9.5 An Interlude

So far in this chapter we have investigated sequences and series, discussed convergence and divergence of infinite series, and examined geometric series in some detail. These concepts, definitions and results will play a fundamental role in understanding the material in the remainder of this chapter — and prove to be important for later work in theoretical and applied mathematics.

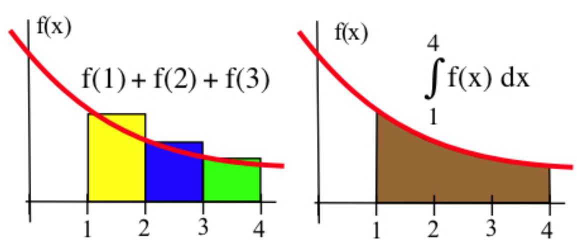
The material in the following sections is of a different sort: it is more technical and specialized. In order to work effectively with power series (the subject of Chapter 10) we need to know where (for which values of x) a power series converges. And to determine that convergence we need additional methods. Over the next several sections we will examine several methods (called "convergence tests") for determining whether particular series converge or diverge. For many of the convergent series we will study, we will be unable to determine the exact sum of the series, even if we know that the series converges.

As mentioned in the previous section, geometric series are a very special type of series for which it is easy to find an exact value for the sum of a convergent series.

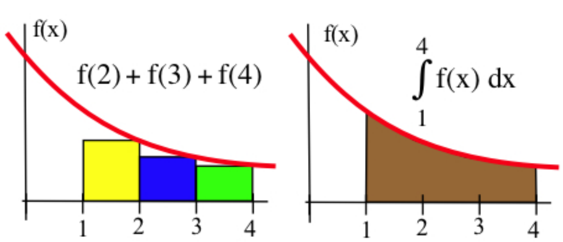
The Problems in this brief section illustrate some of the reasoning that you will need to use throughout the rest of this chapter (but they do not assume you know any information from those later sections). Practicing this reasoning now will pay dividends as you work through the rest of Chapter 9 and into Chapter 10.

9.5 Problems

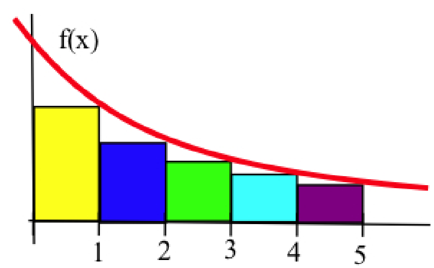
1. Which shaded region in the figure below has the larger area, the sum or the integral?



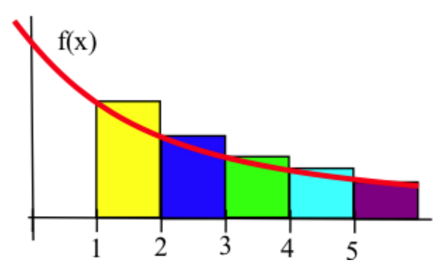
2. Which shaded region in the figure below has the larger area, the sum or the integral?



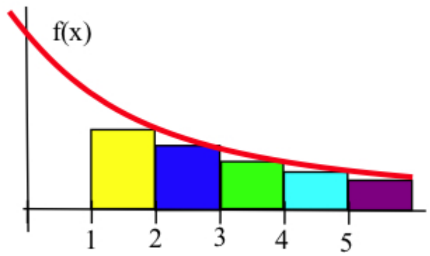
3. Represent the area of the shaded region in the figure below as an infinite series.



4. Represent the area of the shaded region in the figure below as an infinite series.



5. Represent the area of the shaded region in the figure below as an infinite series.

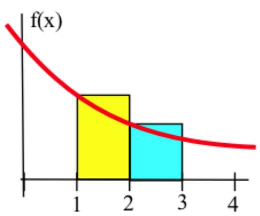


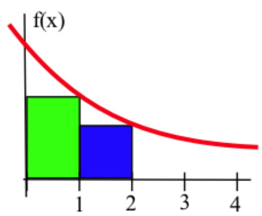
For Problems 6–9, which of the following expressions represents the shaded area in the given figure?

- $f(0)+f(1)$
- $f(1)+f(2)$
- $f(2)+f(3)$
- $f(3)+f(4)$

6. See below left.

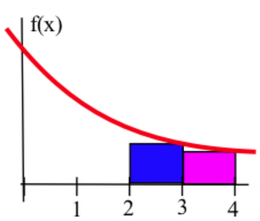
7. See below right.

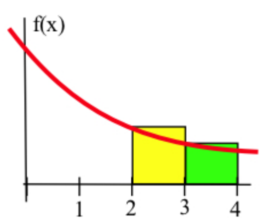




8. See below left.

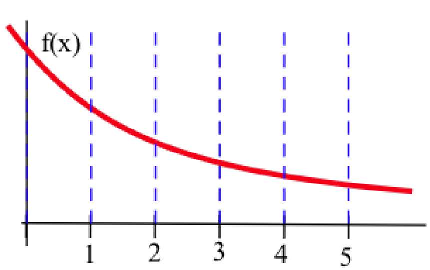
9. See below right.





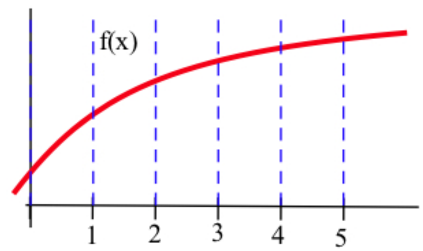
10. Refer to the figure below and arrange the following four values in increasing order:

- $\int_1^3 f(x)\,dx$
- $\int_2^4 f(x)\,dx$
- $f(1)+f(2)$
- $f(2)+f(3)$



11. Refer to the figure below and arrange the following four values in increasing order:

- $\int_1^4 f(x)\,dx$
- $\int_2^5 f(x)\,dx$
- $f(1)+f(2)+f(3)$
- $f(2)+f(3)+f(4)$



12. You want to get a summer job operating heavy equipment and you know there are certain height requirements in order for the operator to fit safely in the cab of the machine. You don't remember what the requirements are, but three of your friends applied. Tom was rejected as too tall. Sam was rejected as too short. Jenny got a job.

   (a) Should you apply for the job if:
      i. you are taller than Tom? Why?
      ii. you are taller than Sam? Why?
      iii. you are shorter than Sam? Why?
      iv. you are shorter than Jenny? Why?
   (b) List the comparisons that indicate you are the wrong height for the job.
   (c) List the comparisons that do not provide enough information to determine whether or not you are an acceptable height for the job.

13. You know Wendy did well on a recent calculus exam and Paula did poorly, but you haven't received your test back yet. If the instructor violates FERPA regulations and provides you the following information, what can you conclude?

   (a) "You did better than Wendy."
   (b) "You did better than Paula."
   (c) "You did worse than Wendy."
   (d) "You did worse than Paula."

14. Planning to take up mountain climbing, you consider a climb of Mt. Baker. You know that Mt. Index is too easy to be challenging, but that Mt. Liberty Bell is too difficult for you.
    (a) State what you can conclude about your plan to climb Mt. Baker if an experienced climber friend tells you that:
        i. "Baker is easier than Index."
        ii. "Baker is more difficult than Index."
        iii. "Baker is easier than Liberty Bell."
        iv. "Baker is more difficult than Liberty Bell."
    (b) Which comparisons indicate that Baker is appropriate: challenging but not too difficult?
    (c) Which comparisons indicate that Baker is not appropriate?

15. You have previously taken classes taught by Professors Good and Bad, and they each lived up to their names. Now you are considering taking a class taught by Prof. Unknown, with whom you are unfamiliar. State what can you expect about Prof. Unknown's class if:
    (a) a classmate who had Good and Unknown says, "Unknown was better than Good."
    (b) a classmate who had Good and Unknown says, "Good was better than Unknown."
    (c) a classmate who had Bad and Unknown says, "Unknown was better than Bad."
    (d) a classmate who had Bad and Unknown says, "Bad was better than Unknown."

In Problems 16–19, all of the series converge. For each pair, which series has the larger sum? (You shouldn't need to compute any of these sums to determine which is bigger. In fact, it is difficult—if not impossible—to find the exact values of these sums.)

16. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ vs. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

17. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^3-5}$ vs. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^3}$

18. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+3k-1}$ vs. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

19. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^2+5k}$ vs. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^3+k-1}$

Problems 20–28 give a formula a_k for the terms of a sequence. For each sequence:

(a) write a formula for a_{k+1}
(b) compute the ratio $\frac{a_{k+1}}{a_k}$
(c) simplify the ratio

20. $a_k = 3k$
21. $a_k = k+3$
22. $a_k = 2k+5$
23. $a_k = \dfrac{3}{k}$
24. $a_k = k^2$
25. $a_k = 2^k$
26. $a_k = \left(\frac{1}{2}\right)^k$
27. $a_k = x^k$
28. $a_k = (x-1)^k$

In Problems 29–36, state whether the series converges or diverges, then calculate the ratio $\dfrac{a_{k+1}}{a_k}$.

29. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$
30. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{5}\right)^k$
31. $\displaystyle\sum_{k=1}^{\infty} 2^k$
32. $\displaystyle\sum_{k=1}^{\infty} (-3)^k$
33. $\displaystyle\sum_{k=1}^{\infty} 4$
34. $\displaystyle\sum_{k=1}^{\infty} (-1)^k$
35. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$
36. $\displaystyle\sum_{k=1}^{\infty} \frac{7}{k}$

For Problems 37–41, s_n represents the n-th partial sum of the series with terms a_k (for example $s_3 = a_1 + a_2 + a_3$). In each problem, write the appropriate symbol ("$<$" or "$=$" or "$>$") in the box provided to make a true statement.

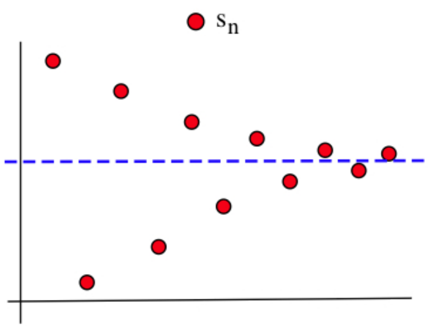
37. If $a_5 > 0$ then s_4 $\boxed{\quad}$ s_5
38. If $a_5 = 0$ then s_4 $\boxed{\quad}$ s_5
39. If $a_5 < 0$ then s_4 $\boxed{\quad}$ s_5
40. If $a_n > 0$ for all n, then s_n $\boxed{\quad}$ s_{n+1} for all n.
41. If $a_n < 0$ for all n, then s_n $\boxed{\quad}$ s_{n+1} for all n.

For Problems 42–45, explain how s_3, s_4, s_5 and s_6 compare, given the information about a_4, a_5 and a_6.

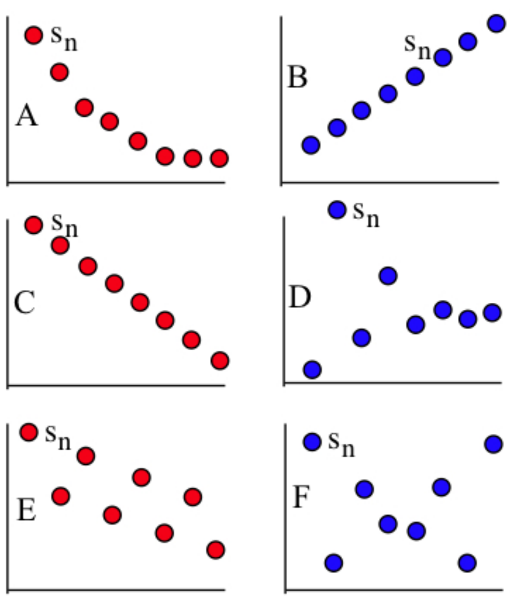
42. $a_4 > 0$, $a_5 > 0$ and $a_6 > 0$
43. $a_4 = 0.2$, $a_5 = -0.1$ and $a_6 = 0.2$
44. $a_4 = -0.3$, $a_5 = 0.2$ and $a_6 = -0.1$
45. $a_4 = -0.3$, $a_5 = -0.2$ and $a_6 = 0.1$

Problems 46–50 list the first eight terms $a_1, a_2, \ldots, a_8$ of a series. Calculate and graph the first eight partial sums $s_1, s_2, \ldots, s_8$ of the series and describe the pattern of the graph of the partial sums.

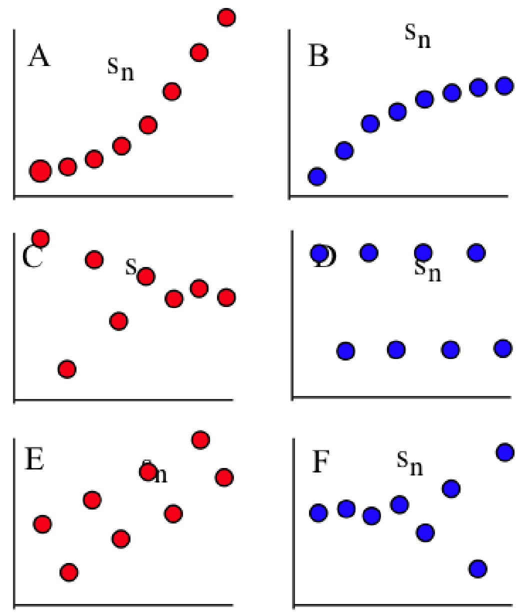
46. 2, −1, 2, −1, 2, −1, 2, −1
47. 2, −1, 0.9, −0.8, 0.7, −0.6, 0.5, −0.4
48. 2, −1, 1, −1, 1, −1, 1, −1
49. −2, 1.5, −0.8, 0.6, −0.4, 0.2, 2, −0.1
50. 5, 1, −0.6, −0.4, 0.2, 0.1, 0.1, −0.2
51. What condition on the terms a_k guarantees that the graph of the partial sums s_n follows an "up-down-up-down" pattern?
52. What condition on the terms a_k guarantees that the graph of the partial sums s_n follows a "narrowing funnel" pattern (see below)?



53. The figure below shows the graphs of several partial sums s_n.
    (a) For which do the terms a_k alternate in sign?
    (b) For which do the values of $|a_k|$ decrease?
    (c) For which graphs do the terms a_k alternate in sign and decrease in absolute value?



54. Redo Problem 53 for the figure below.



55. The geometric series:

$$\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$

converges to:

$$\frac{1}{1-(-\frac{1}{2})} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

Graph the horizontal line $y = \frac{2}{3}$ along with the partial sums s_0 through s_8 of the series.

56. The geometric series:

$$\sum_{k=0}^{\infty} (-0.6)^k = 1 - 0.6 + 0.36 - 0.216 + \cdots$$

converges to:

$$\frac{1}{1-(-0.6)} = \frac{1}{1.6} = 0.625$$

Graph the horizontal line $y = 0.625$ along with the partial sums s_0 through s_8 of the series.

57. The geometric series:

$$\sum_{k=0}^{\infty} (-2)^k = 1 - 2 + 4 - 8 + \cdots$$

diverges because $|r| = |-2| = 2 > 1$. Graph the partial sums s_0 through s_8 of the series.