9.1 *Sequences*

Sequences play important roles in several areas of theoretical and applied mathematics. As you continue your study of mathematics, you will encounter them again and again. In this course, however, their primary role is as a foundation for our study of power series ("big polynomials"). In order to understand how and where it is valid to represent a function such as sine as a power series, we need to examine what it means to add together an infinite number of values. And in order to understand this infinite addition we need to analyze lists of numbers (called **sequences**) and determine whether or not the numbers in the list are converging to a single value. This section examines sequences and how to represent sequences graphically.

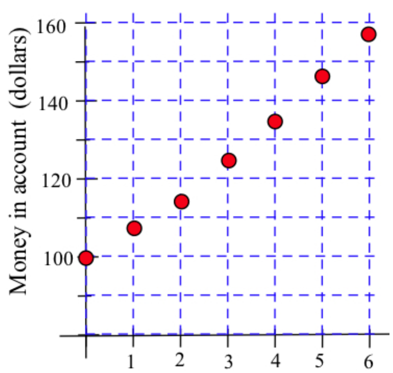
Example 1. You deposit $100 in an account that pays 8% interest at the end of each year. How much money will be in the account at the end of 1 year? 2 years? 3 years? n years?

Solution. After one year, the total amount in the account is the principal (the amount you originally deposited) plus the interest:

$$100 + (0.08)100 = 100(1 + 0.08) = 100(1.08) = \$108$$

or 108% of the amount at the beginning of the first year.

At the end of the second year, the amount is 108% of the amount at the start of the second year:

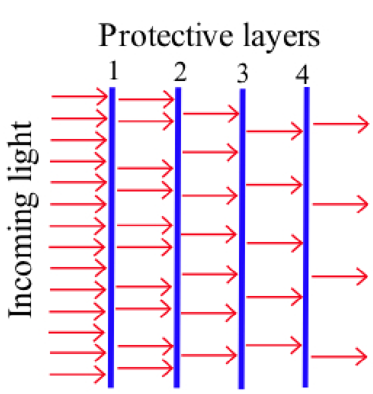


$$(1.08)\,[(1.08)100] = (1.08)^2(100) = \$116.64$$

At the end of the third year, the amount is 108% of the amount at the start of the third year:

$$(1.08)\left[(1.08)^2 100\right] = (1.08)^3(100) = \$125.97$$

See the margin for a graph of these results. In general, at the end of the n-th year, the amount in the account is $(1.08)^n(100)$ dollars. ◀



Practice 1. A layer of protective film transmits two-thirds of the light that reaches that layer (see margin). How much of the incoming light is transmitted through 1 layer? 2 layers? 3 layers? n layers?

The Example and Practice problems above each asked for a list of numbers in a definite order: a first number, then a second number, and so on. Such a list of numbers in a definite order is called a **sequence**. An **infinite sequence** is a sequence that keeps going and has no last number. Often the pattern of a sequence becomes clear from the first few numbers, but in order to precisely specify a sequence, we usually state a rule for finding the value of the n-th term, a_n ("a sub n").

Example 2. List the next two numbers in each sequence and give a rule for calculating the n-th number, a_n:

(a) $1, 4, 9, 16, \ldots$ (b) $-1, 1, -1, 1, \ldots$ (c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Solution. (a) $a_5 = 25$, $a_6 = 36$ and $a_n = n^2$ (b) $a_5 = -1$, $a_6 = 1$ and $a_n = (-1)^n$ (c) $a_5 = \frac{1}{32}$, $a_6 = \frac{1}{64}$ and $a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$. ◀

Practice 2. List the next two numbers in each sequence and give a rule for calculating the n-th number, a_n:

(a) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (b) $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$ (c) $2, 2, 2, 2, \ldots$

Definition and Notation

Because a sequence gives a single value for each integer n, a sequence is a function whose domain is restricted to the integers.

Definition
A **sequence** is a function with a domain consisting of all integers greater than or equal to a starting integer.

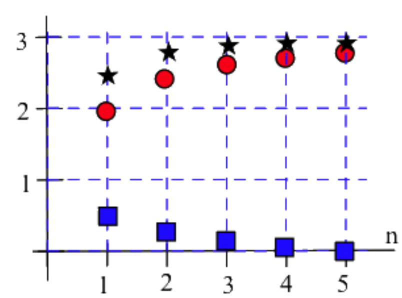
Most sequences we will encounter have a starting integer of 1, but sometimes it is convenient to start with 0 or another integer value.

In many computer languages, arrays (multi-dimensional lists) employ 0-based indexing.

Notation
a_n represents a single number called the n-th term.
$\{a_n\}$ represents the set of all terms in the sequence.
$\{\text{rule}\}$ represents the sequence generated by the rule.
$\{a_n\}_{n=3}$ represents a sequence that starts with $n = 3$.

Because sequences are functions, we can add, subtract, multiply and divide them, and combine them with other functions to form new sequences. We can also graph sequences: these graphs can sometimes help us describe and understand their behavior.

Example 3. For the sequences given by $a_n = 3 - \frac{1}{n}$ and $b_n = \frac{1}{2^n}$, graph the points (n, a_n) and (n, b_n) for $n = 1$ to 5. Compute the first five terms of $c_n = a_n + b_n$ and graph the points (n, c_n).



Solution. $c_1 = \left(3 - \frac{1}{1}\right) + \frac{1}{2^1} = 2.5$, $c_2 = 2.75$, $c_3 \approx 2.792$, $c_4 = 2.8125$ and $c_5 = 2.83125$. The graphs appear in the margin. ◀

Practice 3. For a_n and b_n in the previous example, compute the first five terms of $c_n = a_n - b_n$ and $d_n = (-1)^n b_n$, and graph the points (n, c_n) and (n, d_n).

Recursive Sequences

A **recursive sequence** is a sequence defined by a rule that gives the value of each new term in the sequence as a combination of one or more of the previous terms. We already encountered a recursive sequence when we studied Newton's Method for approximating roots of a function (Section 2.7).

Example 4. Let $f(x) = x^2 - 4$. Take $x_1 = 3$ and apply Newton's method to calculate x_2 and x_3. Give a rule for x_n.

Solution. $f(x) = x^2 - 4 \Rightarrow f'(x) = 2x$; applying Newton's method:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{5}{6} = \frac{13}{6} \approx 2.1667$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{13}{6} - \frac{f\left(\frac{13}{6}\right)}{f'\left(\frac{13}{6}\right)} = \frac{13}{6} - \frac{25}{156} = \frac{313}{156} \approx 2.0064$$

In general:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = x_{n-1} - \frac{(x_{n-1})^2 - 4}{2x_{n-1}}$$

The terms x_1, x_2, x_3, ... appear to be approaching the value 2, one solution of $x^2 - 4 = 0$. The sequence $\{x_n\}$ is a recursive sequence because each x_n is defined as a function of the previous term x_{n-1}. ◄

Successive iterations of a function also generate a recursive sequence.

Practice 4. Let $f(x) = 2x - 1$ and define $a_n = f(f(f(\cdots f(a_0)\cdots)))$, where the function is applied n times. Put $a_0 = 3$ and compute a_1, a_2 and a_3. Note that a_n can be defined recursively as $a_n = f(a_{n-1})$.

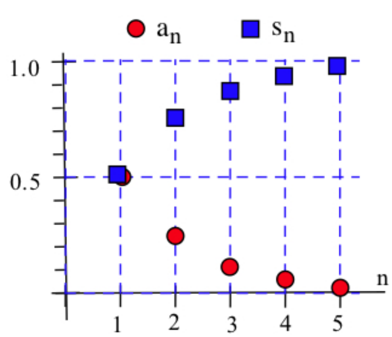
Example 5. Let $a_k = \frac{1}{2^k}$ and define a second sequence $\{s_n\}$ by the rule that s_n is the sum of the first n terms of $\{a_k\}$. Compute the values of s_n for $n = 1$ to 5.

Solution. Computing the first three terms of $\{s_n\}$:

$$s_1 = a_1 = \frac{1}{2}$$

$$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$



Similarly, $s_4 = \frac{15}{16}$ and $s_5 = \frac{31}{32}$. You should notice two patterns in these sums (and in the associated graph—see margin). First, it appears that:

$$s_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

(so we have an example of a sequence defined recursively that can also be written explicitly as a function of n).

Second, you can simplify the addition process: each term s_n is the sum of the previous term s_{n-1} and the a_n term: $s_n = s_{n-1} + a_n$. (For example, check that $s_3 = s_2 + a_3$.) We will meet this second pattern again in Section 9.3. ◀

Practice 5. Let $b_0 = 0$ and, for $n > 0$, define $b_n = b_{n-1} + \frac{1}{3^n}$. Compute b_n for $n = 1$ to 4

9.1 Problems

Problems 1–6 provide the first four numbers a_1, a_2, a_3 and a_4 in a list. (a) Write the next two numbers in the list. (b) Write a formula for the fifth number, a_5, as an expression involving 5. (c) Write a formula for the n-th number, a_n, in terms of n.

1. 2, 4, 8, 16, __, __
2. 3, 9, 27, 81, __, __
3. $-1, 1, -1, 1$, __, __
4. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, __, __
5. 1, 2, 6, 24, __, __
6. 1, 4, 9, 16, __, __

For 7–11, evaluate each of the given expressions and write the next two numbers in the list.

7. $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, __, __
8. $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}, 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}$, __, __
9. $1, 1-\frac{1}{2}, 1-\frac{1}{2}+\frac{1}{4}, 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$, __, __
10. $1, 1+2, 1+2+4, 1+2+4+8$, __, __
11. $1, 1-1, 1-1+1, 1-1+1-1$, __, __

For Problems 12–15, (a) fill in the next two entries in the table for the specified function and (b) plot the values of that function given in the table. (These functions are defined only for integer values of x.)

12. $f(x)$
13. $g(x)$
14. $s(x)$
15. $t(x)$

x	1	2	3	4	5	6
$f(x)$	2	4	8	16		
$g(x)$	-1	1	-1	1		

x	$s(x)$	$t(x)$
1	$1+\frac{1}{2}$	$1-\frac{1}{2}$
2	$1+\frac{1}{2}+\frac{1}{3}$	$1-\frac{1}{2}+\frac{1}{4}$
3	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}$
4	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$	$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$
5		
6		

In Problems 16–21, find a rule that describes the n-th term in the sequence.

16. $1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, \frac{1}{125}, \dots$

17. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots$

18. $-1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \dots$

19. $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

20. $7, 7, 7, 7, 7, \dots$

21. $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \dots$

In 22–33, compute the first six terms (starting with $n = 1$) of each sequence and graph these terms.

22. $\left\{3 + \frac{1}{n^2}\right\}$

23. $\left\{1 - \frac{2}{n}\right\}$

24. $\left\{\frac{\ln(n)}{n}\right\}$

25. $\left\{\frac{n}{2n-1}\right\}$

26. $\{4 + (-1)^n\}$

27. $\left\{3 + \frac{(-1)^n}{n}\right\}$

28. $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}$

29. $\left\{(-1)^n \frac{n-1}{n}\right\}$

30. $\left\{\frac{n+1}{n!}\right\}$

31. $\left\{\frac{1}{n!}\right\}$

32. $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$

33. $\left\{\frac{2^n}{n!}\right\}$

In Problems 34–39, compute the first 10 terms (starting with $n = 1$) of each sequence.

34. $a_1 = 3$ and $a_{n+1} = \frac{1}{a_n}$

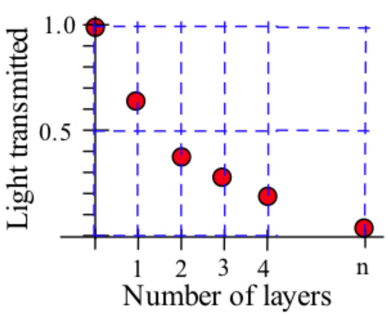
35. $b_1 = 2$ and $b_{n+1} = -b_n$

36. $a_1 = 2$, $a_2 = 3$ and (for $n \geq 3$), $a_n = a_{n-1} - a_{n-2}$

37. $\left\{\sin\left(\frac{2\pi n}{3}\right)\right\}$

38. $c_n =$ the sum of the first n prime numbers (2 is the first prime)

39. $d_n =$ the sum of the first n positive integers



9.1 Practice Answers

1. One layer transmits $\frac{2}{3}$ of the original amount of light. Two layers transmit $\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$ of the original amount of light. Three layers transmit $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$ of the original light, and, in general, n layers transmit $\left(\frac{2}{3}\right)^n$ of the original light.

2. (a) $\frac{1}{5}, \frac{1}{6}, \dots, \frac{1}{n}, \dots$ (b) $-\frac{1}{32}, \frac{1}{64}, \dots, \left(-\frac{1}{2}\right)^n, \dots$ (c) $2, 2, \dots, 2, \dots$

3. $c_1 = \left(3 - \frac{1}{1}\right) - \frac{1}{2} = \frac{3}{2} = 1.5$ $\quad d_1 = (-1)^1\left(\frac{1}{2}\right) = -\frac{1}{2}$

$c_2 = \left(3 - \frac{1}{2}\right) - \frac{1}{2^2} = \frac{9}{4} = 2.25$ $\quad d_2 = (-1)^2\left(\frac{1}{2^2}\right) = \frac{1}{4}$

$c_3 = \left(3 - \frac{1}{3}\right) - \frac{1}{2^3} = \frac{61}{24} \approx 2.542$ $\quad d_3 = (-1)^3\left(\frac{1}{2^3}\right) = -\frac{1}{8}$

$c_4 = \left(3 - \frac{1}{4}\right) - \frac{1}{2^4} = \frac{43}{16} = 2.6875$ $\quad d_4 = \frac{1}{16}, d_5 = -\frac{1}{32}$

$c_5 = \left(3 - \frac{1}{5}\right) - \frac{1}{25} = \frac{443}{160} \approx 2.769$

4. $a_0 = 3$, $a_1 = f(a_0) = 2(3) - 1 = 5$, $a_2 = f(a_1) = 2(5) - 1 = 9$, $a_3 = f(a_2) = 2(9) - 1 = 17$

5. $b_1 = b_0 + \frac{1}{3} = 0 + \frac{1}{3} = \frac{1}{3}$, $b_2 = b_1 + \frac{1}{9} = \frac{4}{9}$, $b_3 = b_2 + \frac{1}{27} = \frac{13}{27}$, $b_4 = b_3 + \frac{1}{81} = \frac{40}{81}$