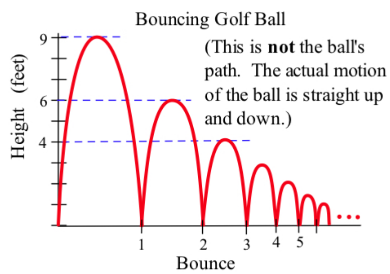
9.3 Infinite Series

What does it mean to add together an infinite number of terms of a sequence? We will define that concept carefully in this section. Is the sum of all the terms in an infinite sequence a finite number? Over the next few sections we will examine a variety of techniques for determining whether the sum of an infinite number of terms is a finite number. If we know the sum of an infinite number of terms is finite, can we determine the value of that sum? In certain special situations, this can be very easy, but often it turns out to be very, very difficult.



Example 1. You throw a golf ball 9 feet straight up into the air, and on each bounce it rebounds to two-thirds of its previous height (see margin). Find a sequence whose terms give the distances the ball travels during each successive bounce. Represent the total distance traveled by the ball as a sum.

Solution. The heights of the successive bounces are 9 feet, $\left(\frac{2}{3}\right) \cdot 9$ feet, $\left(\frac{2}{3}\right) \cdot \left(\frac{2}{3}\right) \cdot 9$ feet, $\left(\frac{2}{3}\right)^3 \cdot 9$ feet, and so forth. On each bounce, the ball rises and falls, so the distance traveled is twice the height of that bounce:

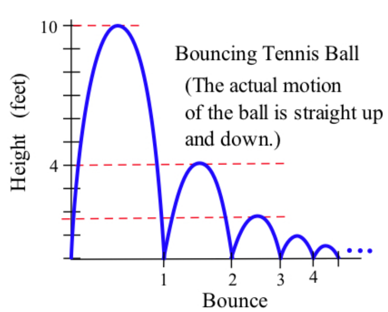
$$18 \text{ ft}, \left(\frac{2}{3}\right) \cdot 18 \text{ ft}, \left(\frac{2}{3}\right) \cdot \left(\frac{2}{3}\right) \cdot 18 \text{ ft}, \left(\frac{2}{3}\right)^3 \cdot 18 \text{ ft}, \ldots$$

The total distance traveled is the sum of these bounce-distances:

$$18 + \left(\frac{2}{3}\right) \cdot 18 + \left(\frac{2}{3}\right)^2 \cdot 18 + \left(\frac{2}{3}\right)^3 \cdot 18 + \cdots$$
$$= 18\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right]$$

At the completion of the first bounce, the ball has traveled 18 feet. After the second bounce, it has traveled 30 feet, a total of 38 feet after the third bounce, $\frac{130}{3} \approx 43.67$ feet after the fourth, and so on. With a calculator — and some patience — you can determine that after the 20th bounce the ball has traveled a total of (approximately) 53.996 feet, after the 30th bounce (approximately) 53.99994 feet, and after the 40th bounce (approximately) 53.9999989 feet. ◀

Do these total distances appear to be approaching a limiting value?



Practice 1. Your friend throws a tennis ball 10 feet straight up, and on each bounce it rebounds to 40% of its previous height (see margin). Represent the total distance traveled by the ball as a sum, and find the total distance traveled by the ball after its third bounce.

Infinite Series

The infinite sums in the preceding Example and Practice problems are called **infinite series**.

Definitions: An **infinite series** is an expression of the form

$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{k=1}^{\infty} a_k$$

The numbers $a_1, a_2, a_3, a_4, \ldots$ are called the **terms** of the series.

Although often used interchangeably in everyday language, the words "sequence" and "series" have precise meanings in mathematics: a sequence is a list of numbers and a series is sum of a sequence.

Example 2. Represent the following series using sigma notation.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

(b) $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

(c) $18\left[1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots\right]$

(d) $0.777\ldots = \frac{7}{10} + \frac{7}{100} + \cdots$

(e) $0.222\ldots = \frac{2}{10} + \frac{2}{100} + \cdots$

While the plural of "sequence" is "sequences," the plural of "series" is "series," so you might say "the series is convergent" when talking about a single series and "the series are convergent" when discussing more than one series.

Solution. (a) The terms are all of the form $\left(\frac{1}{3}\right)^k$, starting with $k = 0$:

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$$

(b) The terms are all of the form $\frac{\pm 1}{k}$ starting with $k = 1$:

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

(c) The terms in the brackets are of the form $\left(\frac{2}{3}\right)^k$, starting with $k = 0$:

$$18\left[1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots\right] = 18 \cdot \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$

(d) The terms are all of the form $\frac{7}{10^k}$, starting with $k = 1$, so:

$$0.777\ldots = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots = \sum_{k=1}^{\infty} \frac{7}{10^k}$$

(e) $0.222\ldots = \sum_{k=1}^{\infty} \frac{2}{10^k}$ ◀

Practice 2. Represent the following series using sigma notation.

(a) $1 + 2 + 3 + 4 + \cdots$

(b) $-1 + 1 - 1 + 1 - \cdots$

(c) $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

(d) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$

(e) $0.111\ldots$

Partial Sums

In Example 1, we computed the total distance traveled by the golf ball after successive bounces: the first term in the sequence, then the sum of the first two terms, then the sum of the first three terms, and so forth. We call these numbers **partial sums** of an infinite series.

> **Definition:** The **partial sums** s_n of the infinite series $\sum_{k=1}^{\infty} a_k$ are:
>
> $$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= \sum_{k=1}^{n} a_k \end{aligned}$$

We can also define the partial sums recursively as $s_n = s_{n-1} + a_n$. The partial sums form a sequence of partial sums $\{s_n\}$.

Example 3. Compute the first four partial sums for the following series.

$$\text{(a) } 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \qquad \text{(b) } \sum_{k=1}^{\infty} (-1)^k \qquad \text{(c) } \sum_{k=1}^{\infty} \frac{1}{k}$$

Solution. (a) $s_1 = 1$, $s_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $s_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$ and $s_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$. Typically, it is easier to use the recursive formulation of s_n:

$$s_3 = s_2 + a_3 = \frac{3}{2} + \frac{1}{4} = \frac{7}{4} \Rightarrow s_4 = s_3 + a_4 = \frac{7}{4} + \frac{1}{8} = \frac{15}{8} \Rightarrow \cdots$$

(b) Using the recursive formulation: $s_1 = (-1)^1 = -1$, $s_2 = s_1 + a_2 = -1 + (-1)^2 = 0$, $s_3 = s_2 + a_3 = 0 + (-1)^3 = -1$ and $s_4 = 0$.

(c) $s_1 = 1$, $s_2 = \frac{3}{2}$, $s_3 = \frac{11}{6}$ and $s_4 = \frac{25}{12}$ ◀

Practice 3. Compute the first four partial sums for the following series.

$$\text{(a) } 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots \qquad \text{(b) } \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k \qquad \text{(c) } \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

If we know the values of the partial sums s_n for an infinite series, we can recover the values of the terms for the series used to build the s_n.

Example 4. If $s_1 = 2.1$, $s_2 = 2.6$, $s_3 = 2.84$ and $s_4 = 2.87$ are the first four partial sums of $\sum_{k=1}^{\infty} a_k$, find the first four terms of $\{a_n\}$.

Solution. $s_1 = a_1$, so $a_1 = 2.1$, while $s_2 = a_1 + a_2$, so $2.6 = 2.1 + a_2 \Rightarrow a_2 = 2.6 - 2.1 = 0.5$. Similarly, $s_3 = a_1 + a_2 + a_3$ so $2.84 = 2.1 + 0.5 + a_3 \Rightarrow a_3 = 2.84 - 2.6 = 0.24$. Finally, $a_4 = 0.03$. ◀

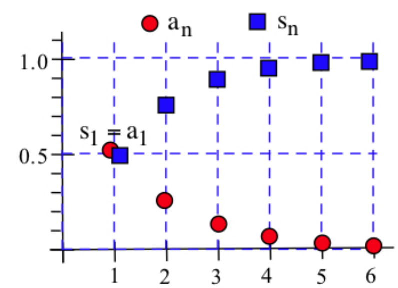
Alternatively, we could use the recursive formula $s_n = s_{n-1} + a_n$ to conclude that $a_n = s_n - s_{n-1}$ and compute:

$$a_2 = s_2 - s_1 = 2.6 - 2.1 = 0.5$$
$$a_3 = s_3 - s_2 = 2.84 - 2.6 = 0.24$$
$$a_4 = s_4 - s_3 = 2.87 - 2.84 = 0.03$$

Practice 4. If $s_1 = 3.2$, $s_2 = 3.6$, $s_3 = 3.5$, $s_4 = 4$, $s_{99} = 7.3$, $s_{100} = 7.6$ and $s_{101} = 7.8$ are partial sums of $\sum_{k=1}^{\infty} a_k$, find a_1, a_2, a_3, a_4 and a_{100}.

Example 5. Graph the first five **terms** of the series $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ and then graph the first five **partial sums**.

Solution. The first few terms are $a_1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$, $a_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$, $a_3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, $a_4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$ and $a_5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$, so $s_1 = a_1 = \frac{1}{2}$, $s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $s_3 = s_2 + a_3 = \frac{3}{4} + \frac{1}{8} = \frac{1}{8}$, $s_4 = \frac{15}{16}$ and $s_5 = \frac{31}{32}$. See margin for a graph of the terms and partial sums. ◀



Practice 5. Graph the first five **terms** of the series $\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k$ and then graph the first five **partial sums**.

Convergence of a Series

In Example 1, you may have noticed that the sequence of total distances traveled by the golf ball after each subsequent bounce:

$$18, 30, 38, 43.67, \ldots, 53.996, \ldots, 53.99994, \ldots, 53.9999989, \ldots$$

appeared to approach a limit. If this sequence of partial sums does in fact approach 54 feet (which appears to be true, but remains a fact that we need to prove) then we could write:

$$18 + \left(\frac{2}{3}\right) \cdot 18 + \left(\frac{2}{3}\right)^2 \cdot 18 + \left(\frac{2}{3}\right)^3 \cdot 18 + \cdots = 54$$

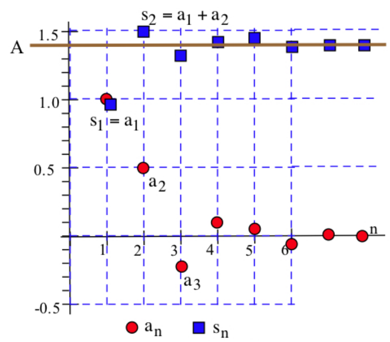
and say that the infinite series $\sum_{k=0}^{\infty} 18 \cdot \left(\frac{2}{3}\right)^k$ **converges** to 54.

If the sequence of partial sums for an infinite series—the sequence obtained by adding up more and more of the terms of the series—approaches a finite number, we say the series **converges** to that finite number. If the sequence of partial sums diverges (does not approach a single finite number), we say that the series **diverges**.

Definition:
Let $\{s_n\}$ be the sequence of partial sums for the series $\sum_{k=1}^{\infty} a_k$.

If the sequence $\{s_n\}$ converges, we say the series $\sum_{k=1}^{\infty} a_k$ **converges**.

If the sequence $\{s_n\}$ diverges, we say the series $\sum_{k=1}^{\infty} a_k$ **diverges**.



If the sequence of partial sums $\{s_n\}$ for an infinite series $\sum_{k=1}^{\infty} a_k$ converges to number A, we say the series **converges to A** or that the sum of the series is A, and we write $\sum_{k=1}^{\infty} a_k = A$.

Example 6. In Example 3(a), it appears that the partial sums of the infinite series $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ are:

$$s_n = \frac{2^{n+1}-1}{2^n} = 2 - \frac{1}{2^n}$$

Use this result to evaluate the limit of the partial sums $\{s_n\}$. Does the series $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ converge? If so, to what value?

Solution. $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[2 - \frac{1}{2^n}\right] = 2$ so $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$. ◀

Practice 6. In Practice 5, the partial sums of $\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k$ are:

We will discover how to arrive at this formula for s_n in the next section.

$$s_n = -\frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n$$

Use this result to evaluate the limit of the partial sums $\{s_n\}$. Does the series $\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k$ converge? If so, to what value?

A Test for Divergence

If a series converges, its partial sums s_n approach a finite limit L, so all of the s_n must be close to L for large values of n. This means that s_n and s_{n-1} must be close to each other (why?), so $a_n = s_n - s_{n-1} \longrightarrow 0$.

Theorem:

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

See Problem 47 for a formal proof.

We can **not** use this theorem to conclude that a series converges. If the terms of the series *do* approach 0, then the series may or may not converge — we will need more information to draw a conclusion above the convergence of the series. An alternate form of the above theorem, called the Test for Divergence, can be very useful for quickly concluding that some series diverge.

Test for Divergence:

If the terms of a series do not approach 0 (as $k \to \infty$)
then the series diverges.

In other words, if $\lim_{k\to\infty} a_k \neq 0$ then $\sum_{k=0}^{\infty} a_k$ must diverge.

Example 7. Which of these series must diverge according to the Test for Divergence?

(a) $\sum_{k=1}^{\infty} (-1)^k$ (b) $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k$ (c) $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$ (d) $\sum_{k=1}^{\infty} \frac{1}{k}$

Solution. (a) $a_k = (-1)^k$ oscillates between -1 and $+1$ and does not approach 0, so $\sum_{k=1}^{\infty} (-1)^k$ diverges. (b) $\lim_{k\to\infty} \left(\frac{3}{4}\right)^k = 0$, so $\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k$ may or may not converge. (c) $\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e \neq 0$, so $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$ diverges. (d) $\lim_{k\to\infty} \frac{1}{k} = 0$, so $\sum_{k=1}^{\infty} \frac{1}{k}$ may or may not converge. We can be certain the series in (a) and (c) diverge. We don't have enough information to decide about (b) and (d). ◀

In the next section, we will show that the series in (b) converges and that the series in (d) diverges.

Practice 7. Which of these series must diverge according to the Test for Divergence?

(a) $\sum_{k=1}^{\infty} (-0.9)^k$ (b) $\sum_{k=1}^{\infty} (1.1)^k$ (c) $\sum_{k=1}^{\infty} \sin(k \cdot \pi)$ (d) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

New Series from Old

If we know whether an infinite series converges or diverges, then we also know whether several related series converge or diverge.

Inserting or deleting an *infinite* number of terms *can* change the convergence or divergence.

- Inserting or deleting a "few" terms (any finite number of terms) does not change the convergence or divergence of a series. The insertions or deletions typically change the sum of the series (the limit of its partial sums), but do not change whether or not it converges.

- Multiplying each term in a series by the same nonzero constant does not change the convergence or divergence of a series. If $c \neq 0$, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} c \cdot a_k$ converges.

Term-by-term multiplication and division of series do not have such nice results.

- Term-by-term addition and subtraction of the terms of two *convergent* series result in a new convergent series.

The proofs of these statements follow directly from the definition of convergence of a series and from results about convergence of sequences (of partial sums). See Problems 44–46.

Theorem:

If $\sum_{k=1}^{\infty} a_k = A$, $\sum_{k=1}^{\infty} b_k = B$ and $c \neq 0$

then $\sum_{k=1}^{\infty} [a_k + b_k] = A + B$, $\sum_{k=1}^{\infty} [a_k - b_k] = A - B$

and $\sum_{k=1}^{\infty} c \cdot a_k = c \cdot A$.

9.3 Problems

In Problems 1–6, rewrite each sum using sigma notation, starting with $k = 1$.

1. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$
2. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$
3. $\frac{2}{3} + \frac{2}{6} + \frac{2}{9} + \frac{2}{12} + \frac{2}{15} + \cdots$
4. $\sin(1) + \sin(8) + \sin(27) + \sin(64) + \cdots$
5. $-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$
6. $-\frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \cdots$

In Problems 7–12, calculate and graph the first four partial sums s_1 through s_4 of the given series.

7. $\sum_{k=1}^{\infty} k^2$
8. $\sum_{k=1}^{\infty} (-1)^k$
9. $\sum_{k=1}^{\infty} \frac{1}{k+2}$
10. $\sum_{k=1}^{\infty} (-\frac{1}{2})^k$
11. $\sum_{k=1}^{\infty} \frac{1}{2^k}$
12. $\sum_{k=2}^{\infty} \left[\frac{1}{k} - \frac{1}{k-1}\right]$

Problems 13–18 give the first five partial sums, s_1 through s_5, of a series $\sum_{k=1}^{\infty} a_k$. Find a_1 through a_4.

13. $s_1 = 3, s_2 = 2, s_3 = 4, s_4 = 5, s_5 = 3$
14. $s_1 = 3, s_2 = 5, s_3 = 4, s_4 = 6, s_5 = 5$
15. $s_1 = 4, s_2 = 4.5, s_3 = 4.3, s_4 = 4.8, s_5 = 5$
16. $s_1 = 4, s_2 = 3.7, s_3 = 3.9, s_4 = 4.1, s_5 = 4$
17. $s_1 = 1, s_2 = 1.1, s_3 = 1.11, s_4 = 1.111, s_5 = 1.1111$
18. $s_1 = 1, s_2 = 0.9, s_3 = 0.93, s_4 = 0.91, s_5 = 0.92$

In Problems 19–28, represent each repeating decimal as a series using sigma notation.

19. $0.888\ldots$
20. $0.333\ldots$
21. $0.555\ldots$
22. $0.111\ldots$
23. $0.aaa\ldots$
24. $0.232323\ldots$
25. $0.171717\ldots$
26. $0.838383\ldots$
27. $0.070707\ldots$
28. $0.ababab\ldots$
29. $0.abcabcabc\ldots$

30. After you throw a golf ball 20 feet straight up into the air, on each bounce it rebounds to 60% of its previous height. Represent the total distance traveled by the ball as an infinite sum.

31. After your friend throws a "super ball" 15 feet straight up, on each bounce it rebounds to 80% of its previous height. Represent the total distance traveled by the ball as an infinite sum.

32. Each special washing of a pair of coveralls removes 80% of the radioactive particles attached to the coveralls. Represent, as a sequence of numbers, the percent of the original radioactive particles that remain after each washing.

33. Each week, 20% of the argon gas in a container leaks out of the container. Represent, as a sequence of numbers, the percent of the original argon gas that remains in the container at the end of the first, second, third and n-th weeks.

34. Eight researchers depart on an expedition by horseback through desolate wilderness. The researchers and their equipment require 12 horses, and those horses require additional horses to carry food for the horses. Each horse can carry enough food to feed 2 horses for the trip. Represent the number of horses needed to carry food as an infinite sum.

Which of the series in Problems 35–43 definitely diverge by the Test for Divergence? What can you conclude about the other series?

35. $\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k$ 36. $\sum_{k=1}^{\infty} \frac{7}{k}$ 37. $\sum_{k=1}^{\infty} \left(\frac{4}{3}\right)^k$

38. $\sum_{k=1}^{\infty} \left(-\frac{7}{4}\right)^2$ 39. $\sum_{k=1}^{\infty} \frac{\sin(k)}{k}$ 40. $\sum_{k=2}^{\infty} \frac{\ln(k)}{k}$

41. $\sum_{k=10}^{\infty} \cos(k)$ 42. $\sum_{k=5}^{\infty} \frac{k^2-20}{k^2+4}$ 43. $\sum_{k=5}^{\infty} \frac{k^2-20}{k^5+4}$

In Problems 44–47, you know that $\sum_{k=1}^{\infty} a_k = A$, $\sum_{k=1}^{\infty} b_k = B$ and $c \neq 0$.

44. Prove that $\sum_{k=1}^{\infty} [a_k + b_k] = A + B$.

45. Prove that $\sum_{k=1}^{\infty} c \cdot a_k = c \cdot A$.

46. Prove that $\sum_{k=1}^{\infty} [a_k - b_k] = A - B$. (Hint: Use the results from the two previous problems, rather than starting over from scratch.)

47. Prove that $\lim_{k\to\infty} a_k = 0$.

9.3 Practice Answers

1. The heights of the bounces are 10, $(0.4)10 = 4$, $(0.4)(0.4)10 = 1.6$, $(0.4)^3 10 = 0.64, \ldots$, so the distances traveled (up and down) by the ball are 20, $(0.4)20 = 8$, $(0.4)(0.4)20 = 3.2$, $(0.4)^3 20 = 1.28, \ldots$. The total distance traveled is:

$$20 + (0.4)20 + (0.4)^2 20 + (0.4)^3 .0 + \cdots$$
$$= 20\left[1 + 0.4 + (0.4)^2 + (0.4)^3 + \cdots\right] = 20\sum_{k=0}^{\infty}(0.4)^k$$

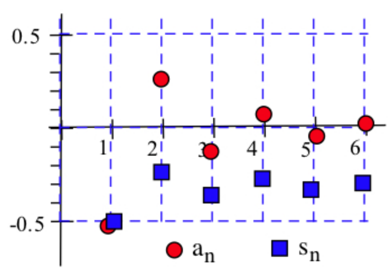
After three bounces, the ball has traveled $20 + 8 + 3.2 = 31.2$ feet.

2. (a) $\sum_{k=1}^{\infty} k$ (b) $\sum_{k=1}^{\infty} (-1)^k$ (c) $\sum_{k=-1}^{\infty} \left(\frac{1}{2}\right)^k = 2\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ (d) $\sum_{k=1}^{\infty} \frac{1}{2k}$
(e) We can write $0.111\ldots$ as:

$$0.111\ldots = 0.1 + 0.01 + 0.001 + \cdots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$
$$= \left(\frac{1}{10}\right)^1 + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots = \sum_{k=1}^{\infty}(\frac{1}{10})^k$$

3. (a) $s_1 = 1$, $s_2 = \frac{1}{2}$, $s_3 = \frac{3}{4}$, $s_4 = \frac{5}{8}$ (b) $s_1 = \frac{1}{3}$, $s_2 = \frac{4}{9}$, $s_3 = \frac{13}{27}$, $s_4 = \frac{40}{81}$
(c) $s_1 = -1$, $s_2 = -1 + \frac{1}{2} = -\frac{1}{2}$, $s_3 = -\frac{1}{2} - \frac{1}{3} = -\frac{5}{6}$, $s_4 = -\frac{7}{12}$

4. $a_1 = s_1 = 3.2$, $a_2 = s_2 - s_1 = 3.6 - 3.2 = 0.4$, $a_3 = s_3 - s_2 = 3.5 - 3.6 = -0.1$, $a_4 = s_4 - s_3 = 4 - 3.5 = 0.5$, $a_{100} = s_{100} - s_{99} = 7.6 - 7.3 = 0.3$



5. See margin graph: $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = -\frac{1}{8}$, $a_4 = \frac{1}{16}$, $a_5 = -\frac{1}{32}$; $s_1 = a_1 = -\frac{1}{2}$, $s_2 = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$, $s_3 = -\frac{1}{4} - \frac{1}{8} = -\frac{3}{8}$, $s_4 = -\frac{5}{16}$, $s_4 = -\frac{11}{32}$.

6. Taking the limit of the partial sums of the series:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[-\frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n\right] = -\frac{1}{3}$$

so the series $\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^k$ converges and its sum is $-\frac{1}{3}$.

7. (a) $\lim_{k\to\infty}(-0.9)^k = 0$, so $\sum_{k=1}^{\infty}(-0.9)^k$ may or may not converge. (b) $\lim_{k\to\infty}(1.1)^k = \infty \neq 0$ so $\sum_{k=1}^{\infty}(1.1)^k$ must diverge. (c) $\sin(k\cdot\pi) = 0$ for any integer k, so all of the terms of the series are 0; the test for Divergence tells us nothing, but because all of the terms are 0, we know that $\sum_{k=1}^{\infty}\sin(k\cdot\pi) = 0$ (so the series converges). (d) $\lim_{k\to\infty}\frac{1}{\sqrt{k}} = 0$, so $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ may or may not converge.

In the next section, we will show that the series in (a) converges and the series in (d) diverges.