8.7 MacLaurin Polynomials

In this chapter you have learned to find antiderivatives of a wide variety of elementary functions, but many more such functions fail to have an antiderivative that can be expressed in terms of other elementary functions. By this point, you should easily be able to find antiderviatives for $\sin(x)$, $x \cdot \sin(x)$, $x \cdot \sin(x^2)$ or $x \cdot e^{-x^2}$, but no matter how hard you try, you won't be able to find antiderivatives for $\sin(x^2)$, $\sin(x^3)$ or e^{-x^2}.

An **elementary function** is a function that can be expressed using a finite number of compositions or combinations of exponential functions, logarithms, trigonometric and inverse trig functions, polynomials and constants, using sums, products and exponentiation.

In Section 2.8, you used linear approximations (tangent lines) to approximate values of complicated functions. In this section, we will investigate how to use polynomials — slightly more complicated, but still relatively "nice" functions — to approximate integrands such as $\sin(x^2)$ and then approximate values of definite integrals such as $\int_0^1 \sin(x^2)\,dx$. This approach will motivate the exploration of many of the concepts to follow in the next two chapters.

Proving that you can't find elementary antiderviatives of integrands such as $\sin(x^2)$ turns out to be quite complicated.

Polynomials

Consider any (non-vertical) line: if you know its y-intercept, b, and its slope, m, you can write down an equation of the line: $y = b + mx$. If we write $y = f(x)$, then $f(0) = b + m \cdot 0 = b$ and $f'(0) = m$, so an equation of *any* linear function is completely determined by its value at $x = 0$ and the value of its first derivative at $x = 0$. It turns out that if you know the values of any polynomial $P(x)$ and all of its derivatives at $x = 0$, you can use those values to find a formula for $P(x)$.

Example 1. If $P(x)$ is a cubic polynomial with $P(0) = 7$, $P'(0) = 5$, $P''(0) = 16$ and $P'''(0) = 18$, find a formula for $P(x)$.

Because $P(x)$ is a cubic, its derivatives of order four and higher are all 0.

Solution. Because $P(x)$ is a cubic polynomial, we can write it as:

$$P(x) = A + Bx + Cx^2 + Dx^3$$

for some numbers A, B, C and D. We know that $P(0) = 7$, and substituting $x = 0$ into the formula above tells us that $P(0) = A$, so $A = 7$. We also know that $P'(0) = 5$, while:

$$P'(x) = B + 2Cx + 3Dx^2 \Rightarrow P'(0) = B$$

so $B = 5$. Similarly, we know that $P''(0) = 16$ while:

$$P''(x) = 2C + 3 \cdot 2Dx \Rightarrow P''(0) = 2C$$

so $2C = 16 \Rightarrow C = \frac{16}{2} = 8$. Finally, we know that $P'''(0) = 18$ while:

$$P'''(x) = 3 \cdot 2 \cdot D \Rightarrow P'''(0) = 6D \Rightarrow 6D = 18 \Rightarrow D = 3$$

Therefore $P(x) = 7 + 5x + 8x^2 + 3x^3$. (You should verify that this cubic polynomial and its derivatives have the values specified above.) ◀

Practice 1. If $P(x)$ is a fourth-degree polynomial with $P(0) =$ ˘3, $P'(0) = 4$, $P''(0) = 10$, $P'''(0) = 12$ and $P^{(4)}(0) = 24$, find a formula for $P(x)$.

Now consider a general polynomial of order 5 with coefficients a_0, a_1, a_2, a_3, a_4, a_5:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

Observe that $P(0) = a_0$ and then differentiate to get:

$$P'(x) = a_1 + 2 \cdot a_2 x + 3 \cdot a_3 x^2 + 4 \cdot a_4 x^3 + 5 \cdot a_5 x^4$$

Putting $x = 0$ into this new equation tells us that $P'(0) = a_1$. Differentiating again yields:

$$P''(x) = 2 \cdot a_2 + 3 \cdot 2 \cdot a_3 x + 4 \cdot 3 \cdot a_4 x^2 + 5 \cdot 4 \cdot a_5 x^3$$

so that $P''(0) = 2 \cdot a_2$. Differentiating again:

$$P'''(x) = 3 \cdot 2 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4 x + 5 \cdot 4 \cdot 3 \cdot a_5 x^2$$

so that $P'''(0) = 3 \cdot 2 \cdot a_3$. Continuing this process, $P^{(4)}(0) = 4 \cdot 3 \cdot 2 \cdot a_4$ and $P^{(5)}(0) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot a_5$. In general, we can write $P^{(k)}(0) = k! \cdot a_k$, where:

We call this expression $k!$ "k **factorial**," defined for any positive integer k.

$$k! = k \cdot (k-1) \cdot (k-2) \cdots 3 \cdot 2 \cdot 1$$

and $k = 1, 2, 3, 4$ or 5. If we define $0! = 1$, then the equation $P^{(k)}(0) = k! \cdot a_k$ holds for $k = 0$ as well (the 0-th derivative of a function is just the function itself). And for any integer $k \geq 6$, $a_k = 0$ and $P^{(k)}(x) = 0$, so for any integer $k \geq 0$ we have:

$$P^{(k)}(0) = k! \cdot a_k \;\Rightarrow\; a_k = \frac{P^{(k)}(0)}{k!}$$

Practice 2. If $P(x)$ is a fourth-degree polynomial with $P(0) =$ ˘3, $P'(0) = 4$, $P''(0) = 10$, $P'''(0) = 12$ and $P^{(4)}(0) = 24$, find a formula for $P(x)$ using the above formula, then compare with your answer to Practice 1.

There was nothing special about the degree $n = 5$ of the polynomial in the preceding discussion. The formula $a_k = \dfrac{P^{(k)}(0)}{k!}$ holds for *any* polynomial of the form:

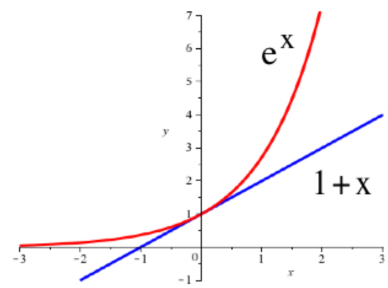
$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

And there was nothing special about the "base point" $x = 0$: we can develop similar formulas if we know the values of a function and all of its derivatives at $x = 1$ or $x = -3$ or $x = \sqrt{7}$ (but we'll stick with $x = 0$ for now, to keep things simple).

Using Polynomials to Approximate Functions

Given any function $f(x)$, we know that a tangent-line approximation to this function at $x = a$ is:

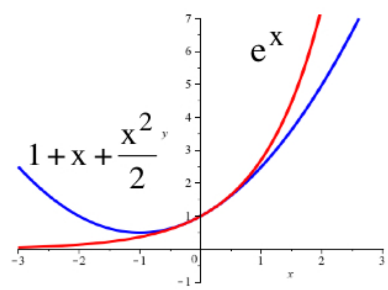
$$L(x) = f(a) + f'(a) \cdot (x - a)$$



If $a = 0$, this becomes $L(x) = f(0) + f'(0) \cdot x$. For $f(x) = e^x$, $f(0) = 1$ and $f'(x) = e^x \Rightarrow f'(0) = 1$, so $L(x) = 1 + x$ (see margin for a graph comparing $L(x)$ and $f(x)$). For values of x very close to 0, we can see that $L(x) = 1 + x \approx e^x = f(x)$ is a decent approximation, but for values of x not close to 0, we need a better approximation.

For the linear approximation $L(x)$, its 0-th derivative agrees with the 0-th derivative of $f(x)$ at $x = 0$: $L(0) = 1 = e^0 = f(0)$. Likewise, the first derivatives agree at $x = 0$: $L'(0) = 1 = e^0 = f'(0)$. But the second derivatives do not agree: $L''(0) = 0 \neq 1 = e^0 = f''(0)$. Can we find a reasonably simple function whose 0-th, first and second derivatives at $x = 0$ match those of $f(x) = e^x$? The next simplest function after a linear function is a quadratic function, so let's try $Q(x) = A + Bx + Cx^2$, for which $Q'(x) = B + 2Cx$ and $Q''(x) = 2C$. We need:

$$\begin{aligned} Q(0) = f(0) &\Rightarrow A = e^0 = 1 \\ Q'(0) = f'(0) &\Rightarrow B = e^0 = 1 \\ Q''(0) = f''(0) &\Rightarrow 2C = e^0 = 1 \Rightarrow C = \frac{1}{2} \end{aligned}$$



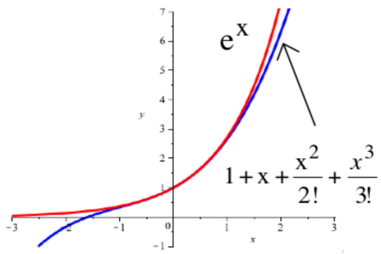
so $Q(x) = 1 + x + \frac{1}{2}x^2$ (see margin for a graph of $Q(x)$ and $f(x)$). While $L(x)$ did a decent job of approximating $f(x) = e^x$ on the interval $[-0.2, 0.2]$, our quadratic approximation $Q(x)$ appears to do a nice job on the interval $[-1, 1]$, but could we do better?

A cubic polynomial is not much more complicated than a quadratic, so let's look for something of the form $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. We want derivatives 0 through 3 of our polynomial to match derivatives 0 through 3 of the function $f(x) = e^x$. We know that:

$$f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow f''(x) = e^x \Rightarrow f'''(x) = e^x$$

so that $f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$ and $f'''(0) = 1$. We therefore need $P(0) = 1$, $P'(0) = 1$, $P''(0) = 1$ and $P'''(0) = 1$. From our work earlier in this section we know this requires that $a_k = \dfrac{P^{(k)}(0)}{k!}$ for $k = 0$, 1, 2 and 3. Thus:

$$\begin{aligned} a_0 &= \frac{P(0)}{0!} = \frac{1}{1} = 1 \\ a_1 &= \frac{P'(0)}{1!} = \frac{1}{1} = 1 \end{aligned}$$



$$a_2 = \frac{P''(0)}{2!} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$
$$a_3 = \frac{P'''(0)}{3!} = \frac{1}{3 \cdot 2 \cdot 1} = \frac{1}{6}$$

which tells us that $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ (see margin for a graphs of $P(x)$ and $f(x)$). This polynomial appears to approximate $f(x) = e^x$ quite well on an even bigger interval.

Could we do even better? You may notice a pattern in our work above. If we used a fourth-order polynomial, then we would also need:

$$a_4 = \frac{P^{(4)}(0)}{4!} = \frac{f^{(4)}(0)}{4!} = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{24}$$

so $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$ would approximate e^x even better. In general, for any positive integer n, we would have:

$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n$$

Named after Scottish mathematician Colin MacLaurin (1698–1746).

We call this a **MacLaurin polynomial** of order n for $f(x) = e^x$.

Example 2. Find a MacLaurin polynomial with three nonzero terms for $f(x) = \sin(x)$.

Solution. We want to find a polynomial of the form:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

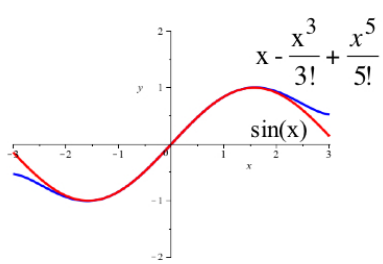
so that $P^{(k)}(0) = f^{(k)}(0)$ for three nonzero values. We know that:

$$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x)$$
$$\Rightarrow f'''(x) = -\cos(x) \Rightarrow f^{(4)}(x) = \sin(x)$$

and that this pattern then repeats. This tells us that:

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{(4)}(0) = 0,\ f^{(5)}(0) = 1$$

so that $P(0) = f(0) = 0$, $P'(0) = f'(0) = 1$, $P''(0) = f''(0) = 0$, $P'''(0) = f'''(0) = -1$, $P^{(4)}(0) = f^{(4)}(0) = 0$ and $P^{(5)}(0) = f^{(5)}(0) = 1$. Using the formula $a_k = \frac{P^{(k)}(0)}{k!}$ yields:



$$P(x) = \frac{0}{0!} + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

so that $P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ should do the job. (See the margin for a graph of this polynomial compared to $\sin(x)$.) ◀

Practice 3. Find a MacLaurin polynomial with five nonzero terms for $f(x) = \sin(x)$.

Practice 4. Find a MacLaurin polynomial with three nonzero terms for $f(x) = \cos(x)$.

Applications of MacLaurin Polynomials

In Section 7.6, we developed a concrete definition of the exponential function $\exp(x) = e^x$, but this definition did not provide us with a useful way to compute values of e^x (other than $e^0 = 1$). To estimate the value of $e = e^1$ using the definition of e^x, we would need to use the numerical techniques of Section 4.9 to evaluate $L(b) = \int_1^b \frac{1}{t}\, dt$ for various values of b until we found a value for which $L(b) \approx 1$.

In this section, however, we have found polynomials that closely approximate e^x, so we can evaluate one of these polynomials at $x = 1$ to approximate e:

$$e = e^1 \approx 1 + 1 + \frac{1}{2!} \cdot 1^2 + \frac{1}{3!} \cdot 1^3 + \frac{1}{4!} \cdot 1^4 + \frac{1}{5!} \cdot 1^5 + \frac{1}{6!} \cdot 1^6 \approx 2.178$$

Using higher-degree MacLaurin polynomials for e^x will result in even better approximations of e.

Practice 5. Use a MacLaurin polynomial to approximate $\frac{1}{e}$ and $\sqrt{e}$.

Practice 6. Use a MacLaurin polynomial to approximate $\sin(1)$ and compare you approximation to what your calculator reports for $\sin(1)$.

In Section 4.9, we learned various numerical integration techniques to approximate values of definite integrals. With enough computing power available, techniques such as the Trapezoidal Rule and Simpson's Rule allow you to approximate values of definite integrals such as $\int_0^1 e^{-x^2}\, dx$ or $\int_0^1 \sin(x^2)\, dx$ (for which it is impossible to use the Fundamental Theorem of Calculus unless we can think of an antiderivative of the integrand). Unfortunately, these approximation methods require you (or a computer) to evaluate the integrand at many different values of x. Now that we know how to approximate transcendental functions with polynomials, we might first approximate a complicated integrand with a "nice" polynomial and then integrate the polynomial instead of the transcendental function.

Example 3. Approximate the value of $\int_0^1 e^{-x^2}\, dx$.

Solution. We don't know how to find an antiderivatve for e^{-x^2}, so we can't use the Fundamental Theorem of Calculus. But we already know a MacLaurin polynomial for e^u:

$$e^u \approx 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \frac{1}{5!}u^5$$

into which we can substitute $u = -x^2$ to get:

$$e^{-x^2} \approx 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10}$$

and use this polynomial to approximate the integrand:

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &\approx \int_0^1 \left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10}\right] dx \\ &= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 - \frac{1}{1320}x^{11}\right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} \approx 0.7467\end{aligned}$$

The Trapezoidal Rule with $n = 25$ yields a similar approximation, but requires many more computations. ◀

Practice 7. Approximate the value of $\int_0^1 \sin(x^2)\,dx$.

How Good Are These Approximations?

Section 4.9 included (without proof) error bounds for the Trapezoidal Rule and Simpson's Rule that provided guarantees for how closely the results of these numerical methods approximated the exact values of the integrals we were trying to compute. We can—and will—state (and prove) a similar error bound for the MacLaurin polynomial approximations we have learned how to use in this section, but we will defer that discussion until Chapter 10.

8.7 Problems

In Problems 1–6, $P(x) = Ax + B$ is a linear polynomial, with the values of $P(0)$ and $P'(0)$ given. Find A and B and then write a formula for $P(x)$.

1. $P(0) = 5$, $P'(0) = 3$
2. $P(0) = -2$, $P'(0) = 7$
3. $P(0) = 4$, $P'(0) = -1$
4. $P(0) = 8$, $P'(0) = 5$
5. $P(0) = 4$, $P'(0) = 0$
6. $P(0) = -3$, $P'(0) = -2$
7. If $P(x) = A + Bx$, how are the values of A and B related to the values of $P(0)$ and $P'(0)$?

In 8–13, $P(x) = A + Bx + Cx^2$ is a quadratic polynomial, with values of $P(0)$, $P'(0)$ and $P''(0)$ given. Find A, B and C, then write a formula for $P(x)$.

8. $P(0) = 5$, $P'(0) = 3$, $P''(0) = 4$
9. $P(0) = -2$, $P'(0) = 7$, $P''(0) = 6$
10. $P(0) = 4$, $P'(0) = -1$, $P''(0) = -2$
11. $P(0) = 8$, $P'(0) = 5$, $P''(0) = 10$
12. $P(0) = 4$, $P'(0) = 0$, $P''(0) = -4$
13. $P(0) = -3$, $P'(0) = -2$, $P''(0) = 4$
14. If $P(x) = A + Bx + Cx^2$, how are the values of A, B and C related to $P(0)$, $P'(0)$ and $P''(0)$?

In Problems 15–20, $P(x) = A + Bx + Cx^2 + Dx^3$ is a cubic polynomial, with values of $P(0)$, $P'(0)$, $P''(0)$ and $P'''(0)$ given. Find the values of A, B, C and D, and then write a formula for $P(x)$.

15. $P(0) = 5$, $P'(0) = 3$, $P''(0) = 4$, $P'''(0) = 6$
16. $P(0) = -2$, $P'(0) = 7$, $P''(0) = 6$, $P'''(0) = 18$
17. $P(0) = 4$, $P'(0) = -1$, $P''(0) = -2$, $P'''(0) = -12$
18. $P(0) = 8$, $P'(0) = 5$, $P''(0) = 10$, $P'''(0) = 12$
19. $P(0) = 4$, $P'(0) = 0$, $P''(0) = -4$, $P'''(0) = 36$
20. $P(0) = -3$, $P'(0) = -2$, $P''(0) = 4$, $P'''(0) = 36$

21. If $P(x) = A + Bx + Cx^2 + Dx^3$, how are the values of A, B, C and D related to the values of $P(0)$, $P'(0)$, $P''(0)$ and $P'''(0)$?

In Problems 22–28, fill in the table below for $f(x)$ and $P(x)$, then graph $f(x)$ and $P(x)$ for $-2 \le x \le 2$.

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x) \rvert$
0.0			
0.1			
0.2			
0.3			
1.0			
2.0			

22. $f(x) = \sin(x)$, $P(x) = x - \frac{1}{2 \cdot 3}x^3$
23. $f(x) = \sin(x)$, $P(x) = x - \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}x^5$
24. $f(x) = \cos(x)$, $P(x) = 1 - \frac{1}{2}x^2$
25. $f(x) = \cos(x)$, $P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 3 \cdot 4}x^4$
26. $f(x) = e^x$, $P(x) = 1 + x + \frac{1}{2}x^2$
27. $f(x) = e^x$, $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2 \cdot 3}x^3$
28. $f(x) = e^x$, $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 3 \cdot 4}x^4$

In 29–38, find a MacLaurin polynomial with four nonzero terms that approximates the given function.

29. e^{2x}
30. $\sin(2x)$
31. $\dfrac{1}{1-x}$
32. $\dfrac{1}{1+x}$
33. $\ln(1+x)$
34. $\sqrt{1+x}$
35. $\cos\left(x^2\right)$
36. e^{-x^2}
37. $x^3 \cdot \sin\left(x^2\right)$
38. $x \cdot e^{-x^2}$

In 39–42, use a MacLaurin polynomial with four nonzero terms to approximate the value of the definite integral.

39. $\displaystyle\int_0^1 \sin\left(x^3\right)\,dx$
40. $\displaystyle\int_0^{\frac{1}{2}} \cos\left(x^2\right)\,dx$
41. $\displaystyle\int_0^{\frac{1}{2}} e^{-x^3}\,dx$
42. $\displaystyle\int_0^1 x \cdot \sin\left(x^3\right)\,dx$

8.7 Practice Answers

1. With $P(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$, $P'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 \Rightarrow P''(x) = 2C + 6Dx + 12Ex^2 \Rightarrow P'''(x) = 6D + 24Ex \Rightarrow P^{(4)}(x) = 24E$ so $3 = P(0) = A \Rightarrow A = 3$, $4 = P'(0) = B \Rightarrow B = 4$, $10 = P''(0) = 2C \Rightarrow C = 5$, $12 = P'''(0) = 6D \Rightarrow D = 2$ and $24 = P^{(4)}(0) = 24E \Rightarrow E = 1$, hence:

$$P(x) = 3 + 4x + 5x^2 + 2x^3 + x^4$$

2. With $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, $a_0 = \frac{P(0)}{0!} = \frac{3}{1} = 3$, $a_1 = \frac{P'(0)}{1!} = \frac{4}{1} = 4$, $a_2 = \frac{P''(0)}{2!} = \frac{10}{2} = 5$, $a_3 = \frac{P'''(0)}{3!} = \frac{12}{6} = 2$ and $a_4 = \frac{P^{(4)}(0)}{4!} = \frac{24}{24} = 1$ so:

$$P(x) = 3 + 4x + 5x^2 + 2x^3 + x^4$$

which agrees with the result of Practice 1.

3. Continuing with the differentiation process from Example 2:

$$f^{(5)}(x) = \cos(x) \Rightarrow f^{(6)}(x) = -\sin(x) \Rightarrow f^{(7)}(x) = -\cos(x)$$
$$\Rightarrow f^{(8)}(x) = \sin(x) \Rightarrow f^{(9)}(x) = \cos(x)$$

so that $f^{(6)}(0) = 0$, $f^{(7)}(0) = -1$, $f^{(8)}(0) = 0$ and $f^{(9)}(0) = 1$. Hence $a_6 = \dfrac{0}{6!} = 0$, $a_7 = \dfrac{-1}{7!} = -\dfrac{1}{5040}$, $a_8 = \dfrac{0}{8!} = 0$ and $a_9 = \dfrac{1}{9!} = 362880$, yielding the polynomial:

$$P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9$$

4. Differentiating $f(x) = \cos(x)$ yields $f'(x) = -\sin(x) \Rightarrow f''(x) = -\cos(x) \Rightarrow f'''(x) = \sin(x) \Rightarrow f^{(4)}(x) = \cos(x)$ so that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$ and $f^{(4)}(0) = 1$, hence $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{-1}{2!} = -\frac{1}{2}$, $a_3 = 0$ and $a_4 = \frac{1}{4!} = \frac{1}{24}$. A MacLaurin polynomial is thus:

Notice that differentiating the result of Example 2 yields the same result.

$$P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

5. Using the MacLaurin polynomial for e^x from the discussion preceding Practice 5 with $x = -1$ yields:

$$\frac{1}{e} = e^{-1} \approx 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \frac{(-1)^5}{5!} + \frac{(-1)^6}{6!}$$
$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = \frac{53}{144} \approx 0.368$$

while using $x = \frac{1}{2}$ approximates $\sqrt{e} = e^{\frac{1}{2}}$ by:

$$1 + \left(\frac{1}{2}\right) + \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \frac{1}{4!}\left(\frac{1}{2}\right)^4 + \frac{1}{5!}\left(\frac{1}{2}\right)^5 + \frac{1}{6!}\left(\frac{1}{2}\right)^6$$
$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840} + \frac{1}{46080} \approx 1.647872$$

6. Using the result of Practice 3:

$$\sin(1) \approx 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \frac{1}{362880} \approx 0.8414710097$$

which agrees to five decimal places with 0.8414709848.

7. Substituting $u = x^2$ into the MacLaurin polynomial from Practice 3 yields:

$$\sin(x^2) \approx x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10} - \frac{1}{5040}x^{14} + \frac{1}{362880}x^{18}$$

Integrating this polynomial from $x = 0$ to $x = 1$ yields:

$$\left[\frac{1}{3}x^3 - \frac{1}{42}x^7 + \frac{1}{1320}x^{11} - \frac{1}{75600}x^{15} + \frac{1}{6894720}x^{19}\right]_0^1$$

which evaluates to (approximately) 0.3102683028.