8.2 *Integration by Parts*

Integration by parts is an integration method that enables us to find antiderivatives of certain functions for which our previous antidifferentiation methods fail, such as $\ln(x)$ and $\arctan(x)$, as well as antiderivatives of certain products of functions, such as $x^2 \ln(x)$ and $e^x \sin(x)$. It leads to many of the general integral formulas in Appendix I and (next to u-substitution) it is one most powerful and frequently used of antidifferentiation techniques.

The Integration by Parts formula for integrals arises from the Product Rule for derivatives. Recall that for functions $u = u(x)$ and $v = v(x)$, the Product Rule says:

$$\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

If we integrate both sides of this equation with respect to x we get:

$$u \cdot v = \int u \cdot \frac{dv}{dx}\, dx + \int v \cdot \frac{du}{dx}\, dx$$

Solving for the first of the two integrals on the right side, yields:

$$\int u \cdot \frac{dv}{dx}\, dx = u \cdot v - \int v \cdot \frac{du}{dx}\, dx$$

At first, this formula may not appear very promising, as it merely exchanges one integration problem for another. But in certain situations, if we choose $u(x)$ and $v(x)$ carefully, this formula exchanges a difficult integral for an easier one. We can restate the formula in slightly more compact form using differentials.

Integration by Parts Formula

If u, v, u' and v' are continuous functions,

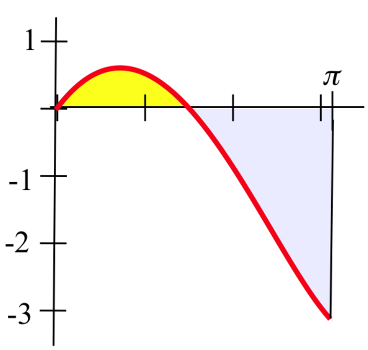
then $\displaystyle\int u\, dv = u \cdot v - \int v\, du$

For definite integrals, the Integration By Parts Formula says:

Integration by Parts Formula (Definite Integrals)

If u, v, u' and v' are continuous functions,

then $\displaystyle\int_a^b u\, dv = \Big[u \cdot v\Big]_a^b - \int_a^b v\, du$



Example 1. Use integration by parts to evaluate $\int x\cos(x)\, dx$ and $\int_0^{\pi} x\cos(x)\, dx$ (see margin for a graphical interpretation).

Solution. Our first step is to write this integral in the form required by the Integration by Parts formula, $\int u\,dv$. If we let $u = x$, then we must have $dv = \cos(x)\,dx$ so that $u\,dv$ completely represents the integrand $x\cos(x)$. We also need to calculate du and v:

$$u = x \ \Rightarrow\ du = dx \qquad \text{and} \qquad dv = \cos(x)\,dx \ \Rightarrow\ v = \sin(x)$$

Putting these pieces into the Integration by Parts formula, we have:

$$\int x \cdot \cos(x)\,dx \ = \ x \cdot \sin(x) - \int \sin(x)\,dx \ = \ x \cdot \sin(x) + \cos(x) + C$$

To check this result, differentiate the answer to verify that:

$$[x\sin(x) + \cos(x)]' = x\cos(x)$$

Now use this result to evaluate the definite integral:

$$\int_0^{\pi} x \cdot \cos(x)\,dx = \Big[x \cdot \sin(x) + \cos(x)\Big]_0^{\pi}$$
$$= [\pi\sin(\pi) + \cos(\pi)] - [0 \cdot \sin(0) + \cos(0)] = -1 - 1$$

or -2, which appears reasonable based on the area interpretation of this integral in the margin graph on the previous page. ◄

Integration by parts allowed us to exchange the problem of evaluating $\int x\cos(x)\,dx$ for the much easier problem of evaluating $\int \sin(x)\,dx$.

Practice 1. Use the Integration by Parts formula on $\int x\cos(x)\,dx$ with $u = \cos(x)$ and $dv = x\,dx$. Why does this lead to a poor exchange?

Example 2. Evaluate $\int xe^{3x}\,dx$ and $\int_0^1 xe^{3x}\,dx$.

Solution. The integrand is a product of two functions, so it is reasonable to use integration by parts to search for an antiderivative. If $u = x \ \Rightarrow du = dx$, then $dv = e^{3x}\,dx \ \Rightarrow\ v = \frac{1}{3}e^{3x}$. Inserting these expressions into the Integration by Parts formula, we get:

$$\int xe^{3x}\,dx = x \cdot \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$$
$$\Rightarrow \int_0^1 xe^{3x}\,dx = \left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_0^1 = \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}\right] = \frac{2}{9}e^3 + \frac{1}{9}$$

or about 4.57. ◄

In practice, when you need to use integration by parts to evaluate a definite integral, it is often safest to first evaluate the corresponding indefinite integral and then use that antiderivative pattern to evaluate the definite integral, as we have done here.

In the previous Example another valid choice would have been $u = e^{3x}$ and $dv = x\,dx$, but that choice results in an integral that is more difficult than the original one: $du = 3e^{3x}\,dx$ and $v = \frac{1}{2}x^2$, so the Integration by Parts formula yields:

$$\int xe^{3x}\,dx = e^{3x} \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot 3e^{3x}\,dx$$

which exchanges $\int xe^{3x}\,dx$ for the more difficult integral $\int \frac{3}{2}x^2e^{3x}\,dx$.

Practice 2. Evaluate $\int x\sin(x)\,dx$ and $\int xe^{5x}\,dx$.

Once you have chosen u and dv to represent the integrand as $u\,dv$, you need to calculate du and v. The du calculation is usually easy, but finding v from dv can be difficult for some choices of dv. In practice, you need to select u so that the remaining dv is simple enough that you can find v, the antiderivative of dv.

Example 3. Evaluate $\int 2x\ln(x)\,dx$.

Solution. The choice $u = 2x$ seems fine until we get a little further into the process. If $u = 2x$, then $dv = \ln(x)\,dx$. We now need to find du and v. Computing $du = 2\,dx$ is simple, but then we face the difficult problem of finding an antiderivative v for our choice $dv = \ln(x)\,dx$.

If you cannot find a v for your original choice of dv, try a different u and dv.

The choice $u = \ln(x)$ results in easier calculations. Let $u = \ln(x)$. Then $dv = 2x\,dx$, so $du = \frac{1}{x}\,dx$ and $v = x^2$. Then the Integration by Parts formula gives:

$$\int 2x\ln(x)\,dx = \ln(x)\cdot x^2 - \int x^2\cdot\frac{1}{x}\,dx = x^2\ln(x) - \int x\,dx$$

so the final result is $x^2\ln(x) - \frac{1}{2}x^2 + C$. ◀

Antiderivatives of Inverse Functions

So far we have applied the Integration by Parts method to products of simple functions, but it also enables us—perhaps surprisingly—to find antiderivatives of the inverse trigonometric functions and of the logarithm (the inverse exponential function).

Example 4. Evaluate $\int \arctan(x)\,dx$.

Solution. Let $u = \arctan(x)$. Then $dv = dx$, so $du = \frac{1}{1+x^2}\,dx$ and $v = x$. Putting these expressions into the Integration by Parts formula:

$$\int \arctan(x)\,dx = x\arctan(x) - \int x\cdot\frac{1}{1+x^2}\,dx$$

We can evaluate the new integral using the substitution $w = 1 + x^2 \Rightarrow dw = 2x\,dx \Rightarrow \frac{1}{2}\,dw = x\,dx$:

Why are absolute value signs not needed in the last term?

$$\int x\cdot\frac{1}{1+x^2}\,dx = \int\frac{1}{2}\cdot\frac{1}{w}\,dw = \frac{1}{2}\ln(|w|) + K \;=\; \frac{1}{2}\ln\left(1+x^2\right) + K$$

Combining these results:

We can now include the antiderivative of $\arctan(x)$ in our list of antiderivatives in Appendix I.

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln\left(1+x^2\right) + C$$

or $x\arctan(x) - \ln\left(\sqrt{1+x^2}\right) + C$. ◀

Practice 3. Evaluate $\int \ln(x)\,dx$ and $\int_1^e \ln(x)\,dx$.

Note the following about the Integration by Parts formula:

- Once you choose u, then dv is completely determined.

dv = rest of the integrand

- Because you need to find an antiderivative of dv to get v, pick u and dv with this in mind.
- Integration by parts allows you to trade one integral for another. If the new integral is more difficult than the original integral, then you have made a poor choice of u and dv. Try a different choice for u and dv (or try a different technique).
- To evaluate the new integral $\int v\,du$ you may need to use substitution, integration by parts again, or some other technique (such as the ones discussed later in this chapter).

General Patterns

Sometimes a single application of integration by parts yields a formula that allows us to integrate an entire family of functions.

Example 5. Evaluate $\int x^p \ln(x)\,dx$ for any number $p \neq -1$.

Solution. Set $u = \ln(x)$ so $dv = x^p\,dx$, $du = \frac{1}{x}\,dx$ and $v = \frac{1}{p+1}x^{p+1}$. Putting all of this into the Integration by Parts formula:

$$\int x^p \ln(x)\,dx = \frac{1}{p+1}x^{p+1} \cdot \ln(x) - \int \frac{1}{p+1}x^{p+1} \cdot \frac{1}{x}\,dx$$

This new integral becomes:

$$\frac{1}{p+1}\int x^p\,dx = \frac{1}{p+1} \cdot \frac{1}{p+1}x^{p+1} + K = \frac{x^{p+1}}{(p+1)^2} + K$$

Combining these results yields:

$$\frac{x^{p+1}}{p+1}\ln(x) - \frac{x^{p+1}}{(p+1)^2} + C = \frac{x^{p+1}}{p+1}\left[\ln(x) - \frac{1}{p+1}\right] + C$$

for any number p as long as $p \neq -1$. ◀

What happens when $p = -1$? Can you evaluate that integral?

Practice 4. Use Example 5 to evaluate $\int x^2 \ln(x)\,dx$ and $\int \ln(x)\,dx$.

Reduction Formulas

Sometimes the result of an integration-by-parts procedure still contains an integral, but a simpler one with a smaller exponent. In these situations we can reuse the resulting **reduction formula** until the remaining integral is simple enough to integrate completely.

Example 6. Evaluate $\int x^n e^x \, dx$ and use the result to evaluate $\int x^2 e^x \, dx$.

Solution. Set $u = x^n$ so $dv = e^x \, dx$, $du = nx^{n-1} \, dx$ and $v = e^x$. The Integration by Parts formula gives

$$\int x^n e^x \, dx \; = \; x^n e^x - n \int x^{n-1} e^x \, dx$$

which is a reduction formula because we have reduced the power of x by 1, trading $\int x^n e^x \, dx$ for the "reduced" integral $\int x^{n-1} e^x \, dx$.

Because $\int x^2 e^x \, dx$ matches the general pattern of $\int x^n e^x \, dx$ with $n = 2$, we know that:

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x^1 e^x \, dx$$

The new integral also matches the pattern in the reduction formula (with $n = 1$ this time) so we know that:

$$\int x^1 e^x \, dx = x^1 e^x - 1 \cdot \int x^0 e^x \, dx$$

This last integral is just $\int e^x \, dx = e^x + K$, so combining our results:

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x^1 e^x \, dx = x^2 e^x - 2 \left[x e^x - \int e^x \, dx \right] \\ &= x^2 e^x - 2x e^x + 2e^x + C \end{aligned}$$

We can also write the answer as $e^x \left[x^2 - 2x + 2 \right] + C$. ◀

Practice 5. Develop the reduction formula:

$$\int x^n \sin(x) \, dx \; = \; -x^n \cos(x) + n \int x^{n-1} \cos(x) \, dx$$

using integration by parts.

The Reappearing Integral

Sometimes the integral we are trying to evaluate shows up on both sides of the equation during our calculations in such a way that we can solve for the desired integral algebraically.

Example 7. Evaluate $\int e^x \cos(x) \, dx$.

Solution. Let $u = e^x$, so $dv = \cos(x) \, dx$, $du = e^x \, dx$ and $v = \sin(x)$:

$$\int e^x \cos(x) \, dx \; = \; e^x \sin(x) - \int e^x \sin(x) \, dx$$

The new integral does not look any easier than the original one, but it doesn't look any worse, so let's try to evaluate the new integral using integration by parts again. To evaluate $\int e^x \sin(x)\,dx$, let $u = e^x$ and $dv = \sin(x)\,dx$ so that $du = e^x\,dx$ and $v = -\cos(x)$, giving us:

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx$$

Putting this result back into the original problem, we get:

$$\begin{aligned}\int e^x \cos(x)\,dx &= e^x \sin(x) - \int e^x \sin(x)\,dx \\ &= e^x \sin(x) - \left[-e^x \cos(x) + \int e^x \cos(x)\,dx\right] \\ &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx\end{aligned}$$

Note that $\int e^x \cos(x)\,dx$ appears on both sides of this last equation, so we can solve for that expression algebraically:

$$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) + K$$

We need an arbitrary constant on the right side of this equation because the left side is an indefinite integral.

and, finally, divide both sides by 2 to get:

$$\int e^x \cos(x)\,dx = \frac{1}{2}\left[e^x \sin(x) + e^x \cos(x)\right] + C$$

$C = \frac{K}{2}$

which we can also write as $\frac{1}{2}e^x\left[\sin(x) + \cos(x)\right] + C$. ◀

Practice 6. Evaluate $\int e^x \sin(x)\,dx$.

A Useful Shortcut

Repeated application of integration by parts, as in Example 6, can quickly become tedious, even with the aid of a reduction formula. For integrals of the form $\int x^n \cdot f(x)\,dx$ where n is an integer, it is often possible to arrange the integration-by-parts ingredients in a table to allow much speedier computation.

Example 8. Evaluate $\int x^2 e^{3x}\,dx$.

Solution. Make a table (see margin) with two columns. In the second entry of the left column, start with $u = x^2$ and below it list the successive derivatives of x^2: $2x$, 2 and 0 (you can stop when you get to 0). In the right column start with $v' = e^{3x}$ and below it list successive antiderivatives of e^{3x}: $\frac{1}{3}e^{3x}$, $\frac{1}{9}e^{3x}$ and $\frac{1}{27}e^{3x}$. Now multiply the functions in each row and add the results, alternating signs:

		e^{3x}	$= v'$
$u =$	x^2	$\frac{1}{3}e^{3x}$	$(+)$
	$2x$	$\frac{1}{9}e^{3x}$	$(-)$
	2	$\frac{1}{27}e^{3x}$	$(+)$
	0	$\frac{1}{81}e^{3x}$	$(-)$

$$x^2 \cdot \frac{1}{3}e^{3x} - 2x \cdot \frac{1}{9}e^{3x} + 2 \cdot \frac{1}{27}e^{3x} - 0 \cdot \frac{1}{81}e^{3x}$$

We can stop here, because all of the remaining terms will be 0. Now add a constant and you have the result: $\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$. ◀

Apply this new technique to the second integral in Example 6 to verify that you get the same result with the new method. Which method is faster?

This shortcut method only works when the chosen u in the initial integration-by-parts setup is x^n (so that taking several derivatives results in 0) and when the chosen dv is a function like e^{ax} or $\sin(bx)$ so that repeated antidifferentiation is fairly easy. But when this shortcut does apply, it can save you a great deal of time.

Practice 7. Evaluate $\int x^5 \cos(x)\,dx$ and $\int x^3 e^{-2x}\,dx$.

8.2 Problems

Problems 1–6 list one part (u or dv) needed for integration by parts. Find the other part (dv or u), calculate du and v, and apply the Integration by Parts formula to evaluate the integral.

1. $\int 12x \ln(x)\,dx, \quad u = \ln(x)$
2. $\int xe^{-x}\,dx, \quad u = x$
3. $\int x^4 \ln(x)\,dx, \quad dv = x^4\,dx$
4. $\int x \sec^2(3x)\,dx, \quad u = x$
5. $\int x \arctan(x)\,dx, \quad dv = x\,dx$
6. $\int x(5x+1)^{19}\,dx, \quad u = x$

In Problems 7–24, evaluate the integral.

7. $\int_0^1 \frac{x}{e^{3x}}\,dx$
8. $\int_0^1 10xe^{3x}\,dx$
9. $\int x \sec(x)\tan(x)\,dx$
10. $\int_0^\pi 5x \sin(2x)\,dx$
11. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 7x \cos(3x)\,dx$
12. $\int 6x \sin(x^2+1)\,dx$
13. $\int 12x \cos(3x^2)\,dx$
14. $\int x^2 \cos(x)\,dx$
15. $\int_1^3 \ln(2x+5)\,dx$
16. $\int x^3 \ln(5x)\,dx$
17. $\int_1^e (\ln(x))^2\,dx$
18. $\int_1^e \sqrt{x}\ln(x)\,dx$
19. $\int \arcsin(x)\,dx$
20. $\int x^2 e^{5x}\,dx$
21. $\int x \arctan(3x)\,dx$
22. $\int x \ln(x+1)\,dx$
23. $\int_1^2 \frac{\ln(x)}{x}\,dx$
24. $\int_1^2 \frac{\ln(x)}{x^2}\,dx$

25. Write $\sin^n(x) = \sin^{n-1}(x)\cdot\sin(x)$ and use integration by parts to obtain the following reduction formula for $\int \sin^n(x)\,dx$:
$$\frac{1}{n}\left[-\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx\right]$$
26. Write $\cos^n(x) = \cos^{n-1}(x)\cdot\cos(x)$ and use integration by parts to obtain the following reduction formula for $\int \cos^n(x)\,dx$:
$$\frac{1}{n}\left[\cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx\right]$$
27. Use integration by parts to obtain a reduction formula for $\int \sec^n(x)\,dx$.
28. Use integration by parts to obtain a reduction formula for $\int \tan^n(x)\,dx$.

In Problems 29–40, use a result from Problems 25–28 to evaluate the integral.

29. $\int \sin^3(x)\,dx$
30. $\int \sin^4(x)\,dx$
31. $\int \sin^5(x)\,dx$
32. $\int \cos^3(x)\,dx$
33. $\int \cos^4(x)\,dx$
34. $\int \cos^5(x)\,dx$
35. $\int \sec^3(x)\,dx$
36. $\int \sec^4(x)\,dx$
37. $\int \sec^5(x)\,dx$
38. $\int \sin^3(5x-2)\,dx$
39. $\int \cos^3(2x+3)\,dx$
40. $\int \sec^3(7x-1)\,dx$

In Problems 41–44, obtain a reduction formula using integration by parts.

41. $\int x^n e^{ax}\,dx$

42. $\int x^n \sin(ax)\,dx$

43. $\int x\,(\ln(x))^n\,dx$

44. $\int x^n \cos(ax)\,dx$

45. The integral $\int x(2x+5)^{19}\,dx$ can be evaluated with integration by parts or by substitution.
   (a) Evaluate the integral using integration by parts with $u = x$ and $dv = (2x+5)^{19}\,dx$.
   (b) Evaluate the integral using a change of variable with $w = 2x+5$.
   (c) Which method is easier?

46. The integral $\int \frac{x}{\sqrt{1+x}}\,dx$ can be evaluated with integration by parts or by substitution.
   (a) Evaluate the integral using integration by parts with $u = x$ and $dv = \frac{1}{\sqrt{1+x}}\,dx$.
   (b) Evaluate the integral using a change of variable with $w = 1+x$.
   (c) Which method is easier?

In Problems 47–68, evaluate the integral using any appropriate method.

47. $\int x\,(\ln(x))^2\,dx$

48. $\int x^2 \arctan(x)\,dx$

49. $\int_0^1 e^{-x}\sin(x)\,dx$

50. $\int_0^1 \frac{\cos(x)}{e^x}\,dx$

51. $\int \sin(\ln(x))\,dx$

52. $\int \cos(\ln(x))\,dx$

53. $\int \cos\left(\sqrt{x}\right)\,dx$

54. $\int \sin\left(\sqrt{x}\right)\,dx$

55. $\int e^{3x}\sin(x)\,dx$

56. $\int e^x \cos(3x)\,dx$

57. $\int_0^\infty xe^{-x}\,dx$

58. $\int_0^\infty x^2 e^{-3x}\,dx$

59. $\int_0^\infty e^{-x}\sin(x)\,dx$

60. $\int_0^\infty e^{-2x}\cos(3x)\,dx$

61. $\int x\sqrt{x+1}\,dx$

62. $\int x\sqrt{x^2+1}\,dx$

63. $\int x\cos(x^2)\,dx$

64. $\int x^2\sqrt{x^3+1}\,dx$

65. $\int x^2\cos(x)\,dx$

66. $\int x^3\sqrt{x^2+1}\,dx$

67. $\int x^3\sqrt[3]{x^2+1}\,dx$

68. $\int x^2\sin(x)\,dx$

In Problems 69–72, solve the initial value problem.

69. $y' = x\sin(x),\ y(0) = 0$

70. $y' = xe^{7x},\ y(0) = 1$

71. $y' = \frac{x}{e^{x+y}},\ y(0) = 1$

72. $y' = x\sin(x)\cos^2(y),\ y(0) = \frac{\pi}{4}$

73. Consider $\int_0^1 x\sin(x)\,dx$ and $\int_0^1 \sin(x)\,dx$.
   (a) Before evaluating the integrals, which do you think is larger? Why?
   (b) Evaluate both integrals. Was your prediction in part (a) correct?

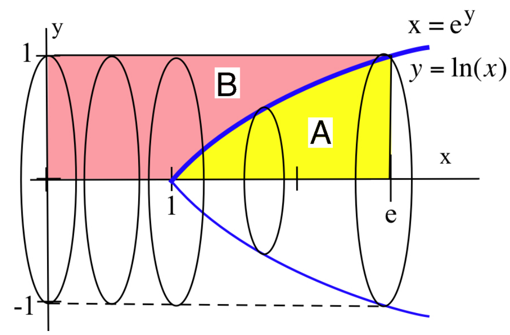
74. Consider $\int_0^\pi x\sin(x)\,dx$ and $\int_0^\pi \sin(x)\,dx$.
   (a) Before evaluating the integrals, which do you think is larger? Why?
   (b) Evaluate both integrals. Was your prediction in part (a) correct?

75. The figure below shows two regions, A and B. The volume swept out when region A is revolved about the x-axis is (using the disk method):

$$\int_{x=1}^{x=e} \pi\,(\ln(x))^2\,dx$$

and the volume swept out when region B is revolved about the x-xis is (using the tube method):

$$\int_{y=0}^{y=1} 2\pi y e^y\,dy$$



   (a) Before evaluating the integrals, which volume do you think is larger? Why?
   (b) Evaluate the integrals. Was your prediction in part (a) correct?

76. Refer to regions A and B from Problem 75.
   (a) Compute the volume of the solid generated when A is revolved around the y-axis.
   (b) Compute the volume of the solid generated when B is revolved around the y-axis.

77. Calculate the volume swept out when the region between the x-axis and the graph of $y = \sin(x)$ for $0 \le x \le \pi$ is rotated about the y-axis.

78. Calculate the volume swept out when the region between the x-axis and the graph of $y = \cos(x)$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the y-axis.

79. Calculate the volume swept out when the region between the x-axis and the graph of $y = x\sin(x)$ for $0 \le x \le \pi$ is rotated about the y-axis.

80. Calculate the volume swept out when the region between the x-axis and the graph of $y = x\cos(x)$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the y-axis.

81. Determine if the area of the region between the graph of $y = xe^{-x}$ and the positive x-axis is finite. (If so, compute the area.)

82. Determine if the area of the region between the graph of $y = x^2e^{-x}$ and the positive x-axis is finite. (If so, compute the area.)

83. Determine if the volume of the solid obtained by revolving the region between the graph of $y = xe^{-x}$ and the positive x-axis about the x-axis is finite. (If so, compute the volume.)

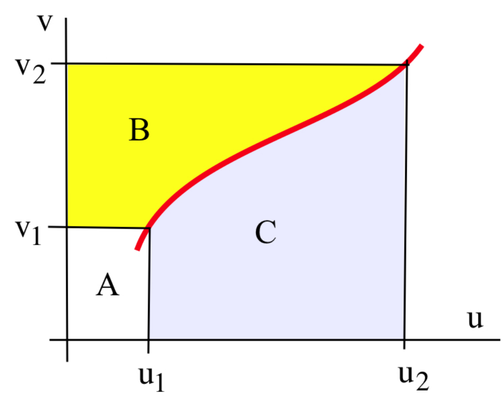
84. Determine if the volume of the solid obtained by revolving the region between the graph of $y = x^2e^{-x}$ and the positive x-axis about the x-axis is finite. (If so, compute the volume.)

85. Determine if the volume of the solid obtained by revolving the region between the graph of $y = xe^{-x}$ and the positive x-axis about the y-axis is finite. (If so, compute the volume.)

86. Determine if the volume of the solid obtained by revolving the region between the graph of $y = x^2e^{-x}$ and the positive x-axis about the y-axis is finite. (If so, compute the volume.)

87. We obtained the Integration by Parts formula analytically, starting with the Product Rule, but the formula also has a geometric interpretation. In the figure below, let D be the large rectangle formed by the regions A, B and C so that:

$$(\text{area of } C) = (\text{area of } D) - (\text{area of } A) - (\text{area of } B)$$



   (a) Represent the area of the large rectangle D as a function of u_2 and v_2.
   (b) Represent the area of the small rectangle A as a function of u_1 and v_1.
   (c) Represent the area of region C as an integral with respect to the variable u.
   (d) Represent the area of region B as an integral with respect to the variable v.
   (e) Rewrite the area equation using the results of parts (a)–(d). This should look familiar.

88. Suppose f and f' are continuous and bounded on the interval $[0, 2\pi]$, meaning that $|f(x)| < M$ and $|f'(x)| < M$ when $0 \le x \le 2\pi$. The n-th Fourier Sine Coefficient of f is defined as:

$$S_n = \int_0^{2\pi} f(x)\sin(nx)\,dx$$

   (a) Use the Integration By Parts Formula with $u = f(x)$ and $dv = \sin(nx)\,dx$ to represent the formula for S_n in a different way.
   (b) Use the new representation of S_n from part (a) to determine what happens to the values of S_n when n is very large ($n \to \infty$). (Hint: $|f'(x)\cos(nx)| = |f'(x)| \cdot |\cos(nx)| < M \cdot 1$.)
   (c) What happens to the n-th Fourier Cosine Coefficients $C_n = \displaystyle\int_0^{2\pi} f(x)\cos(nx)\,dx$ as $n \to \infty$?

8.2 Practice Answers

1. $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$ and $dv = x\,dx \Rightarrow v = \frac{1}{2}x^2$, so:

$$\begin{aligned}\int x\cos(x)\,dx = \int \cos(x)\cdot x\,dx = \int u\,dv = u\cdot v - \int v\,du \\ = \cos(x)\cdot\frac{1}{2}x^2 - \int \frac{1}{2}x^2\left[-\sin(x)\right]\,dx \\ = \frac{1}{2}x^2\cos(x) + \int \frac{1}{2}x^2\sin(x)\,dx\end{aligned}$$

resulting in a new integral worse than the original integral.

2. (a) With $u = x$, $dv = \sin(x)\,dx$ so $du = dx$ and $v = -\cos(x)$:

$$\begin{aligned}\int x\sin(x)\,dx = \int u\,dv = u\cdot v - \int v\,du \\ = x\cdot(-\cos(x)) - \int \left[-\cos(x)\right]\,dx \\ = -x\cos(x) + \int \cos(x)\,dx = -x\cos(x) + \sin(x) + C\end{aligned}$$

(b) With $u = x$, $dv = e^{5x}\,dx$ so $du = dx$ and $v = \frac{1}{5}e^{5x}$:

$$\begin{aligned}\int xe^{5x}\,dx = \int u\,dv = u\cdot v - \int v\,du \\ = x\cdot\frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x}\,dx = \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C\end{aligned}$$

3. Let $u = \ln(x)$ and $dv = dx$ so that $du = \frac{1}{x}\,dx$ and $v = x$:

$$\begin{aligned}\int \ln x\,dx = \int u\,dv = u\cdot v - \int v\,du \\ = \ln(x)\cdot x - \int x\cdot\frac{1}{x}\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C\end{aligned}$$

Using this result:

$$\int_1^e \ln x\,dx = \left[x\ln(x) - x\right]_1^e = \left[e\ln(e) - e\right] - \left[1\ln(1) - 1\right] = 1$$

4. With $n = 2$: $\displaystyle\int x^2\ln(x)\,dx = \frac{x^3}{3}\left[\ln(x) - \frac{1}{3}\right] + C$

With $n = 0$: $\displaystyle\int \ln(x)\,dx = \frac{x^1}{1}\left[\ln(x) - \frac{1}{1}\right] + C = x\ln(x) - x + C$

5. Set $u = x^n$ and $dv = \sin(x)\,dx$ so $du = nx^{n-1}\,dx$ and $v = -\cos(x)$:

$$\begin{aligned}\int x^n\sin(x)\,dx = \int u\,dv = uv - \int v\,du \\ = -x^n\cos(x) - \int \left[-\cos(x)\right]\cdot nx^{n-1}\,dx \\ = -x^n\cos(x) + n\int x^{n-1}\cos(x)\,dx\end{aligned}$$

6. Proceeding as in Example 7, set $u = e^x$ so that $dv = \sin(x)\,dx$. Then $du = e^x\,dx$ and $v = -\cos(x)\,dx$, yielding:

$$\begin{aligned}\int e^x \sin(x)\,dx &= \int u\,dv = uv - \int v\,du \\ &= e^x \cdot (-\cos(x)) - \int (-\cos(x)) \cdot e^x\,dx \\ &= -e^x \cos(x) + \int e^x \cos(x)\,dx\end{aligned}$$

For this new integral, set $u = e^x$ so that $dv = \cos(x)\,dx$. Then $du = e^x\,dx$ and $v = \sin(x)\,dx$, yielding:

$$\int e^x \cos(x)\,dx = \int u\,dv = uv - \int v\,du = e^x \sin(x) - \int e^x \sin(x)\,dx$$

Combining these results we get:

$$\begin{aligned}\int e^x \sin(x)\,dx &= -e^x \cos(x) + \int e^x \cos(x)\,dx \\ &= -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,dx\end{aligned}$$

and solving for the integral we started with yields:

$$2\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x) + K$$

so that:

$$\int e^x \sin(x)\,dx = \frac{1}{2}\left[-e^x \cos(x) + e^x \sin(x)\right] + C$$

7. For the first integral, let $u = x^5$ and take derivatives until you get 0:

$$x^5 \longrightarrow 5x^4 \longrightarrow 20x^3 \longrightarrow 60x^2 \longrightarrow 120x \longrightarrow 120 \longrightarrow 0$$

		$\cos(x)$	$= v'$
$u =$	x^5	$\sin(x)$	$(+)$
	$5x^4$	$-\cos(x)$	$(-)$
	$20x^3$	$-\sin(x)$	$(+)$
	$60x^2$	$\cos(x)$	$(-)$
	$120x$	$\sin(x)$	$(+)$
	120	$-\cos(x)$	$(-)$
	0	$-\sin(x)$	$(+)$

Put these in the left column of the margin table. Then set $v' = \cos(x)$ and start taking antiderivatives:

$$\cos(x) \longrightarrow \sin(x) \longrightarrow -\cos(x) \longrightarrow -\sin(x) \longrightarrow \cos(x)$$

and put these in the right column of the margin table. Now multiply the entries in each and add, using alternating signs:

$$\begin{aligned}\int x^5 \cos(x)\,dx =& x^5 \sin(x) - 5x^4\left[-\cos(x)\right] + 20x^3\left[-\sin(x)\right] \\ & - 60x^2 \cos(x) + 120x \sin(x) - 120\left[-\cos(x)\right] + C\end{aligned}$$

which we can rewrite as:

$$\left[x^5 - 20x^3 + 120x\right]\sin(x) + \left[5x^4 - 60x^2 + 120\right]\cos(x) + C$$

Using the same method with the second integral yields:

$$x^3\left[-\frac{1}{2}e^{-2x}\right] - 3x^2\left[\frac{1}{4}e^{-2x}\right] + 6x\left[-\frac{1}{8}e^{-2x}\right] - 6\left[\frac{1}{16}e^{-2x}\right] + C$$

so $\displaystyle\int x^3 e^{-2x}\,dx = -e^{-2x}\left[\frac{1}{2}x^3 + \frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{8}\right] + C.$