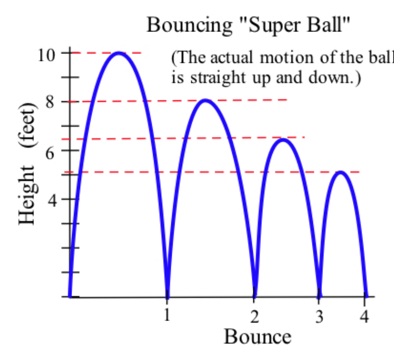
9.4 *Geometric and Harmonic Series*

This section investigates three special types of series. Geometric series appear throughout mathematics and arise in a variety of applications. Much of the early work in the 17th century with series focused on geometric series and generalized them. And many of the ideas used later in this chapter (and into the next) originated with geometric series. It is very easy to determine whether a geometric series converges or diverges — and when one does converge, we can easily find its sum.

The harmonic series is important as an example of a *divergent* series whose terms approach 0.

The last part of this section briefly discusses a third type of series, called "telescoping": these are relatively uncommon, but their partial sums exhibit a particularly nice pattern.

Geometric Series



Example 1. Your friend throws a "super ball" 10 feet straight up into the air. On each bounce, it rebounds to 80% of its previous height (see margin) so the sequence of heights the ball attains is 10 feet, 8 feet, 6.4 feet, 5.12 feet, and so on.

(a) How far does the ball travel (up and down) during its n-th bounce?

(b) Use a sum to represent the total distance traveled by the ball.

Solution. (a) Because the ball travels up and down on each bounce, the distance traveled during each bounce is twice the height the ball attains on that bounce. So the distance the ball travels prior to its first bound is $d_1 = 2\,(10 \text{ feet}) = 20$ feet, the distance it travels between the first and second bounces is $d_2 = (0.8)(20) = 16$ feet, $d_3 = (0.8)(16) = 12.8$ feet and, in general, $d_n = 0.8 d_{n-1}$. Looking at these values in another way:

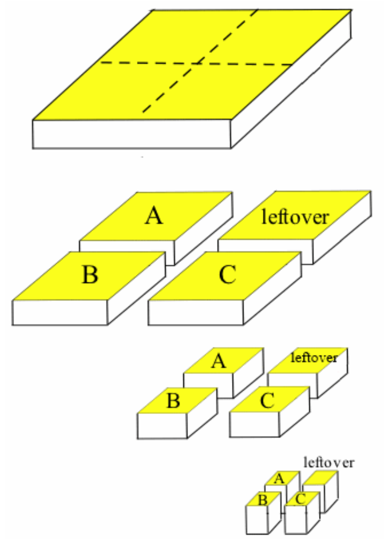
$$d_1 = 20,\ d_2 = 0.8(20),\ d_3 = (0.8)d_2 = (0.8)^2(20),\ d_4 = (0.8)^3(20)$$

and, in general, $d_n = (0.8)^{n-1}(20)$.

(b) In theory, the ball bounces up and down forever, and the total distance traveled by the ball is the sum of the distances traveled during each bounce (an up-and-down flight):

$$\begin{aligned} 20 + (0.8)(20) + (0.8)^2(20) + (0.8)^3(20) + (0.8)^4(20) + \cdots \\ = 20\left[1 + 0.8 + (0.8)^2 + (0.8)^3 + (0.8)^4 + \cdots\right] \\ = 20 \cdot \sum_{k=0}^{\infty} (0.8)^k \end{aligned}$$

(In practice, the ball will eventually top bouncing.) ◀



Practice 1. Three calculus students want to share a small square cake equally, but they go about it in a rather strange way. First they cut the cake into four equal square pieces, then each person takes one square, leaving one square (see margin). Then they cut the leftover square piece into four equal square pieces, each person takes one square, leaving one square. And they keep repeating this process.

(a) What fraction of the total cake does each student "eventually" eat?

(b) Represent the amount of cake each person gets as an infinite series.

The infinite series in the previous Example and Practice problems are both **geometric series**, a type of series in which each term is a fixed multiple of the previous term. Geometric series have the form:

$$\sum_{k=0}^{\infty} C \cdot r^k = C + C \cdot r + C \cdot r^2 + C \cdot r^3 + \cdots$$
$$= C\left[1 + r + r^2 + r^3 + \cdots\right] = C \cdot \sum_{k=0}^{\infty} r^k$$

with $C \neq 0$ and $r \neq 0$ representing fixed numbers. Each term in the series is r times the previous term. Geometric series are among the most common series we will encounter, and among the easiest to work with. A simple test determines whether a geometric series converges, and we can even determine the sum of any convergent geometric series.

Geometric Series Theorem

The geometric series $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots$

- converges to $\dfrac{1}{1-r}$ if $|r| < 1$
- diverges if $|r| \geq 1$

Proof. If $|r| = 1$, then $\lim_{k\to\infty} \left|r^k\right| = 1$; if $|r| > 1$, then $\lim_{k\to\infty} \left|r^k\right| = \infty$. Either way, $\lim_{k\to\infty} r^k \neq 0$, so $\sum_{k=0}^{\infty} r^k$ diverges by the Test for Divergence.

If $|r| < 1$, then $\lim_{k\to\infty} r^k = 0$, so the Test for Divergence says that $\sum_{k=0}^{\infty} r^k$ may or may not converge.

Examining the partial sums $s_n = 1 + r + r^2 + r^3 + \cdots + r^n$ of the geometric series when $|r| < 1$, a clever insight allows us to find a simple formula for those partial sums:

$$(1-r)\cdot s_n = (1-r)\cdot\left(1 + r + r^2 + r^3 + \cdots + r^n\right)$$
$$= \left(1 + r + r^2 + r^3 + \cdots + r^n\right) - r \cdot \left(1 + r + r^2 + r^3 + \cdots + r^n\right)$$
$$= \left(1 + r + r^2 + r^3 + \cdots + r^n\right) - \left(r + r^2 + r^3 + \cdots + r^n + r^{n+1}\right)$$
$$= 1 - r^{n+1}$$

Because $|r| < 1$, we know that $r \neq 1$ so that $1 - r \neq 0$, meaning we can divide the preceding equality by $1 - r$ to get:

$$s_n = 1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

This formula for the n-th partial sum of a geometric series is sometimes useful; at the moment, we are interested in $\lim_{n\to\infty} s_n$.

Because $|r| < 1$, $\lim_{n\to\infty} r^{n+1} = 0$, so we can conclude that:

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n} r^k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}$$

giving a formula for the sum of any (convergent) geometric series. □

More generally, for any $C \neq 0$ and any r with $|r| < 1$ we can write:

$$\sum_{k=0}^{\infty} C \cdot r^k = \frac{C}{1 - r}$$

allowing us to quickly find the sum of any convergent geometric series.

Example 2. How far did the ball in Example 1 travel?

Solution. In Example 1, we expressed the total distance the ball travels as a geometric series, so:

$$20 \cdot \sum_{k=0}^{\infty} (0.8)^k = 20 \cdot \frac{1}{1 - 0.8} = \frac{20}{0.2} = 100$$

so the ball (theoretically) travels a total distance of 100 feet. ◀

Repeating decimal numbers are really geometric series in disguise, so we can now represent their exact values as fractions.

Example 3. Represent the repeating decimals $0.\overline{4}$ and $0.\overline{13}$ as geometric series and find their sums.

Solution. We can rewrite $0.\overline{4} = 0.444\ldots$ as:

$$\begin{aligned} 0.444\ldots &= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \cdots = \frac{4}{10} + \frac{4}{10^2} + \frac{4}{10^3} + \cdots \\ &= \frac{4}{10}\left[1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right] = \frac{4}{10} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k \\ &= \frac{4}{10} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{4}{10} \cdot \frac{1}{\frac{9}{10}} = \frac{4}{10} \cdot \frac{10}{9} = \frac{4}{9} \end{aligned}$$

so $0.\overline{4} = \frac{4}{9}$. Proceeding similarly with $0.\overline{13} = 0.131313\ldots$:

$$\begin{aligned} 0.131313\ldots &= \frac{13}{100} + \frac{13}{10000} + \frac{13}{1000000} + \cdots = \frac{13}{100} + \frac{13}{100^2} + \frac{13}{100^3} + \cdots \\ &= \frac{13}{100}\left[1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right] = \frac{13}{100} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{100}\right)^k \\ &= \frac{13}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{13}{100} \cdot \frac{1}{\frac{99}{100}} = \frac{13}{100} \cdot \frac{100}{99} = \frac{13}{99} \end{aligned}$$

so we can express $0.\overline{13}$ as $\frac{13}{99}$. ◀

Practice 2. Represent the repeating decimals $0.\overline{3}$ and $0.\overline{432}$ as geometric series and find their sums.

Replacing the number r with an expression involving x allows us to create a function defined as an infinite series.

Example 4. Given the functions $f(x) = \sum_{k=0}^{\infty} 3x^k = 3 + 3x + 3x^2 + \cdots$ and $g(x) = \sum_{k=0}^{\infty} (2x-5)^k = 1 + (2x-5) + (2x-5)^2 + \cdots$, find the domains of each function (that is, determine the values of x for which each infinite series converges).

Solution. We know that a geometric series converges if and only if $|r| < 1$, and the series defining $f(x)$ has ratio $r = x$, so it converges if and only if $|x| < 1$. The sum of this first series is:

$$f(x) = \sum_{k=0}^{\infty} 3x^k = 3 \cdot \sum_{k=0}^{\infty} x^k = 3 \cdot \frac{1}{1-x} = \frac{3}{1-x}$$

provided that $|x| < 1$, or, equivalently, that $-1 < x < 1$. Notice that the domain of the function $\frac{3}{1-x}$ consists of all real numbers except $x = 1$, but that $\sum_{k=0}^{\infty} 3x^k$ converges only when $-1 < x < 1$, so $f(x) = \frac{3}{1-x}$ holds only on the interval $(-1, 1)$.

In the series defining $g(x)$, the ratio is $r = 2x - 5$ so the series converges if and only if $|2x - 5| < 1$. Removing the absolute values and solving for x, we get:

$$|2x-5| < 1 \quad \Rightarrow \; -1 < 2x - 5 < 1 \quad \Rightarrow \; 2 < x < 3$$

so this second series converges precisely on the interval $(2, 3)$. The sum of the second series is:

$$g(x) = \sum_{k=0}^{\infty} (2x-5)^k = \frac{1}{1-(2x-5)} = \frac{1}{6-2x}$$

as long as $2 < x < 3$. ◀

Practice 3. Given $F(x) = \sum_{k=0}^{\infty} (2x)^k$ and $G(x) = \sum_{k=0}^{\infty} (3x-4)^k$, determine the values of x for which these series converge.

We call the type of series in the previous Example and Practice problems "**power series**" because they involve powers of the variable x. In the next chapter, we will embark on an extensive investigation of other power series, including many non-geometric series, such as:

These series are extensions of the MacLaurin polynomials from Section 8.7 to "infinite polynomials" with an unlimited number of terms.

$$1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

For each power series we will attempt to determine the values of x for which the series is guaranteed to converge.

The Harmonic Series: $\sum_{k=1}^{\infty} \frac{1}{k}$

The series $\sum_{k=1}^{\infty} \frac{1}{k}$ ranks among the best-known and most important divergent series. We call it the **harmonic series** because of its ties to music (see the discussion following the Practice Answers for additional background). Because $\lim_{k\to\infty} \frac{1}{k} = 0$, the Test for Divergence tells us nothing about the convergence or divergence of this series. Calculating partial sums of the harmonic series (see margin table) reveals that the partial sums s_n increase very, very slowly. By taking $n \approx 2,000,000$, we can make $s_n > 15$, but does it ever exceed 16? The answer to that question turns out to be "yes," but in our examination of the divergence of the harmonic series, brain power will prove to be much more effective than lots and lots of computing power.

n	s_n
31	4.0224519544
83	5.00206827268
227	6.00436670835
1,674	8.00048557200
12,367	10.00004300827
1,835,421	15.00000378267

Theorem:
The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

Proof. Assume the harmonic series converges, and let S be its sum:

$$S = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

Next, group the terms of the series as indicated by the brackets:

$$S = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] + \left[\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right] + \cdots$$

This proof is essentially due to Oresme in 1630 (12 years before Newton's birth). In 1821, Cauchy included Oresme's proof in his *Course in Analysis*, after which it became known as "Cauchy's argument."

so each set of brackets includes twice as many terms as the previous set. Notice that:

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

Looking at the next set of terms:

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

The sum of the terms in the next set of brackets exceeds $\frac{8}{16} = \frac{1}{2}$, the sum after that exceeds $\frac{16}{32} = \frac{1}{2}$, and so on. We can therefore write:

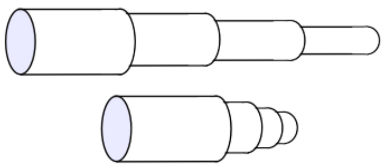
$$S > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

By adding enough sets of terms, you can make this lower bound for S arbitrarily large: S cannot be finite, so the harmonic series diverges. □

The harmonic series provides an example of a *divergent* series whose terms, $a_k = \dfrac{1}{k}$, approach 0. Any geometric series with $|r| < 1$ provides an example of a *convergent* series whose terms approach 0. This illustrates why the Test for Divergence says that a series with terms approaching 0 may or may not converge.

The Test for Divergence is typically a good first step when investigating the converge or divergence of an infinite series, but merely works as a "screening test" to identify certain series that definitely diverge.

Telescoping Series



During the 17th and 18th centuries, sailors used telescopes (see margin) that could be extended for viewing and collapsed for storage. Telescoping series get their name because they exhibit a similar "collapsing" property. Telescoping series arise infrequently, but they are easy to analyze and it can be useful to recognize them.

Example 5. Determine a formula for the partial sum s_n of the series $\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]$ and then compute $\displaystyle\lim_{n\to\infty} s_n$.

Solution. It is tempting to rewrite the formula for a_k as a single fraction, but the pattern becomes clearer if you begin writing out all of the terms:

$$s_1 = \left[1-\frac{1}{2}\right] = 1-\frac{1}{2}$$
$$s_2 = \left[1-\frac{1}{2}\right]+\left[\frac{1}{2}-\frac{1}{3}\right] = 1-\frac{1}{3}$$
$$s_3 = \left[1-\frac{1}{2}\right]+\left[\frac{1}{2}-\frac{1}{3}\right]+\left[\frac{1}{3}-\frac{1}{4}\right] = 1-\frac{1}{4}$$

In these partial sums, all terms cancel except the first and last terms, so we can write:

$$s_n = \left[1-\frac{1}{2}\right]+\left[\frac{1}{2}-\frac{1}{3}\right]+\left[\frac{1}{3}-\frac{1}{4}\right]+\cdots+\left[\frac{1}{n}-\frac{1}{n+1}\right] = 1-\frac{1}{n+1}$$

It should now be obvious that $\displaystyle\lim_{n\to\infty} s_n = 1$ so that the original series converges and equals 1. ◀

Practice 4. Find the sum of the series $\displaystyle\sum_{k=3}^{\infty}\left[\sin\left(\frac{1}{k}\right)-\sin\left(\frac{1}{k+1}\right)\right]$.

9.4 Problems

In Problems 1–12, calculate the value of the sum or explain why the series diverges.

1. $\displaystyle\sum_{k=0}^{\infty}\left(\frac{2}{7}\right)^k$
2. $\displaystyle\sum_{k=0}^{\infty}\left(\frac{4}{7}\right)^k$
3. $\displaystyle\sum_{k=0}^{\infty}\left(-\frac{4}{7}\right)^k$
4. $\displaystyle\sum_{k=0}^{\infty}\left(-\frac{2}{7}\right)^k$
5. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{2}{7}\right)^k$
6. $\displaystyle\sum_{k=2}^{\infty}\left(\frac{4}{7}\right)^k$
7. $\displaystyle\sum_{k=3}^{\infty}\left(-\frac{7}{4}\right)^k$
8. $\displaystyle\sum_{k=4}^{\infty}\left(-\frac{7}{2}\right)^k$
9. $\displaystyle\sum_{k=5}^{\infty}\left(-\frac{2}{7}\right)^k$

10. $\sum_{k=0}^{\infty}\left(\frac{e}{3}\right)^k$ 11. $\sum_{k=0}^{\infty}\left(\frac{\pi}{3}\right)^k$ 12. $\sum_{k=0}^{\infty}\left(\frac{e}{\pi}\right)^k$

In Problems 13–18, rewrite each geometric series using sigma notation, then compute the sum.

13. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$ 14. $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\cdots$

15. $\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots$

16. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$

17. $-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\frac{16}{81}-\cdots$

18. $1+\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}+\cdots$

19. Show that:
    (a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$
    (b) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$
    (c) $\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}+\cdots=\frac{1}{a-1}$ (for $a>1$)

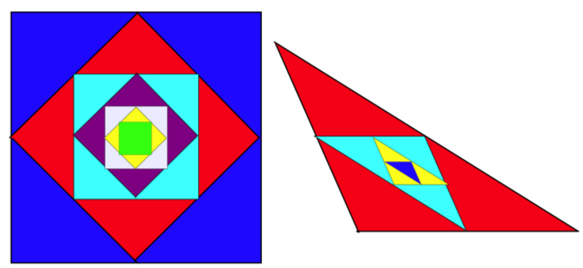
20. You throw a ball 10 feet straight up into the air, and on each bounce it rebounds to 60% of its previous height.
    (a) How far does the ball travel (up and down) during its n-th bounce?
    (b) Use an infinite sum to represent the total distance traveled by the ball.
    (c) Find the total distance traveled by the ball.

21. Your friend throws an old tennis ball 20 feet straight up into the air, and on each bounce it rebounds to 40% of its previous height.
    (a) How far does the ball travel (up and down) during its n-th bounce?
    (b) Use an infinite sum to represent the total distance traveled by the ball.
    (c) Find the total distance traveled by the ball.

22. Eighty people embark on an expedition by horseback through desolate country. The people and their gear require 90 horses, with additional horses needed to carry food for the original 90 horses. Each additional horse can carry enough food to feed three horses for the trip. How many total horses are needed for the trip?

23. Your friend follows a mathematical diet that says he can eat "half of whatever is on the plate," so he bites off half of a cake and puts the other half back on the plate. Then he picks up the remaining half from the plate (it's "on the plate"), bites off half of that and returns the rest to the plate. He continues this silly process of picking up the remaining piece from the plate, biting off half, and returning the rest to the plate.
    (a) Use an infinite sum to represent the total amount of cake he eats.
    (b) How much cake is left after one bite? Two bites? n bites?
    (c) "Eventually," how much does he eat?

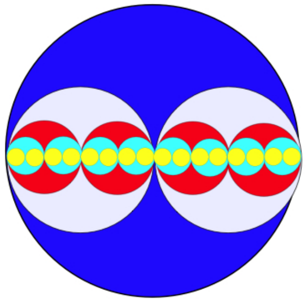
24. As indicated in the figure below left, begin with a square with sides of length 1 (so its area is 1). Construct another square inside the first one by connecting the midpoints of the sides of the first square, so the new square has area $\frac{1}{2}$. Continue this process of constructing each new square by connecting the midpoints of the sides of the previous square to get a sequence of squares, each of which has half the area of the previous square. Find the total area of all of the squares.

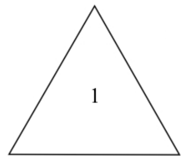


25. As indicated in the figure above right, begin with a triangle with area 1. Construct another triangle inside the first triangle by connecting the midpoints of the sides of the first triangle, so this new triangle will have area $\frac{1}{4}$. Imagine that you continue this construction process "forever" and find the total area of all of the triangles.

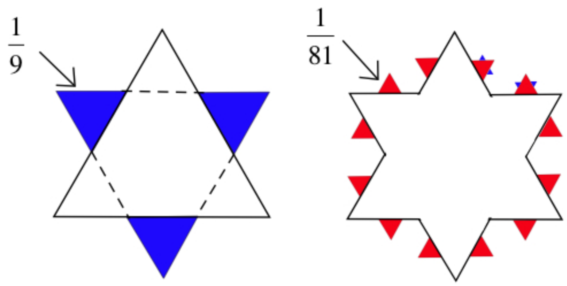
26. Begin with a circle of radius 1. Construct two more circles inside the first one, each with radius $\frac{1}{2}$. Continue this process "forever," constructing two new circles inside each previous circle (see

below left). Find the total area of all the circles.



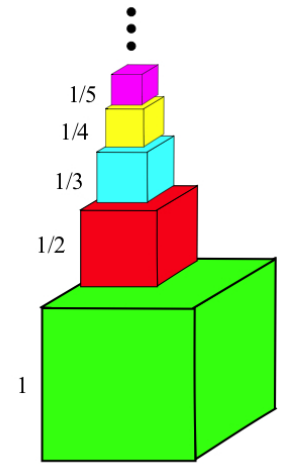


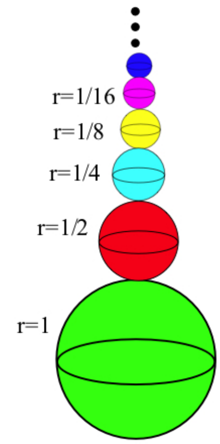
27. Swedish mathematician Helge von Koch (1870-1924) described one of the earliest examples of a fractal, now known as the **Koch snowflake**. Beginning with an equilateral triangle of area 1 (above right), subdivide each edge into three equal lengths, then build three equilateral triangles, each with area $\frac{1}{9}$, on the "middle thirds" of sides of the original triangle, adding a total of $3 \cdot \frac{1}{9} = \frac{1}{3}$ to the original area (below left). Now repeat this process: at the next stage, build $3 \cdot 4$ equilateral triangles, each with area $\frac{1}{81}$ on the new "middle thirds," adding $3 \cdot 4 \cdot \frac{1}{81}$ to the total area.
   (a) Find the total area that results when you repeat this process "forever."
   (b) Express the perimeter of the Koch Snowflake as a geometric series and find its sum.



28. The base of a "harmonic tower" is a cube with edges one foot long. Sitting on top of the bases are cubes with edges of length $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and so on.
   (a) Represent the total height of the tower as a series. Is the height finite?
   (b) Represent the total surface area of the cubes as an infinite series.
   (c) Represent the total volume of the cubes as an infinite series.

(See below left for a picture of the harmonic tower. In the next section, we will be able to determine whether its surface area and volume are finite or infinite.)





29. The base of a tower is a sphere with radius one foot. On top of each sphere sits another sphere with a radius half the radius of the sphere immediately beneath it (as in the figure above right).
   (a) Represent the total height of the tower as a series and evaluate the sum.
   (b) Represent the total surface area of the spheres as an infinite series and evaluate the sum.
   (c) Represent the total volume of the spheres as an infinite series and evaluate the sum.

30. Represent the repeating decimals $0.\overline{6}$ and $0.\overline{63}$ as geometric series and express the value of each series as a fraction in lowest terms.

31. Represent the repeating decimals $0.\overline{8}$, $0.\overline{9}$ and $0.\overline{285714}$ as geometric series and express the value of each series as a fraction in lowest terms.

32. Represent the repeating decimals $0.\overline{a}$, $0.\overline{ab}$ and $0.\overline{abc}$ as geometric series and express the value of each series as a fraction. What do you think a fractional representation for $0.\overline{abcd}$ would be?

In Problems 33–44, find all values of x for which the geometric series converges.

33. $\sum_{k=1}^{\infty} (2x+1)^k$ 34. $\sum_{k=1}^{\infty} (3-x)^k$ 35. $\sum_{k=1}^{\infty} (1-2x)^k$

36. $\sum_{k=1}^{\infty} 5x^k$ 37. $\sum_{k=1}^{\infty} (7x)^k$ 38. $\sum_{k=1}^{\infty} \left(\frac{x}{3}\right)^k$

39. $1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots$

40. $1+\frac{2}{x}+\frac{4}{x^2}+\frac{8}{x^3}+\cdots$

41. $1+2x+4x^2+8x^3+\cdots$

42. $\sum_{k=1}^{\infty} \left(\frac{2x}{3}\right)^k$ 43. $\sum_{k=1}^{\infty} \sin^k(x)$ 44. $\sum_{k=1}^{\infty} e^{kx}$

45. A student thought she remembered the formula for a geometric series as:

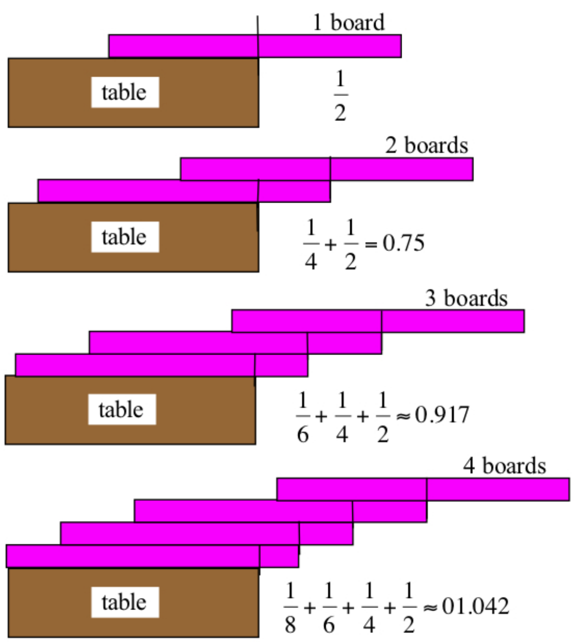
$$1+x+x^2+x^3+\cdots=\frac{1}{1-x}$$

Her friend said, "That can't be right. If we replace x with 2, then the formula says the sum of the positive numbers $1+2+4+8+\cdots$ is a negative number: $\frac{1}{1-2}=-1$." Who was right? Why?

46. If you have many identical 1-foot-long boards, you can arrange them so that they hang over the edge of a table. One board can extend $\frac{1}{2}$ foot beyond the edge, two boards can extend $\frac{1}{2}+\frac{1}{4}$ feet and, in general, n boards can extend:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}$$

feet beyond the edge (see below).



(a) How many boards do you need for an arrangement in which the entire top board sits beyond the edge of the table?

(b) How many boards do you need for an arrangement in which the entire top *two* boards sit beyond the edge of the table?

(c) How far can any such arrangement extend beyond the edge of the table?

In Problems 47–52, compute the value of the partial sums s_4 and s_5 for the given series, then find a formula for s_n. (The patterns may be more obvious if you do not simplify each term.)

47. $\sum_{k=3}^{\infty} \left[\frac{1}{k}-\frac{1}{k+1}\right]$

48. $\sum_{k=1}^{\infty} \left[\cos\left(\frac{1}{k}\right)-\cos\left(\frac{1}{k+2}\right)\right]$

49. $\sum_{k=1}^{\infty} \left[k^3-(k+1)^3\right]$ 50. $\sum_{k=1}^{\infty} \left[\ln\left(\frac{k}{k+1}\right)\right]$

51. $\sum_{k=1}^{\infty} [f(k)-f(k+1)]$ 52. $\sum_{k=1}^{\infty} [g(k)-g(k+1)]$

In 53–56, compute s_4 and s_5 for each series and then $\lim_{n\to\infty} s_n$. (If the limit is a finite value, it represents the value of the corresponding infinite series.)

53. $\sum_{k=1}^{\infty} \left[\sin\left(\frac{1}{k}\right)-\sin\left(\frac{1}{k+1}\right)\right]$

54. $\sum_{k=2}^{\infty} \left[\cos\left(\frac{1}{k}\right)-\cos\left(\frac{1}{k+1}\right)\right]$

55. $\sum_{k=2}^{\infty} \left[\frac{1}{k^2}-\frac{1}{(k+1)^2}\right]$ 56. $\sum_{k=3}^{\infty} \ln\left(1-\frac{1}{k^2}\right)$

Problems 57–58 outline two "proofs by contradiction" that the harmonic series diverges. Each proof begins with the assumption that the "sum" of the harmonic series is a finite number and then obtains an obviously false conclusion based on this assumption. Verify that each step follows from the assumption and the previous steps, then explain why the conclusion is false.

57. Assume that $H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is a finite number. Let $O = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$ be the sum of the "odd reciprocals" and $E = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ be the sum of the "even reciprocals." Then:
   - $H = O + E$
   - each O term $>$ the corresponding E term
   - $O > E$
   - $E = \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\right] = \frac{1}{2}H$
   - $H = O + E > 2E > 2 \cdot H > H$

58. Assume that $H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is a finite number. Starting with the second term, group the terms into groups of three. Using:

$$\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} > \frac{3}{n}$$

(show this inequality is true) conclude that:

$$\begin{aligned} H &= 1 + \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right] + \cdots \\ &> 1 + [1] + \left[\frac{1}{2}\right] + \left[\frac{1}{3}\right] + \cdots \\ &= 1 + \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots\right] = 1 + H \end{aligned}$$

so that $H > 1 + H$.

59. Jacob Bernoulli (1654–1705) was a master of understanding and manipulating series by breaking a difficult series into easier pieces. In his 1713 book *Ars Conjectandi*, he used such a technique to find the sum of the non-geometric series:

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots$$

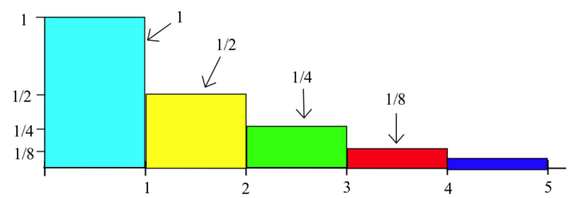
Show that you can write:

$$\begin{aligned} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots &= b_1 \\ \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots &= b_2 \\ \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots &= b_3 \\ \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots &= b_4 \end{aligned}$$

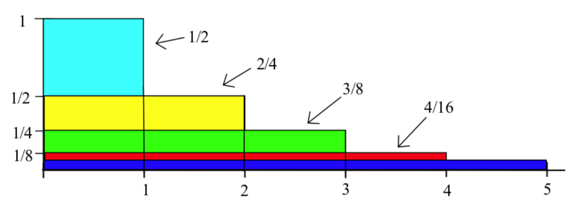
and so forth, so that $\sum_{k=1}^{\infty} \frac{k}{2^k} = \sum_{n=1}^{\infty} b_n$. Find the values of the geometric series b_n, and then find $\sum_{n=1}^{\infty} b_n$ (which will be another geometric series).

60. We can also interpret Bernoulli's approach in the previous problem as a geometric argument for representing the area of an infinitely long region in two different ways.

   (a) Represent the total area in the figure below as a (geometric) sum of areas of side-by-side rectangles, then find the sum of the series.



   (b) Represent the total area of the stacked rectangles in the figure below as a sum of the areas of the horizontal slices.



   (c) Explain why the series must be equal.

61. Use the approach of Problem 59 to find:

   (a) the value of the non-geometric series:

$$\sum_{k=1}^{\infty} \frac{k}{3^k} = \frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \cdots$$

   (b) a formula (when $c > 1$) for the value of:

$$\sum_{k=1}^{\infty} \frac{k}{c^k} = \frac{1}{c} + \frac{2}{c^2} + \frac{3}{c^3} + \frac{4}{c^4} + \cdots$$

9.4 Practice Answers

1. (a) Because each student gets an equal share, and because they eventually eat all of the cake, they each get $\frac{1}{3}$ of the cake. More precisely, after first first step, $\frac{1}{4}$ of the cake remains, with $\frac{3}{4}$ having been eaten by the students. After the second step, $\left(\frac{1}{4}\right)^2$ of the cake remains, with $1-\left(\frac{1}{4}\right)^2$ having been eaten. After the n-th step, $\left(\frac{1}{4}\right)^n$ of the cake remains, with $1-\left(\frac{1}{4}\right)^n$ having been eaten. So after the n-th step, each student has eaten $\frac{1}{3}\left[1-\left(\frac{1}{4}\right)^n\right]$ of the cake. "Eventually," each student gets (almost) $\frac{1}{3}$ of the cake.

   (b) As an infinite series each person gets:

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\cdots=\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k$$

2. We can rewrite $0.\overline{3} = 0.333\ldots$ as:

$$\begin{aligned}\frac{3}{10}+\frac{3}{100}+\frac{3}{1000} &= \frac{3}{10}\left[1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\right]\\ &= \frac{3}{10}\left[1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots\right]\end{aligned}$$

   which is a geometric series with $C=\frac{3}{10}$ and $r=\frac{1}{10}$. Because $|r|=\frac{1}{10}<1$, the series converges to:

$$\frac{C}{1-r}=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{\frac{3}{10}}{\frac{9}{10}}=\frac{3}{9}=\frac{1}{3}$$

   Similarly, we can rewrite $0.\overline{432} = 0.432432432\ldots$ as:

$$\begin{aligned}\frac{432}{1000}+\frac{432}{1000000}+\frac{432}{1000000000} &= \frac{432}{1000}\left[1+\frac{1}{1000}+\frac{1}{1000000}+\cdots\right]\\ &= \frac{432}{1000}\left[1+\frac{1}{1000}+\frac{1}{1000^2}+\cdots\right]\end{aligned}$$

   which is a geometric series with $C=\frac{432}{1000}$ and $r=\frac{1}{1000}$. Becasue $|r|=\frac{1}{1000}<1$, the series converges to:

$$\frac{C}{1-r}=\frac{\frac{432}{1000}}{1-\frac{1}{1000}}=\frac{\frac{432}{1000}}{\frac{999}{1000}}=\frac{432}{999}=\frac{16}{37}$$

3. The ratio for $F(x)$ is $r=2x$, so for the series to converge we need:

$$|2x|<1 \quad\Rightarrow\quad -1<2x<1 \quad\Rightarrow\quad -\frac{1}{2}<x<\frac{1}{2}$$

The series $\sum_{k=0}^{\infty}(2x)^k$ therefore converges to $\dfrac{1}{1-2x}$ when $-\dfrac{1}{2}<x<\dfrac{1}{2}$.
For $G(x)$, the ratio is $r=3x-4$, so we need:

$$|3x-4|<1 \;\Rightarrow\; -1<3x-4<1 \;\Rightarrow\; 3<3x<5 \;\Rightarrow 1<x<\frac{5}{3}$$

The series $\sum_{k=0}^{\infty}(3x-4)^k$ converges to $\dfrac{1}{1-(3x-4)}=\dfrac{1}{5-3x}$ when $1<x<\dfrac{5}{3}$.

4. Let $s_n=\sum_{k=3}^{n}\left[\sin\left(\frac{1}{k}\right)-\sin\left(\frac{1}{k+1}\right)\right]$ so that:

$$\begin{aligned} s_n &= \left[\sin\left(\frac{1}{3}\right)-\sin\left(\frac{1}{4}\right)\right]+\left[\sin\left(\frac{1}{4}\right)-\sin\left(\frac{1}{5}\right)\right] \\ &\qquad +\left[\sin\left(\frac{1}{5}\right)-\sin\left(\frac{1}{6}\right)\right]+\cdots+\left[\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right] \\ &= \left[\sin\left(\frac{1}{3}\right)-\sin\left(\frac{1}{n+1}\right)\right] \end{aligned}$$

allowing us to see that $\lim_{n\to\infty} s_n=\sin\left(\frac{1}{3}\right)$ and hence:

$$\sum_{k=3}^{\infty}\left[\sin\left(\frac{1}{k}\right)-\sin\left(\frac{1}{k+1}\right)\right]=\sin\left(\frac{1}{3}\right)\approx 0.327$$

Background on the Harmonic Series

A taut piece of string, such as a guitar string or piano wire, can only vibrate in such a way that it forms an integer number of waves. The fundamental mode determines the note being played, while the number and intensity of the harmonics (overtones) determine the characteristic quality of the sound. Because of these characteristic qualities, a listener can distinguish between a middle C (264 vibrations per second) played on a piano versus the same note played on a guitar or clarinet.