}\,dx$

48. $\int_1^\infty \frac{e^{-x}}{1+e^{-2x}}\,dx$

In 49–54, complete the square in the denominator, make an appropriate substitution, then integrate.

49. $\int \frac{7}{x^2+4x+5}\,dx$

50. $\int \frac{3}{x^2+4x+29}\,dx$

51. $\int \frac{2}{x^2-6x+58}\,dx$

52. $\int \frac{11}{x^2-2x+10}\,dx$

53. $\int \frac{3}{x^2+10x+29}\,dx$

54. $\int \frac{5}{x^2+2x+5}\,dx$

In 55–60, first split the integral into two integrals. (Hint: In Problem 55, $2x+11=(2x+4)+7$.)

55. $\int \frac{2x+11}{x^2+4x+5}\,dx$

56. $\int \frac{4x+11}{x^2+4x+5}\,dx$

57. $\int \frac{4x+7}{x^2-6x+10}\,dx$

58. $\int \frac{6x+28}{x^2+10x+34}\,dx$

59. $\int \frac{6x+5}{x^2-4x+13}\,dx$

60. $\int \frac{4x+9}{x^2+6x+13}\,dx$

In 61–66, remember that completing the square only helps with irreducible quadratic denominators.

61. $\int \frac{1}{x^2+4x+4}\,dx$

62. $\int \frac{x+2}{x^2+4x+4}\,dx$

63. $\int \frac{x+3}{x^2-6x+9}\,dx$

64. $\int_4^\infty \frac{1}{x^2-6x+9}\,dx$

65. $\int_3^\infty \frac{1}{x^2-6x+9}\,dx$

66. $\int_4^\infty \frac{8x-24}{x^2-6x+9}\,dx$

8.1 Practice Answers

1. First use the "multiply by 1" trick to write:

$$\csc(\theta) = \frac{\csc(\theta)}{1} \cdot \frac{\csc(\theta) + \cot(\theta)}{\csc(\theta) + \cot(\theta)} = \frac{\csc^2(\theta) + \csc(\theta)\cot(\theta)}{\csc(\theta) + \cot(\theta)}$$

and then make the substitution $u = \csc(\theta) + \cot(\theta)$ so that $du = \left(-\csc(\theta)\cot(\theta) - \csc^2(\theta)\right) d\theta$ and:

$$\int \csc(\theta)\, d\theta = \int \frac{\csc^2(\theta) + \csc(\theta)\cot(\theta)}{\csc(\theta) + \cot(\theta)}\, d\theta = \int \frac{-1}{u}\, du$$
$$= -\ln(|u|) + C = -\ln(|\csc(\theta) + \cot(\theta)|) + C$$

2. First complete the square in the denominator:

$$x^2 + 8x + 25 = \left(x^2 + 8x + 16\right) + (25 - 16) = (x+4)^2 + 9$$

so the integral becomes:

$$\int \frac{5}{x^2 + 8x + 25}\, dx = \int \frac{5}{(x+4)^2 + 9}\, dx$$

Now make the substitution $u = x + 4 \Rightarrow du = dx$ to get:

$$5 \int \frac{1}{u^2 + 3^2}\, du = 5 \cdot \frac{1}{3} \arctan\left(\frac{u}{3}\right) + C = \frac{5}{3} \arctan\left(\frac{x+4}{3}\right) + C$$

3. If we substitute $u = x^2 + 8x + 25$ then $du = (2x + 8)\, dx$. We would be in good shape if the numerator of the integrand were $4x + 16 = 2(2x + 8)$, so split the numerator into $4x + 21 = (4x + 16) + 5$ to get:

$$\int \frac{4x + 21}{x^2 + 8x + 25}\, dx = \int \frac{(4x+16) + 5}{x^2 + 8x + 25}\, dx = \int \frac{4x + 16}{x^2 + 8x + 25}\, dx + \int \frac{5}{x^2 + 8x + 25}\, dx$$

The first integral (with $u = x^2 + 8x + 25$) now becomes:

$$\int \frac{4x + 16}{x^2 + 8x + 25}\, dx = \int \frac{2}{u}\, du = 2\ln(|u|) + C_1 = 2\ln\left(x^2 + 8x + 25\right) + C_1$$

The second integral is just the integral from Practice 2:

$$\int \frac{5}{x^2 + 8x + 25}\, dx = \frac{5}{3} \arctan\left(\frac{x+4}{3}\right) + C_2$$

Combining these results yields:

$$\int \frac{4x + 21}{x^2 + 8x + 25}\, dx = \ln\left(x^2 + 8x + 25\right)^2 + \frac{5}{3} \arctan\left(\frac{x+4}{3}\right) + C$$