

8.3 Partial Fraction Decomposition

Rational functions (polynomials divided by polynomials) and their integrals play important roles in mathematics and applications, but if you look through an integral table (such as Appendix I) you will find very few formulas for antiderivatives of rational functions. Partly this is because the general formulas are rather complicated and have many special cases, and partly it is because they can all be reduced to just a few cases using the algebraic technique discussed in this section: Partial Fraction Decomposition.

In algebra you learned to add rational functions to get a single rational function (just as in arithmetic you learned how to add many fractions to get a single fraction). Partial Fraction Decomposition is a technique for reversing that procedure to “decompose” a single rational function into a sum of simpler rational functions. This allows us to turn the integral of a single rational function into the sum of integrals of simpler functions.

Example 1. Verify that the algebraic decomposition:

$$\frac{17x - 35}{2x^2 - 5x} = \frac{7}{x} + \frac{3}{2x - 5}$$

is true and use this fact to evaluate $\int \frac{17x - 35}{2x^2 - 5x} dx$.

Solution. Working from the right side of the above equality, combine the two rational expressions by converting to a common denominator:

$$\frac{7}{x} + \frac{3}{2x - 5} = \frac{7}{x} \cdot \frac{(2x - 5)}{(2x - 5)} + \frac{3}{(2x - 5)} \cdot \frac{x}{x} = \frac{14x - 35 + 3x}{2x^2 - 5x} = \frac{17x - 35}{2x^2 - 5x}$$

which is the expression on the left side. This decomposition allows us to exchange the original integral for two much easier ones:

$$\int \frac{17x - 35}{2x^2 - 5x} dx = \int \frac{7}{x} dx + \int \frac{3}{2x - 5} dx = 7 \ln(|x|) + \frac{3}{2} \ln(|2x - 5|) + C$$

which can also be written as $\ln(|x|^7 \cdot |2x - 5|^{\frac{3}{2}}) + C$. ◀

Practice 1. Verify that the algebraic decomposition:

$$\frac{7x - 11}{3x^2 - 8x - 3} = \frac{4}{3x + 1} + \frac{1}{x - 3}$$

is true and use this fact to evaluate $\int \frac{7x - 11}{3x^2 - 8x - 3} dx$.

Example 1 illustrates how to use a “decomposed” fraction to find an antiderivative of a rational function, but it does not show how to achieve this decomposition. The algebraic basis for the Partial Fraction Decomposition technique relies on the Fundamental Theorem of

You might think that the Fundamental of Algebra would be easier to prove than the Fundamental Theorem of Calculus, because you studied algebra before calculus. Although we have already proved the Fundamental Theorem of Calculus in Chapter 4, a proof of the Fundamental Theorem of Algebra requires some higher-powered math.

Now might be a good time to review polynomial division if you have not used it recently.

Algebra, which guarantees that every polynomial with integer coefficients can be factored into a product of linear factors of the form $ax + b$ and irreducible quadratic factors of the form $ax^2 + bx + c$ (with $b^2 - 4ac < 0$). Unfortunately, the Fundamental Theorem of Algebra does not tell us *how* to find these factors, which typically will be more complicated than the examples in this section, but every polynomial has such factors. Before we apply the Partial Fraction Decomposition technique, the fraction must have the following form:

- (degree of the numerator) < (degree of the denominator)
- The denominator has been factored into a product of linear factors and irreducible quadratic factors.

If the first assumption is not true, we can use polynomial division until we get a remainder with a smaller degree than the denominator. If the second assumption is not true, we simply cannot use the Partial Fraction Decomposition technique.

Example 2. Put each fraction into a form ready for Partial Fraction Decomposition:

$$(a) \frac{2x^2 + 4x - 6}{x^2 - 2x} \quad (b) \frac{3x^3 - 3x^2 - 9x + 8}{x^2 - x - 6} \quad (c) \frac{7x^2 + 12x - 12}{x^3 - 4x}$$

Solution. (a) The degree of the numerator is not lower than the degree of the denominator, so we need to use polynomial division to rewrite the rational function:

$$\frac{2x^2 + 4x - 6}{x^2 - 2x} = 2 + \frac{8x - 6}{x^2 - 2x} = 2 + \frac{8x - 6}{x(x - 2)}$$

and conclude by factoring the denominator.

(b) The degree of the numerator is bigger than the degree of the denominator, so we need to use polynomial division to rewrite the rational function:

$$\frac{3x^3 - 3x^2 - 9x + 8}{x^2 - x - 6} = 3x + \frac{9x + 8}{x^2 - x - 6} = 3x + \frac{9x + 8}{(x + 2)(x - 3)}$$

and conclude by factoring the denominator.

(c) Here the degree of the numerator is smaller than the degree of the denominator so we need only factor the denominator:

$$\frac{7x^2 + 12x - 12}{x^3 - 4x} = \frac{7x^2 + 12x - 12}{x(x + 2)(x - 2)}$$

and no polynomial division is required. ◀

Distinct Linear Factors

If you can factor the denominator into a product of distinct linear factors, then it turns out you can write the original fraction as the sum of fractions of the form $\frac{\text{number}}{\text{linear factor}}$. Your job is to find the numbers in the numerators, and that requires solving a system of equations.

We won't prove you can always do this, but you should be able to understand *why* this technique works by examining the solutions of Examples 3 and 4.

Example 3. Find values for A and B so that:

$$\frac{17x - 35}{x(2x - 5)} = \frac{A}{x} + \frac{B}{2x - 5}$$

Solution. Multiply both sides of this equation by the common denominator $x(2x - 5)$ to get:

$$17x - 35 = A(2x - 5) + Bx = 2Ax - 5A + Bx = (2A + B)x - 5A$$

The leftmost and rightmost expressions in this equality are both linear functions, and we want them to be equal for all values of x , so the coefficients of x must be the same: $17 = 2A + B$. Similarly, the constant terms must be the same: $-5A = -35 \Rightarrow A = 7$. Putting this into the previous equation:

$$17 = 2A + B \Rightarrow 17 = 2(7) + B \Rightarrow B = 3$$

so we now know that:

$$\frac{17x - 35}{x(2x - 5)} = \frac{7}{x} + \frac{3}{2x - 5}$$

which agrees with what we saw in Example 1. ◀

Practice 2. Find values of A and B so that:

$$\frac{6x - 7}{(x + 3)(x - 2)} = \frac{A}{x + 3} + \frac{B}{x - 2}$$

In general, there will be one unknown coefficient for each distinct linear factor of the denominator. If the number of distinct linear factors is large, we would need to solve a large system of equations for the unknowns. For any situation involving only distinct linear factors, however, a useful shortcut exists.

Example 4. Find values for A , B and C so that:

$$\frac{2x^2 + 7x + 9}{x(x + 1)(x + 3)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x + 3}$$

Solution. Multiplying both sides of the above equation by the common denominator $x(x + 1)(x + 3)$ yields:

$$2x^2 + 7x + 9 = A(x + 1)(x + 3) + Bx(x + 3) + Cx(x + 1)$$

At this point, you could combine the expressions on the right side of this new equation into one big quadratic polynomial and compare its coefficients to those of the quadratic polynomial on the left side, resulting in three equations in three unknowns. Or you could plug in some well-chosen values of x to avoid much of that algebra. If $x = 0$:

$$2 \cdot 0^2 + 7 \cdot 0 + 9 = A(0+1)(0+3) + B \cdot 0 \cdot (0+3) + C \cdot 0 \cdot (0+1)$$

so $9 = 3A \Rightarrow A = 3$. Similarly, if $x = -1$:

$$2(-1)^2 + 7(-1) + 9 = A(0)(2) + B(-1)(2) + C(-1)(0) \Rightarrow 4 = -2B$$

so $B = -2$. And if $x = -3$:

$$2(-3)^2 + 7(-3) + 9 = A(-2)(0) + B(-3)(0) + C(-3)(-2) \Rightarrow 6 = 6C$$

so $C = 1$. We now know that:

$$\frac{2x^2 + 7x + 9}{x(x+1)(x+3)} = \frac{3}{x} - \frac{2}{x+1} + \frac{1}{x+3}$$

and can easily integrate each of these three simpler fractions. ◀

Practice 3. Evaluate the integral $\int \frac{2x^2 + 7x + 9}{x(x+1)(x+3)} dx$.

Practice 4. Apply the method of Example 4 to the Partial Fraction Decomposition in Example 3.

For fractions whose denominators contain irreducible quadratic factors or repeated factors, the form of the decomposition becomes more complicated—and with it, the algebra required to find the values of the constants. We will not discuss why the following suggestions work to decompose more general rational functions, except to note that they provide enough (but not too many) unknown coefficients.

Distinct Irreducible Quadratic Factors

If the factored denominator includes a distinct irreducible quadratic factor, then the Partial Fraction Decomposition sum contains a fraction of the form:

$$\frac{\text{linear polynomial}}{\text{irreducible quadratic factor}} \quad \text{or} \quad \frac{Ax + B}{x^2 + px + q}$$

where we typically need to solve a system of equations to find the values of the unknown coefficients A and B , given p and q with $p^2 - 4q < 0$.

Example 5. Find values for A , B and C so that:

$$\frac{x^2 + 3x - 15}{(x^2 + 2x + 5)x} = \frac{Ax + B}{x^2 + 2x + 5} + \frac{C}{x}$$

Solution. Multiply the equation on both sides by the common denominator $(x^2 + 2x + 5)x$ to get:

$$x^2 + 3x - 15 = (Ax + B)x + C(x^2 + 2x + 5)$$

Put $x = 0$ into the equation to get:

$$-15 = (B)(0) + C(5) \Rightarrow C = -3$$

Unfortunately, there are no other numbers we can use for x to make terms on the right side vanish, but we can plug in any other “nice” number we want. The “nicest” numbers after 0 are 1 and -1 :

$$\begin{aligned} x = 1: \quad -11 &= (A + B) - 3(8) && \Rightarrow A + B = 13 \\ x = -1: \quad -17 &= (-A + B)(-1) - 3(4) && \Rightarrow A - B = -5 \end{aligned}$$

We now have two equations in two unknowns. Adding these equations yields $2A = 8 \Rightarrow A = 4$ and subtracting the second equation from the first yields $2B = 18 \Rightarrow B = 9$. This tells us that:

$$\frac{x^2 + 3x - 15}{(x^2 + 2x + 5)x} = \frac{4x + 9}{x^2 + 2x + 5} - \frac{3}{x}$$

The original function is now in a form that is ready for integration. ◀

Practice 5. Evaluate the integral $\int \frac{x^2 + 3x - 15}{(x^2 + 2x + 5)x} dx$.

In general, there are two unknown coefficients for each distinct irreducible quadratic factor in the denominator.

Example 6. Decompose the rational function $\frac{6x^3 + 36x^2 + 50x + 53}{(x^2 + 4)(x^2 + 4x + 5)}$.

Solution. The degree of the denominator (4) is bigger than the degree of the numerator (3) and fortunately the denominator has already been factored into two irreducible quadratics. We can write:

$$\frac{6x^3 + 36x^2 + 50x + 53}{(x^2 + 4)(x^2 + 4x + 5)} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{x^2 + 4x + 5}$$

Multiplying by the common denominator $(x^2 + 4)(x^2 + 4x + 5)$ yields:

$$\begin{aligned} 6x^3 + 36x^2 + 50x + 53 &= (Ax + B)(x^2 + 4x + 5) + (Cx + D)(x^2 + 4) \\ &= Ax^3 + 4Ax^2 + 5Ax + Bx^2 + 4Bx + 5B + Cx^3 + 4Cx + Dx^2 + 4D \\ &= (A + C)x^3 + (4A + B + D)x^2 + (5A + 4B + 4C)x + (5B + 4D) \end{aligned}$$

Compare coefficients between the first and last polynomials to see that $A + C = 6$, $4A + B + D = 36$, $5A + 4B + 4C = 50$ and $5B + 4D = 53$. After much algebra (see margin note), this system of four equations in four unknowns reduces to $A = 6$, $B = 5$, $C = 0$ and $D = 7$. ◀

The denominator in unfactored form is:

$$x^4 + 4x^3 + 9x^2 + 16x + 20$$

Would you be able to factor this into two irreducible quadratic factors if it had not already been factored for you?

To simplify the algebra, note that:

$$C = 6 - A \text{ and } D = \frac{1}{4}(53 - 5B)$$

then substitute these expressions into the second and third equations to reduce a system of four equations in four unknowns to a system of two equations in two unknowns.

Repeated Factors

If the factored denominator contains a linear factor raised to a power (greater than one), the decomposition requires one unknown coefficient for each power of the linear factor. For example:

$$\frac{\text{something}}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

Similarly, if the factored denominator contains an irreducible quadratic factor raised to a power (greater than one), then the decomposition requires one linear term (with two unknown coefficients) for each power of the irreducible quadratic. For example:

$$\frac{\text{something}}{x^2(x^2+9)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9} + \frac{Ex+F}{(x^2+9)^2} + \frac{Gx+H}{(x^2+9)^3}$$

This leads to a system of 8 equations with 8 unknowns.

Example 7. Evaluate $\int \frac{-4x^2 + 5x + 3}{x^3 - 2x^2 + x} dx$.

Solution. Before we can integrate, we need to decompose the integrand into simpler rational functions. The degree of the denominator (3) is bigger than the degree of the numerator (2) but we do need to factor:

$$x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$$

so that we can write:

$$\frac{-4x^2 + 5x + 3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiply this equation by the common denominator $x(x-1)^2$ to get:

$$-4x^2 + 5x + 3 = A(x-1)^2 + Bx(x-1) + Cx$$

Inserting the values $x = 0$ or $x = 1$ yields:

$$x = 0: \quad 3 = (A)(-1)^2 + B(0)(-1) + C(0) \Rightarrow A = 3$$

$$x = 1: \quad 4 = A(0) + B(1)(0) + C(1) \Rightarrow C = 4$$

Using $A = 3$ and $C = 4$ with $x = -1$ results in:

$$-6 = 3(-2)^2 + B(-1)(-2) - 4 \Rightarrow 2B = -14 \Rightarrow B = -7$$

We can now integrate:

$$\begin{aligned} \int \frac{-4x^2 + 5x + 3}{x^3 - 2x^2 + x} dx &= \int \left[\frac{3}{x} - \frac{7}{x-1} + \frac{4}{(x-1)^2} \right] dx \\ &= 3 \ln(|x|) - 7 \ln(|x-1|) - \frac{4}{x-1} + C \end{aligned}$$

which can also be written $\ln \left(\frac{|x|^3}{|x-1|^7} \right) - \frac{4}{x-1} + C$. ◀

Practice 6. Evaluate $\int \frac{2x^2 + 27x + 85}{(x + 5)^2} dx$.

Be careful: Partial Fraction Decomposition only works with *rational* functions, although you may be able to turn another type of integrand into a rational function using substitution. For example, we can write:

$$\int \sec(\theta) d\theta = \int \frac{1}{\cos(\theta)} d\theta = \int \frac{\cos(\theta)}{\cos^2(\theta)} d\theta = \int \frac{\cos(\theta)}{1 - \sin^2(\theta)} d\theta$$

and substitute $u = \sin(\theta) \Rightarrow du = \cos(\theta) d\theta$ to convert this to:

$$\int \sec(\theta) d\theta = \int \frac{\cos(\theta)}{1 - \sin^2(\theta)} d\theta = \int \frac{1}{1 - u^2} du$$

which is now a job for partial fractions.

Problem 41 asks you to complete this integration and resubstitute to obtain the antiderivative pattern for $\sec(\theta)$.

Additional Applications

The primary use of the partial fraction technique in this course is to convert rational functions into a form that makes them easier to integrate, but this algebraic technique can also be used to simplify the differentiation of certain rational functions. (In a later math course, you will also use partial fractions when computing certain inverse Laplace transforms.) The next Example illustrates the use of partial fractions to make a differentiation problem easier.

Example 8. For $f(x) = \frac{2x + 13}{x^2 + x - 2}$, calculate $f'(x)$, $f''(x)$ and $f'''(x)$.

Solution. You already know how to calculate these derivatives using the Quotient Rule, but that process becomes rather tedious for the second and third derivatives of this function. Instead, we can use the partial fraction technique to rewrite f as:

$$f(x) = \frac{5}{x - 1} - \frac{3}{x + 2} = 5(x - 1)^{-1} - 3(x + 2)^{-1}$$

Computing the derivatives is now quite straightforward:

$$\begin{aligned} f'(x) &= -5(x - 1)^{-2} + 3(x + 2)^{-2} \\ f''(x) &= 10(x - 1)^{-3} - 6(x + 2)^{-3} \\ f'''(x) &= -30(x - 1)^{-4} + 18(x + 2)^{-4} \end{aligned}$$

You will appreciate the value of this shortcut if you attempt to compute $f'''(x)$ using the Quotient Rule. ◀

Practice 7. Use a partial fraction decomposition of $g(x) = \frac{9x + 1}{x^2 - 2x - 3}$ to calculate $g'(x)$, $g''(x)$ and $g^{(4)}(x)$.

The rational functions in the Examples and Problems in this section have been carefully constructed to allow you to easily factor denominators and to decompose fractions with a minimum of effort. In practice, factoring higher-order polynomials can require numerical methods (like Newton's Method) to approximate roots and decomposing rational functions whose denominators include repeated linear irreducible quadratic factors can lead to systems of many equations in many unknowns that benefit from the tools available in linear algebra. As a result, computers are best suited to handle the partial fraction decomposition of more complicated rational functions.

8.3 Problems

In Problems 1–32, decompose the integrand and then evaluate the integral.

$$\begin{array}{ll}
 1. \int \frac{7x+2}{x(x+1)} dx & 2. \int \frac{7x+9}{(x+3)(x-1)} dx \\
 3. \int \frac{11x+25}{x^2+9x+8} dx & 4. \int \frac{3x+7}{x^2-1} dx \\
 5. \int \frac{2x^2+15x+25}{x^2+5x} dx & 6. \int \frac{3x^3+3x^2}{x^2+x-2} dx \\
 7. \int \frac{6x^2+9x-15}{x(x+5)(x-1)} dx & 8. \int \frac{6x^2-x-1}{x^3-x} dx \\
 9. \int \frac{8x^2-x+3}{x^3+x} dx & 10. \int \frac{9x^2+13x+15}{x^3+2x^2-3x} dx \\
 11. \int \frac{11x^2+23x+6}{x^2(x+2)} dx & 12. \int \frac{6x^2+14x-9}{x(x+3)^2} dx \\
 13. \int \frac{3x+13}{(x+2)(x-5)} dx & 14. \int \frac{2x+11}{(x-7)(x-2)} dx \\
 15. \int_2^5 \frac{2}{x^2-1} dx & 16. \int_1^3 \frac{5x^2+5x+3}{x^3+x} dx \\
 17. \int \frac{2x^2+5x-3}{x^2-1} dx & 18. \int \frac{2x^2+19x+22}{x^2+x-12} dx \\
 19. \int \frac{3x^2+19x+24}{x^2+6x+5} dx & 20. \int \frac{7x^2+8x-2}{x^2+2x} dx \\
 21. \int \frac{3x^2-1}{x^3+x} dx & 22. \int \frac{x^4+5x^3+x-15}{x^2+5x} dx \\
 23. \int \frac{x^3+3x^2-4x+30}{x^2+3x-10} dx &
 \end{array}$$

$$\begin{array}{ll}
 24. \int \frac{7x^3+x^2+7x+10}{x^4+2x^3} dx & \\
 25. \int \frac{12x^2+19x-6}{x^3+3x^2} dx & 26. \int \frac{2x+5}{(x+1)^2} dx \\
 27. \int \frac{7x^2+3x+7}{x^3+x} dx & 28. \int \frac{7x^2-4x+4}{x^3+1} dx \\
 29. \int_2^\infty \frac{2}{x^2-1} dx & 30. \int_2^\infty \frac{7x+2}{6x^2-13x+16} dx \\
 31. \int \frac{6x^2+5x+61}{(x-1)(x^2+4x+13)} dx & \\
 32. \int \frac{x-84}{(x+5)(x^2-6x+34)} dx &
 \end{array}$$

Integrals can be very sensitive to small changes in the integrand. In 33–34, notice how similar functions require vastly different integration methods.

$$\begin{array}{ll}
 33. (a) \int \frac{1}{x^2+2x+2} dx & 34. (a) \int \frac{1}{x^2-6x+8} dx \\
 (b) \int \frac{1}{x^2+2x+1} dx & (b) \int \frac{1}{x^2-6x+9} dx \\
 (c) \int \frac{1}{x^2+2x+0} dx & (c) \int \frac{1}{x^2-6x+10} dx
 \end{array}$$

In Problems 35–40, use a partial fraction decomposition to compute the first and second derivatives of the given function.

$$\begin{array}{ll}
 35. f(x) = \frac{7x+2}{x(x+1)} & 36. F(x) = \frac{7x+9}{(x+3)(x-1)} \\
 37. g(x) = \frac{11x+25}{x^2+9x+8} & 38. G(x) = \frac{3x+7}{x^2-1} \\
 39. h(x) = \frac{2x^2+15x+25}{x^2+5x} & 40. H(x) = \frac{9x^2+13x+15}{x^3+2x^2-3x}
 \end{array}$$

41. Obtain a formula for $\int \sec(\theta) d\theta$ by writing:

$$\int \frac{1}{\cos(\theta)} d\theta = \int \frac{\cos(\theta)}{\cos^2(\theta)} d\theta = \int \frac{\cos(\theta)}{1 - \sin^2(\theta)} d\theta$$

and using the substitution $u = \sin(\theta)$ to turn this integrand into a rational function.

42. Obtain a formula for $\int \csc(\theta) d\theta$ by writing:

$$\int \frac{1}{\sin(\theta)} d\theta = \int \frac{\sin(\theta)}{\sin^2(\theta)} d\theta = \int \frac{\sin(\theta)}{1 - \cos^2(\theta)} d\theta$$

and using the substitution $u = \cos(\theta)$ to turn this integrand into a rational function.

Logistic Growth

The following two applications involve a type of differential equation that can be solved by separating variables and then using a partial fraction decomposition to calculate the antiderivatives. The same type of differential equation is also used to model the spread of rumors and diseases, as well as the growth of some populations and chemical reactions.

The growth rate of many different types of populations depends not only on the number of individuals currently in the population (leading to exponential growth) but also on a “carrying capacity” of the environment. (As a population grows, it might deplete a food source, slowing the growth of the population.) If x is the size of a population at time t and the growth rate of x is proportional only to the size of the population, we get the differential equation:

$$\frac{dx}{dt} = kx$$

which we investigated in Section 6.4. If we want to create a model in which the growth slows as the size of the population nears the carrying capacity M , we can multiply this model for $\frac{dx}{dt}$ by factor $\left(1 - \frac{x}{M}\right)$:

$$\frac{dx}{dt} = kx \left(1 - \frac{x}{M}\right)$$

When x is small (relative to M), this new factor is close to 1 so that:

$$\frac{dx}{dt} = kx \left(1 - \frac{x}{M}\right) \approx kx$$

leading to growth that is “almost exponential.” When x gets close to M , this new factor is close to 0:

$$\frac{dx}{dt} = kx \left(1 - \frac{x}{M}\right) \approx kx \left(1 - \frac{M}{M}\right) = 0$$

so that the growth of x slows down and x is (roughly) constant. We call this differential equation the **logistic equation** and its solution a **logistic function**.

43. Let $k = 1$ and $M = 100$, and assume the initial population is $x(0) = 5$.
- Create a direction field for the ODE.
 - Solve the corresponding logistic IVP.
 - Graph the population $x(t)$ for $0 \leq t \leq 20$.
 - When will the population be 20? 50? 90? 100?
 - What is the population after a "long" time? (Find the limit of x , as $t \rightarrow \infty$.)
 - Explain the shape of the graph in (c) in the context of a population of bacteria.
 - When is the growth rate largest?
 - What is the population when the growth rate is largest?
 - What would happen if $x(0) > 100$?
44. Let $k = 1$ and $M = 100$, and assume the initial population is $x(0) = 150$.
- Solve the corresponding logistic IVP.
 - Graph the population $x(t)$ for $0 \leq t \leq 20$.
 - When will the population be 120? 110? 100?
 - What is the population after a "long" time?
 - Explain the shape of the graph in (b).
45. Let k and M be positive constants, and assume the initial population is $x(0) = x_0$.
- Solve the corresponding logistic IVP.
 - What is the population after a "long" time?
 - When is the growth rate largest?
 - What is the population at that time?

Chemical Reactions

In certain chemical reactions, a new material X is formed from materials A and B , and the rate at which X forms is proportional to the product of the amount of A and the amount of B remaining. Let x represent the amount of material X present at time t , and assume that the reaction begins with a grams of A , b grams of B and no material X (so that $x(0) = 0$). Then the rate of formation of material X can be described by the differential equation:

$$\frac{dx}{dt} = k(a - x)(b - x)$$

46. Solve the IVP given above for x if $k = 1$ and the reaction begins with
- 7 grams of A and 5 grams of B .
 - 6 grams of A and 6 grams of B .
47. Solve the IVP given above for x if $k = 1$ and the reaction begins with
- a grams of A and b grams of B with $a \neq b$.
 - c grams of A and c grams of B (for $c > 0$).

8.3 Practice Answers

1. To verify the identity:

$$\frac{4}{(3x+1)} \cdot \frac{(x-3)}{(x-3)} + \frac{1}{(x-3)} \cdot \frac{(3x+1)}{(3x+1)} = \frac{(4x-12) + (3x+1)}{(3x+1)(x-3)} = \frac{7x-11}{3x^2-8x-3}$$

To evaluate the integral:

$$\int \frac{7x-11}{3x^2-8x-3} dx = \int \left[\frac{4}{(3x+1)} + \frac{1}{(x-3)} \right] dx = \frac{4}{3} \ln(|3x+1|) + \ln(|x-3|) + C$$

2. Multiply both sides of the equality by the common denominator
- $(x+3)(x-2)$
- to get:

$$6x-7 = A(x-2) + B(x+3) = Ax-2A+Bx+3B = (A+B)x + (-2A+3B)$$

so $A+B=6$ and $-2A+3B=-7$. Solving this system of equations yields $A=5$ and $B=1$.

3. Using the result of Example 4:

$$\int \left[\frac{3}{x} - \frac{2}{x+1} + \frac{1}{x+3} \right] dx = 3 \ln(|x|) - 2 \ln(|x+1|) + \ln(|x+3|) + C$$

4. Multiply each side of the equation by the common denominator
- $x(2x-5)$
- to get:

$$\frac{17x-35}{x(2x-5)} = \frac{A}{x} + \frac{B}{2x-5} \Rightarrow 17x-35 = A(2x-5) + Bx$$

Now put $x=0$ and $x=\frac{5}{2}$ into this new equation to get:

$$x=0: \quad -35 = -5A + 0 \Rightarrow A = 7$$

$$x=\frac{5}{2}: \quad \frac{15}{2} = A(0) + B\left(\frac{5}{2}\right) \Rightarrow B = 3$$

5. Using the result of Example 5:

$$\begin{aligned} \int \left[\frac{4x+9}{x^2+2x+5} - \frac{3}{x} \right] dx &= \int \left[\frac{4x+8+1}{(x+1)^2+1} - \frac{3}{x} \right] dx \\ &= \int \left[\frac{2(2x+2)}{x^2+2x+5} + \frac{5}{(x+1)^2+4} - \frac{3}{x} \right] dx \\ &= 2 \ln(x^2+2x+5) + \frac{1}{2} \arctan\left(\frac{x+1}{2}\right) - 3 \ln(|x|) + C \end{aligned}$$

6. Because the degree of the numerator equals the degree of the denominator, we must use polynomial division to rewrite the integrand as:

$$\frac{2x^2+27x+85}{x^2+10x+25} = 2 + \frac{7x+35}{x^2+10x+25} = 2 + \frac{7(x+5)}{(x+5)^2} = 2 + \frac{7}{x+5}$$

This integrand does not require partial fraction decomposition:

$$\int \frac{2x^2+27x+85}{(x+5)^2} dx = \int \left[2 + \frac{7}{x+5} \right] dx = 2x + 5 \ln(|x+5|) + C$$

7. First write:

$$g(x) = \frac{9x+1}{x^2-2x-3} = \frac{9x+1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

and then multiply through the last equality by the common denominator $(x-3)(x+1)$ to get:

$$9x+1 = A(x+1) + B(x-3)$$

Now put $x = 3$ and $x = -1$ into this new equation to get:

$$x = 3: \quad 28 = 4A + B(0) \Rightarrow A = 7$$

$$x = -1: \quad -8 = A(0) + B(-4) \Rightarrow B = 2$$

We can now rewrite $g(x)$ as $g(x) = 7(x-3)^{-1} + 2(x+1)^{-1}$ so that:

$$g'(x) = -7(x-3)^{-2} - 2(x+1)^{-2}$$

$$g''(x) = 14(x-3)^{-3} + 4(x+1)^{-3}$$

$$g'''(x) = -42(x-3)^{-4} - 12(x+1)^{-4}$$

$$g^{(4)}(x) = 168(x-3)^{-5} + 48(x+1)^{-5}$$