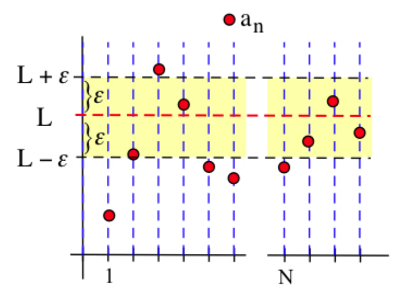
9.2 *Limits of Sequences*

Because sequences are discrete functions defined only for integers, many calculus concepts that hold for continuous functions do not apply to sequences, with one important exception: the limit as n approaches infinity. Do the values a_n eventually approach (or equal) some number?

We say that the **limit** of a sequence $\{a_n\}$ is L if the terms a_n are all arbitrarily close to L for sufficiently large values of n. The terms at the beginning of the sequence can take on any values, but for large values of n, the a_n terms must all be close to L. The following definition makes these concepts of "close" and "large" more precise.

Definition
$\lim_{n\to\infty} a_n = L$ means that for any $\epsilon > 0$ ("epsilon > 0") there is an index N (typically depending on ϵ) so that a_n is within ϵ of L whenever n is larger than N:

$$n > N \;\Rightarrow\; |a_n - L| < \epsilon$$



If a sequence has a finite limit L, we say that the sequence "**converges** to L." If a sequence does not have a finite limit, we say the sequence "**diverges**." Typically a sequence diverges because its terms grow infinitely large (positively or negatively) or because the terms oscillate and do not approach a single value.

Example 1. For $a_n = 3 + \frac{1}{n^2}$ show that $\lim_{n\to\infty} a_n = 3$.

Solution. Given $\epsilon > 0$, we need to find a number N so that $|a_n - L|$ is less than ϵ whenever n is larger than N:

$$n > N \quad\Rightarrow\quad \left|\left(3 + \frac{1}{n^2}\right) - 3\right| < \epsilon$$

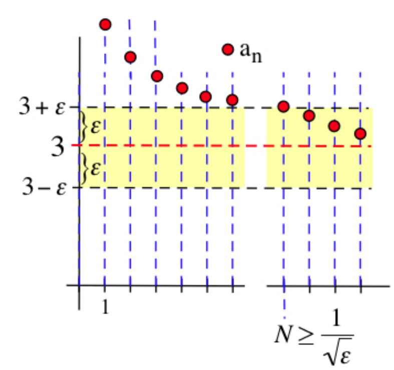
To determine how large N must be, solve this inequality for n:

$$\left|\frac{1}{n^2}\right| < \epsilon \quad\Rightarrow\quad \frac{1}{\epsilon} < n^2 \quad\Rightarrow\quad n > \frac{1}{\sqrt{\epsilon}}$$

Taking $N = \left\lfloor \frac{1}{\sqrt{\epsilon}} \right\rfloor + 1$ (the next integer after $\frac{1}{\sqrt{\epsilon}}$):

$$n > N \quad\Rightarrow\quad n > \left\lfloor \frac{1}{\sqrt{\epsilon}} \right\rfloor + 1 \geq \frac{1}{\sqrt{\epsilon}} \quad\Rightarrow\quad \frac{1}{n^2} < \epsilon$$

so $\left|\left(3 + \frac{1}{n^2}\right) - 3\right| < \epsilon$, as desired. ◀



Practice 1. Show that $\lim_{n\to\infty} \frac{n+1}{n} = 1$.

The limit of a sequence, as n approaches infinity, depends only on the behavior of the terms of the sequence corresponding to "large" values of n (the "tail end" of the sequence) and not on the values of the first few (or few thousand, or few billion) terms. Consequently, we can insert or delete any *finite* number of terms without changing the convergence behavior of the sequence.

Because sequences are functions, limits of sequences possess certain properties of limits of more general functions.

Uniqueness Theorem

If	a sequence converges to a limit
then	the limit is unique:
	a sequence cannot converge to two different values.

Proof. We will use the "proof by contradiction" technique.

Suppose that a sequence $\{a_n\}$ converges to two different limits. Call these limits L_1 and L_2. Let $d = |L_1 - L_2| > 0$ be the distance between L_1 and L_2. (We know $d > 0$ because $L_1 \neq L_2$.)

We chose $\epsilon = \frac{d}{3}$ because it "works" for our purpose by leading to a contradiction. Many other choices (any ϵ less than $\frac{d}{2}$) also lead to the contradiction. Because the definition of limit says "for any $\epsilon > 0$," we picked one we wanted.

Because $\{a_n\}$ converges to L_1, for any $\epsilon > 0$ there is an N_1 so that $n > N_1 \Rightarrow |a_n - L_1| < \epsilon$. If $\epsilon = \dfrac{d}{3}$, then there is an N_1 so that $n > N_1 \Rightarrow |a_n - L_1| < \frac{d}{3}$. Similarly, there is an N_2 so that $n > N_2 \Rightarrow |a_n - L_2| < \frac{d}{3}$.

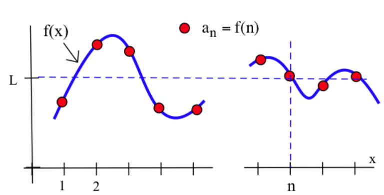
Now let N be the larger of N_1 and N_2. When $n > N$, both $|a_n - L_1| < \dfrac{d}{3}$ and $|a_n - L_2| < \dfrac{d}{3}$ hold, so:

$$d = |L_1 - L_2| = |(a_n - L_2) - (a_n - L_1)|$$

Using the triangle inequality:

$$d \leq |a_n - L_2| + |a_n - L_1| < \frac{d}{3} + \frac{d}{3} = \frac{2}{3}d$$

We have shown that $0 < d < \frac{2}{3}d$, which is impossible, so we can conclude that our original assumption, that $L_1 \neq L_2$, must be false. □



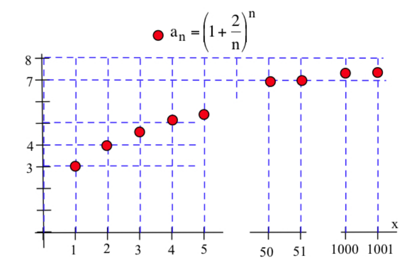
Sometimes it is useful to replace a sequence $\{a_n\}$, a function whose domain consists of integers, with a function f whose domain consists of the real numbers and satisfying $a_n = f(n)$. If $f(x)$ has a limit as "$x \to \infty$" then $f(n)$ has the same limit as "$n \to \infty$." This replacement of x with n allows us to use earlier results about limits of functions, particularly L'Hôpital's Rule, to compute limits of sequences.

Theorem:

If	$a_n = f(n)$ and $\lim\limits_{x\to\infty} f(x) = L$
then	$\{a_n\}$ converges to L: $\lim\limits_{n\to\infty} a_n = L$

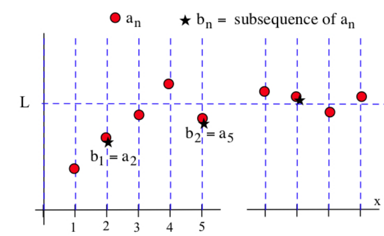
Example 2. Compute $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$.

Solution. With $a_n = \left(1+\frac{2}{n}\right)^n$, we can define $f(x) = \left(1+\frac{2}{x}\right)^x$ so that $a_n = f(n)$. Using L'Hôpital's Rule (as in Example 7 from Section 3.7), we can show that $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2$ and conclude that $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2 \approx 7.389$. ◄



Practice 2. Compute $\lim_{n\to\infty}\frac{\ln(n)}{n}$.

A **subsequence** is an infinite subset of terms from a sequence that occur in the same order as they appear in the original sequence. The sequence of even integers $\{2,4,6,\ldots\}$ is a subsequence of the sequence of all positive integers $\{1,2,3,4,\ldots\}$. The sequence of reciprocals of primes $\left\{\frac{1}{2},\frac{1}{3},\frac{1}{5},\frac{1}{7},\ldots\right\}$ is a subsequence of the sequence of the reciprocals of all positive integers $\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots\right\}$. Subsequences inherit certain properties from their original sequences.



Subsequence Theorem

Every subsequence of a convergent sequence converges to the same limit as the original sequence:

If $\lim_{n\to\infty} a_n = L$ and $\{b_n\}$ is a subsequence of $\{a_n\}$

then $\lim_{n\to\infty} b_n = L$.

The proof is left to you. Use proof by contradiction and start by assuming that $\lim_{n\to\infty} b_n \neq L$. Then for some $\epsilon > 0$, infinitely many terms of $\{b_n\}$ satisfy $|b_n - L| > \epsilon$, hence infinitely many terms of $\{a_n\}$ satisfy $|a_n - L| > \epsilon$.

If the sequence $\{a_n\}$ does *not* converge, then the subsequence $\{b_n\}$ **may or may not** converge.

Corollary

If two subsequences of $\{a_n\}$ converge to two different limits

then $\{a_n\}$ diverges.

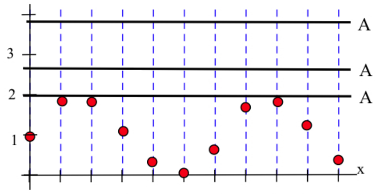
The proof is left to you. Use proof by contradiction and start by assuming that $\lim_{n\to\infty} a_n \neq L$. By the Subsequence Theorem, any two subsequences must also converge to L.

Example 3. Show that the sequence $\left\{(-1)^n\frac{n}{n+1}\right\}$ diverges.

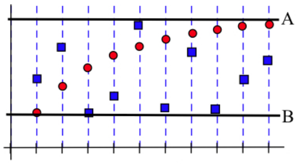
Solution. For the even-indexed terms, $a_n = (-1)^n\frac{n}{n+1} = \frac{n}{n+1}$, so the subsequence of even-indexed terms converges to 1. For the odd-indexed terms, $a_n = (-1)^n\frac{n}{n+1} = \frac{-n}{n+1}$, so the subsequence of odd-indexed terms converges to -1. Because two subsequences converge to different values, the original sequence diverges. ◄

Practice 3. Show that the sequence $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$ diverges.

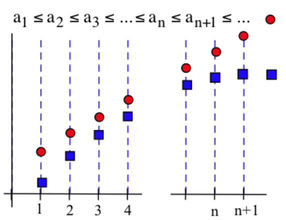
Bounded and Monotonic Sequences



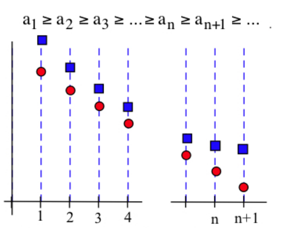
Several upper bounds of the sequence $a_n = 1 + \sin(n)$



Two bounded sequences



Two monotonic increasing sequences



Two monotonic decreasing sequences

A sequence $\{a_n\}$ is **bounded above** if there is a value A so that $a_n \leq A$ for all values of n: we call A an **upper bound** of $\{a_n\}$. Similarly, $\{a_n\}$ is **bounded below** if there is a value B so that $a_n \geq B$ for all values of n: we call B a **lower bound** of the sequence. A sequence is **bounded** if it has an upper bound and a lower bound. All of the terms of a bounded sequence are between (or equal to) the upper and lower bounds.

A **monotonically increasing** sequence is a sequence for which each term is at least as big as the previous term:

$$a_1 \leq a_2 \leq a_3 \leq \ldots$$

A **monotonically decreasing** sequence is one for which each term is no bigger than the previous term:

$$a_1 \geq a_2 \geq a_3 \geq \ldots$$

A monotonic sequence does not oscillate: if one term is larger than a previous term and another term is smaller than a previous term, then the sequence is neither monotonic increasing nor monotonic decreasing.

There are three common ways to demonstrate a sequence is monotonically increasing. You can do this by showing that:

- $a_{n+1} \geq a_n$ for all n
- $a_n > 0$ and that $\frac{a_{n+1}}{a_n} \geq 1$ for all n
- $a_n = f(n)$ for integer values n and some differentiable function f for which $f'(x) \geq 0$ for all $x > 0$

Practice 4. List three ways to demonstrate that a sequence is monotonically decreasing.

Example 4. Show that $\{a_n\} = \left\{\frac{2^n}{n!}\right\}$ is monotonically decreasing.

Solution. We will use the second technique from Practice 4 (we can't use the third technique—why not?) and write:

$$a_n = \frac{2^n}{n!} = \frac{2^n}{1 \cdot 2 \cdot 3 \cdots n}$$

$$a_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2^n \cdot 2}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)}$$

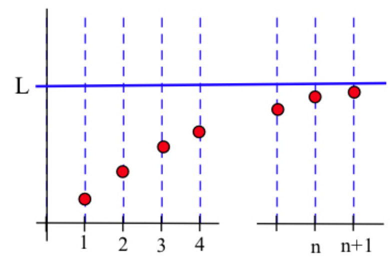
so that:

$$\frac{a_{n+1}}{a_n} = \frac{2^n \cdot 2}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)} \cdot \frac{1 \cdot 2 \cdot 3 \cdots n}{2^n} = \frac{2}{n+1}$$

When $n \geq 1$, then $n+1 \geq 2$ so that $\frac{2}{n+1} \leq 1$. ◀

Practice 5. Show that $\left\{ \left(\frac{2}{3} \right)^n \right\}$ is monotonically decreasing.

Because the behavior of a monotonic sequence is so "regular," with a bit more information we can determine whether or not a monotonic sequence has a finite limit: we just need to show that it is bounded.

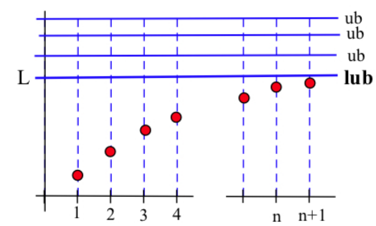


The idea for a monotonically decreasing sequence is similar.

Monotone Convergence Theorem

If	a monotonic sequence is bounded
then	the sequence converges.

Idea for a proof. If a monotonically increasing sequence $\{a_n\}$ is bounded above, then $\{a_n\}$ has an infinite number of upper bounds, each of them larger than every a_n. Let L be the smallest of these upper bounds (the **least upper bound**) of $\{a_n\}$. Then there is a value a_N as close as we want to L (otherwise there would be an upper bound smaller than L), so given $\epsilon > 0$, find an N so that $|a_N - L| < \epsilon$. Because $\{a_n\}$ is increasing, all of the later values in the sequence (with $n \geq N$), must satisfy $a_n \geq a_N$; because L is an upper bound for $\{a_n\}$, it must also be true that $\{a_n\} \leq L$ for $n > N$. So if $n > N$, then $a_N \leq a_n \leq L$, hence:



$$a_N - L \leq a_n - L \leq 0 \quad \Rightarrow \quad |a_n - L| \leq |a_N - L| < \epsilon$$

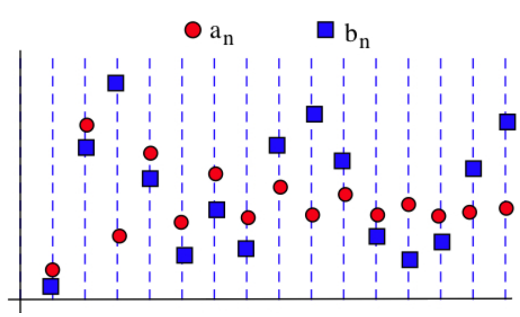
Cauchy and other mathematicians accepted this theorem on intuitive and geometric grounds similar to the "idea for a proof" given above, but later mathematicians felt more rigor was needed. Yet even the mathematician Dedekind, who supplied much of that rigor, recognized the usefulness of geometric intuition.

"Even now such resort to geometric intuition in a first presentation of differential calculus, I regard as exceedingly useful, from a didactic standpoint, and indeed indispensable if one does not wish to lose too much time." (Dedekind, *Essays on the Theory of Numbers*, 1901)

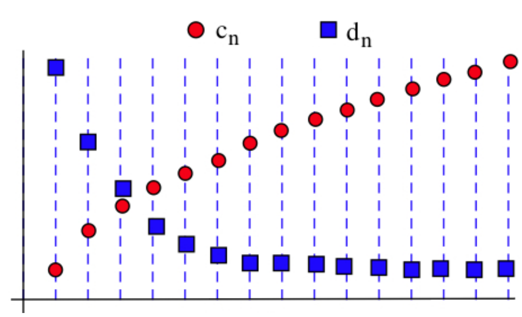
9.2 Problems

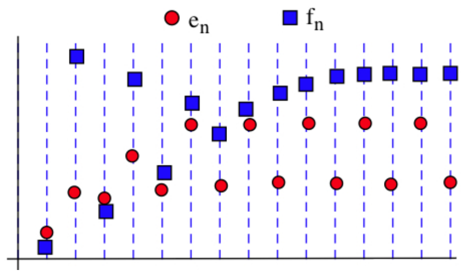
In Problems 1–4, state whether each sequence appears to be converging or diverging. If you think the sequence is converging, mark its limit as a value on the vertical axis. (Important: The behavior of a sequence can change drastically after the first terms, which have no influence on whether or not the sequence converges, but sometimes we need to reach a tentative conclusion based on the few sequence values we know.)

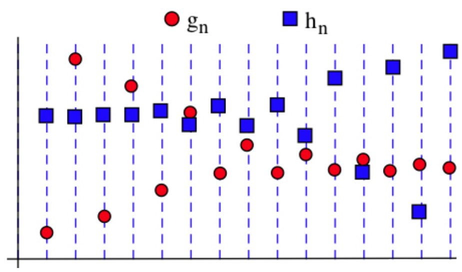
1.



2.



3. 

4. 

In 5–19, state whether each sequence converges or diverges. If the sequence converges, find its limit.

5. $\left\{1-\frac{2}{n}\right\}$

6. $\left\{n^2\right\}$

7. $\left\{\frac{n^2}{n+1}\right\}$

8. $\left\{3+\frac{1}{n^2}\right\}$

9. $\left\{\frac{n}{2n-1}\right\}$

10. $\left\{\frac{\ln(n)}{n}\right\}$

11. $\left\{\ln\left(3+\frac{7}{n}\right)\right\}$

12. $\left\{3+\frac{(-1)^{n+1}}{n}\right\}$

13. $\{4+(-1)^n\}$

14. $\left\{(-1)^n\frac{n-1}{n}\right\}$

15. $\left\{\frac{1}{n!}\right\}$

16. $\left\{\left(1+\frac{3}{n}\right)^n\right\}$

17. $\left\{\left(1-\frac{1}{n}\right)^n\right\}$

18. $\left\{\frac{\sqrt{n}-3}{\sqrt{n}+3}\right\}$

19. $\left\{\frac{(n+2)(n+5)}{n^2}\right\}$

In 20–23, prove that the sequence converges to the given limit by showing that, for any $\epsilon > 0$, you can find an N that satisfies the definition of convergence.

20. $\lim_{n\to\infty}\left[2-\frac{3}{n}\right]=2$

21. $\lim_{n\to\infty}\frac{3}{n^2}=0$

22. $\lim_{n\to\infty}\frac{7}{n+1}=0$

23. $\lim_{n\to\infty}\frac{3n-1}{n}=3$

In Problems 24–29, use subsequences to help determine whether the given sequence converges or diverges. If the sequence converges, find its limit.

24. $\{(-1)^n\cdot 3\}$

25. $\left\{\frac{1}{n\text{-th prime}}\right\}$

26. $\left\{(-1)^n\cdot\frac{n+1}{n}\right\}$

27. $\left\{(-2)^n\cdot\left(\frac{1}{2}\right)^n\right\}$

28. $\left\{\left(1+\frac{1}{3n}\right)^{3n}\right\}$

29. $\left\{\left(1+\frac{5}{n^2}\right)^{n^2}\right\}$

In problems 30–35, calculate $a_{n+1}-a_n$ and use that value to determine whether $\{a_n\}$ is monotonic increasing, monotonic decreasing, or neither.

30. $\left\{\frac{3}{n}\right\}$

31. $\left\{7-\frac{2}{n}\right\}$

32. $\left\{\frac{n-1}{2n}\right\}$

33. $\{2^n\}$

34. $\left\{1-\frac{1}{2^n}\right\}$

35. $\left\{5+\frac{7}{3^n}\right\}$

In Problems 36–41, calculate $\frac{a_{n+1}}{a_n}$ and use that value to determine whether each sequence is monotonic increasing, monotonic decreasing, or neither.

36. $\left\{\frac{n}{n+1}\right\}$

37. $\left\{\frac{n+1}{n!}\right\}$

38. $\left\{\frac{n^2}{n!}\right\}$

39. $\left\{\left(\frac{5}{4}\right)^n\right\}$

40. $\left\{\frac{n^2}{2^n}\right\}$

41. $\left\{\frac{n}{e^n}\right\}$

In Problems 42–46, use derivatives to determine whether each sequence is monotonic increasing, monotonic decreasing, or neither.

42. $\left\{\frac{n+1}{n}\right\}$

43. $\left\{5-\frac{3}{n}\right\}$

44. $\{n\cdot e^{-n}\}$

45. $\left\{\cos\left(\frac{1}{n}\right)\right\}$

46. $\left\{\left(1+\frac{1}{n}\right)^3\right\}$

In 47–51, show that each sequence is monotonic.

47. $\left\{\frac{n+3}{n!}\right\}$

48. $\left\{\frac{n}{n+1}\right\}$

49. $\left\{1-\frac{1}{2^n}\right\}$

50. $\left\{\sin\left(\frac{1}{n}\right)\right\}$

51. $\left\{\frac{n+1}{e^n}\right\}$

52. The **Fibonacci sequence**, named after Leonardo Fibonacci (1170–1250), who used it to model a population of rabbits, is obtained by setting the first two terms equal to 1 and then defining each new term as the sum of the two previous terms: $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. (a) Write the first seven terms of this sequence. (b) Compute the successive ratios of the terms, $\frac{a_n}{a_{n-1}}$. (These ratios approach the "golden ratio," $\frac{1+\sqrt{5}}{2} \approx 1.618$.)

53. To approximate the square root of a positive number N, **Heron's method** puts $a_1 = N$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{N}{a_n}\right)$. Then $\{a_n\}$ converges to $\sqrt{N}$. Compute a_1 through a_4 for $N = 4$, 9 and 5. (Heron's method is equivalent to Newton's method applied to the function $f(x) = x^2 - N$.)

54. Using an initial or "seed" value h_0, define the **hailstone sequence** by the rule:

$$h_n = \begin{cases} 3 \cdot h_{n-1} + 1 & \text{if } h_{n-1} \text{ is odd} \\ \frac{1}{2} \cdot h_{n-1} & \text{if } h_{n-1} \text{ is even} \end{cases}$$

Define the **length** of the sequence to be the first value of n so that $h_n = 1$. If the seed value is $h_0 = 3$, then $h_1 = 3(3) + 1 = 10$, $h_2 = \frac{1}{2} \cdot 10 = 5$, $h_3 = 3(5) + 1 = 16$, $h_4 = \frac{1}{2} \cdot 16 = 8$, $h_5 = 4$, $h_6 = 2$ and $h_7 = 1$, so the length is 7.

(a) Find the length of the hailstone sequence for each seed value from 2 through 10.

(b) Find the length of the hailstone sequence for a seed value of the form $h_0 = 2^n$. .

(c) Is the length of the hailstone sequence finite for every seed value? (This is an open question — no one has yet been able to answer it.)

Called the hailstone sequence because (for some seed values) the terms of the sequence rise and drop like the path of a hailstone as it forms, this sequence has been attributed to Lothar Collatz. The open question in part (c) is also called Ulam's conjecture, the Syracuse problem, Kakutani's problem and Hasse's algorithm.

55. Suppose that individuals with the gene combination "aa" do not reproduce and those with the combinations "aA" and "AA" do reproduce. When the initial proportion of individuals with "aa" is $a_0 = p$ (typically a small number), then the proportion of individuals with "aa" in the k-th generation is $a_k = \frac{p}{kp+1}$. Use this formula for a_k to answer the following questions.

(a) If 2% of a population initially have the combination "aa" and these individuals do not reproduce, then how many generations will it take for the proportion of individuals with "aa" to drop below 1%?

(b) In general, find the number of generations until the proportion of individuals with "aa" is half of the initial proportion.

"Negative eugenics" was a strategy proposed during the early 20th century in which individuals with an undesirable trait were discouraged or forcibly prevented from reproducing. Mathematics shows that this is not an effective strategy for reducing the occurrence of traits carried by recessive genes (the above example) that are uncommon (p small) in a species that reproduces slowly (people).

56. The fractional part of a number is the difference between the number and its integer part: $x - \lfloor x \rfloor$. The sequence of fractional parts of multiples of a number x is the sequence with terms $a_n = n \cdot x - \lfloor n \cdot x \rfloor$. The behavior of the sequence of fractional parts of the multiples of a number is one way in which rational numbers differ from irrational numbers.

(a) Let $a_n = n \cdot x - \lfloor n \cdot x \rfloor$ be the fractional part of the n-th multiple of x. Calculate a_1 through a_6 for $x = \frac{1}{3}$. These are the fractional parts of the first six multiples of $\frac{1}{3}$.

(b) Calculate the fractional parts of the first nine multiples of $\frac{3}{4}$ and $\frac{2}{5}$.

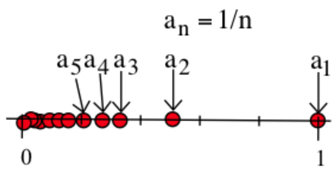
(c) Calculate the fractional parts of the first five multiples of π.

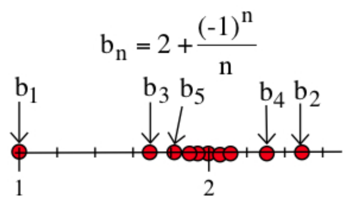
(d) Let $a_n = n \cdot \pi - \lfloor n \cdot \pi \rfloor$ be the fractional part of the n-th multiple of π. Can two different multiples of π have the same fractional part? (Suggestion: Assume the answer is yes and obtain a contradiction.)

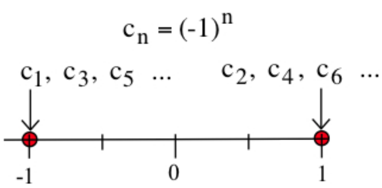
An Alternative Way to Visualize Sequences and Convergence

A sequence is a function, and we have graphed sequences in the xy-plane in the same way we graphed other functions: because $a_n = f(n)$, we plotted the point (n, a_n). If the sequence $\{a_n\}$ converges to L, then the points (n, a_n) eventually (for big values of n) all lie close to — or on — the horizontal line $y = L$.

We can also graph a sequence $\{a_n\}$ in one dimension, using the x-axis alone. For each value of n, plot the point $x = a_n$. Then the graph of $\{a_n\}$ consists of a collection of points on the x-axis. The margin figures show the one-dimensional graphs of $a_n = \frac{1}{n}$, $b_n = 2 + \frac{(-1)^n}{n}$ and $c_n = (-1)^n$.







If $\{a_n\}$ converges to L, then the points $x = a_n$ eventually (for big values of n) all lie close to — or on — the point $x = L$. If we build a narrow box, with width $2\epsilon > 0$, and center the box at the point $x = L$, then all of the points a_n will sit inside the box once n becomes larger than some value N.

In more advanced courses, these points $x = L$ about which the terms in the sequence eventually "cluster" are called **cluster points** or **accumulation points**. A convergent sequence has exactly one accumulation point; a divergent sequence with more than one convergent subsequence has multiple accumulation points.

57. Suppose that the sequence $\{a_n\}$ converges to 3 and that you place a single grain of sand at each point $x = a_n$ on the x-axis. Describe the likely result (a) after you have placed a few grains and (b) after you have placed a lot (thousands or millions) of grains.

58. Suppose the sequence $\{a_n\}$ converges to 3, the sequence $\{b_n\}$ converges to 1, and that you place a single grain of sand at each point $(x, y) = (a_n, b_n)$ on the xy-plane. Describe the likely result (a) after you have placed a few grains and (b) after you have placed a lot (thousands or millions) of grains.

59. Suppose that $a_n = \sin(n)$ for positive integers n and that you place a single grain of sand at each point $x = a_n$ on the x-axis. (a) Describe the likely result after you have placed a few grains. (b) After you have placed a lot (thousands or millions) of grains. (c) Do two grains ever end up on the same point?

60. Suppose that $a_n = \cos(n)$ and $b_n = \sin(n)$ for positive integers n. If you place a single grain of sand at each point $(x, y) = (a_n, b_n)$ on the xy-plane, describe the likely result (a) after you have placed a few grains and (b) after you have placed millions of grains.

9.2 Practice Answers

1. We need to show that, for any positive ϵ, there is a number N so that the distance from $a_n = \dfrac{n+1}{n}$ to $L = 1$, $|a_n - L|$, is less than ϵ whenever n is larger than N. For this particular sequence:

$$|a_n - L| = \left|\frac{n+1}{n} - 1\right| = \left|\frac{n}{n} + \frac{1}{n} - 1\right| = \left|\frac{1}{n}\right|$$

needs to be less than ϵ. To determine how large N needs to be, solve the inequality:

$$\left|\frac{1}{n}\right| < \epsilon \quad \Rightarrow n > \frac{1}{\epsilon}$$

So, for any positive ϵ, we can take $N = \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1$. Then:

We're "rounding down" and then adding 1 to ensure that N is an integer.

$$n > N = \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1 \geq \frac{1}{\epsilon} \quad \Rightarrow \quad \frac{1}{n} < \epsilon \quad \Rightarrow \quad \left|\frac{n+1}{n} - 1\right| < \epsilon$$

2. Replacing n with x and applying L'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

3. We can show that $a_n = \left\{\sin\left(\frac{\pi n}{2}\right)\right\}$ diverges by finding two subsequences b_n and c_n of a_n that converge to different limits. Let $\{b_n\}$ consist of the terms $\{a_1, a_5, a_9, \ldots, a_{4k-3}, \ldots\}$:

$$\left\{\sin\left(\frac{\pi}{2}\right), \sin\left(\frac{5\pi}{2}\right), \sin\left(\frac{9\pi}{2}\right), \ldots\right\} = \{1, 1, 1, \ldots\}$$

so that $\lim\limits_{n\to\infty} b_n = 1$. Then let $\{c_n\}$ consist of $\{a_2, a_4, a_6, \ldots, a_{2k}, \ldots\}$:

$$\left\{\sin\left(\frac{2\pi}{2}\right), \sin\left(\frac{4\pi}{2}\right), \sin\left(\frac{6\pi}{2}\right), \ldots\right\} = \{0, 0, 0, \ldots\}$$

so that $\lim\limits_{n\to\infty} c_n = 0$. Because the subsequences $\{b_n\}$ and $\{c_n\}$ have different limits, the original sequence $\{a_n\}$ must diverge.

Many other pairs of subsequences will also work in place of the $\{b_n\}$ and $\{c_n\}$ that we used.

4. By showing that:
   - $a_{n+1} \leq a_n$ for all n
   - $a_n > 0$ and that $\dfrac{a_{n+1}}{a_n} \leq 1$ for all n
   - $a_n = f(n)$ for integer values n and some differentiable function f for which $f'(x) \leq 0$ for all $x > 0$

5. Using the second method from Practice 9:

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^{n}} = \frac{2}{3} < 1$$

so $\left\{\left(\dfrac{2}{3}\right)^n\right\}$ is monotonically decreasing.