9.7 *Comparison Tests*

In the previous section we compared the value of an infinite sum to the value of an improper integral and used the convergence or divergence of the integral to determine whether the series converged or diverged. In this section, we compare an infinite series to another infinite series already known to be convergent or divergent in order to determine the convergence or divergence of the new series. As with the Integral Test, the methods of this section only apply to series with positive terms.

Basic Comparison Test

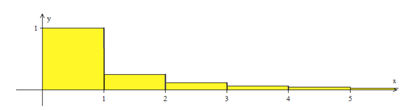
Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$. The first series is a p-series with $p = 2 > 1$, so we know it converges by the P-Test. The second series also converges, but it is not a p-series or a geometric series or a harmonic series, so we need to appeal to the Integral Test. Because:

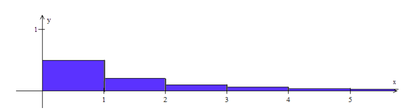
$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{M\to\infty} \int_1^{M} \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \Big[\arctan(x)\Big]_1^M$$
$$= \lim_{M\to\infty} \left[\arctan(M) - \arctan(1)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

the series $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ converges as well. Is there an easier way to see that this second series converges? For large values of k:

$$k^2+1 \approx k^2 \;\Rightarrow\; \frac{1}{k^2+1} \approx \frac{1}{k^2}$$

so we might suspect that the convergence of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ are related even if these series do not converge to the same sum. Consider a graph of rectangles with areas corresponding to $\sum_{k=1}^{\infty} \frac{1}{k^2}$ (see first margin figure) and another graph with rectangles corresponding to $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ (second margin figure). Comparing these graphs, it appears that $\sum_{k=1}^{\infty} \frac{1}{k^2+1} \le \sum_{k=1}^{\infty} \frac{1}{k^2}$. For any $k \ge 1$ and $n \ge 1$:





$$0 < \frac{1}{k^2+1} < \frac{1}{k^2} \Rightarrow 0 < \sum_{k=1}^{n} \frac{1}{k^2+1} < \sum_{k=1}^{n} \frac{1}{k^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2+1} \le \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Because we know the series on the right in the last inequality converges (as discussed above), the series on the left, being term-by-term smaller, should converge as well. The following test formalizes and generalizes this result.

To be precise, we know that:

$$\sum_{k=1}^{n} \frac{1}{k^2+1} \le \sum_{k=1}^{n} \frac{1}{k^2} \le \sum_{k=1}^{\infty} \frac{1}{k^2}$$

holds for all $n \ge 1$, so the partial sums $\sum_{k=1}^{n} \frac{1}{k^2+1}$ are increasing and bounded above, hence convergent by the Monotone Convergence Theorem.

The Basic Comparison Test (BCT) also works when the hypotheses hold for all $k \geq N$ for some positive integer N.

Basic Comparison Test (BCT)

If $0 < d_k \leq a_k \leq c_k$ for all $k \geq 1$, then:

- $\sum_{k=1}^{\infty} c_k$ converges $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ converges
- $\sum_{k=1}^{\infty} d_k$ diverges $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ diverges

Proof. If $0 < a_k \leq c_k$ and $\sum_{k=1}^{\infty} c_k$ converges, then:

$$s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} c_k \leq \sum_{k=1}^{\infty} c_k = C$$

for some finite number C, so the sequence $\{s_n\}$ is bounded above. Because $a_k > 0$, $\{s_n\}$ is monotonically increasing, so by the Monotone Converge Theorem $\{s_n\}$ converges, hence $\sum_{k=1}^{\infty} a_k$ converges.

It's important to note what the BCT does *not* say: if terms of $\sum_{k=1}^{\infty} a_k$ are each smaller than the corresponding terms of a divergent series, or each bigger than the corresponding terms of a convergent series, the BCT does not allow us to conclude anything about the convergence or divergence of $\sum_{k=1}^{\infty} a_k$.

If $0 < d_k \leq a_k$ and $\sum_{k=1}^{\infty} d_k$ diverges, then $s_n = \sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} d_k = D_n$. Because $d_k > 0$, $\{D_n\}$ is monotonically increasing, and because $\sum_{k=1}^{\infty} d_k$ diverges, the partial sums D_n must not be bounded above, hence the bigger partial sums s_n (which are also increasing) are not bounded above and $\lim_{n\to\infty} s_n = \infty$ so that $\sum_{k=1}^{\infty} a_k$ diverges. □

The Basic Comparison Test requires that we compare a given series to a series whose convergence or divergence we already know. This requires that we have at our disposal a collection of series that converge and some that diverge. Often we select a p-series or a geometric series to compare with the new series, but making this choice quickly requires some experience and practice.

Example 1. Use the Basic Comparison Test to determine the convergence or divergence of $\sum_{k=1}^{\infty} \frac{1}{k^2+3}$ and $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$.

Solution. We can compare each of these series with a p-series for an appropriate value of p. For the first series:

$$k^2 + 3 > k^2 \ \Rightarrow\ 0 < \frac{1}{k^2+3} < \frac{1}{k^2}$$

holds for all k. We know that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (it's a p-series with $p = 2 > 1$), so $\sum_{k=1}^{\infty} \frac{1}{k^2+3}$ must also converge.

For the second series, we know that (for all $k \geq 1$):

$$\frac{k+1}{k^2} > \frac{k}{k^2} = \frac{1}{k} > 0$$

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ must also diverge. ◀

Practice 1. Use the Basic Comparison Test to determine the convergence or divergence of $\sum_{k=3}^{\infty} \frac{1}{\sqrt{k-2}}$ and $\sum_{k=1}^{\infty} \frac{1}{2^k+7}$.

Example 2. Your classmate has shown that $\frac{1}{k^2} < \frac{1}{k^2-1} < \frac{1}{k}$ for all $k \geq 2$. Using this information and the Basic Comparison Test, what can you conclude about the convergence of the series $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$?

Solution. Nothing. The Basic Comparison Test only provides a definitive answer about a series if that series is smaller than a convergent series or larger than a divergent series. In this situation, the series $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ is larger than a convergent series, $\sum_{k=2}^{\infty} \frac{1}{k^2}$, and smaller than a divergent series, $\sum_{k=2}^{\infty} \frac{1}{k}$, so the Basic Comparison Test does not allow us to conclude anything about the convergence of $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$. ◀

Although the inequalities in the previous Example did not allow us to determine whether $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges or diverges, some clever algebra does allow us to employ the BCT with this series. For $k \geq 2$:

$$k^2 \geq 4 \Rightarrow \frac{1}{4}k^2 \geq 1 \Rightarrow -\frac{1}{4}k^2 \leq -1 \Rightarrow k^2 - \frac{1}{4}k^2 \leq k^2 - 1$$
$$\Rightarrow \frac{3}{4}k^2 \leq k^2 - 1 \Rightarrow \frac{\frac{4}{3}}{k^2} \geq \frac{1}{k^2-1}$$

Because $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, the BCT (together with the above inequality) tells us that $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ must also converge.

We could also have applied the Integral Test, which requires the use of Partial Fraction Decomposition:

$$\int_2^{\infty} \frac{1}{x^2-1}\, dx = \lim_{M\to\infty} \int_2^M \left[\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1} \right] dx = \frac{1}{2}\ln(3)$$

We've left out some steps here; you should fill them in.

to show that the series $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges, or we could have used the same partial fraction decomposition to rewrite the series:

$$\sum_{k=2}^{\infty} \frac{1}{k^2-1} = \frac{1}{2}\sum_{k=2}^{\infty}\left[\frac{1}{k-1} - \frac{1}{k+1}\right]$$
$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots\right] = \frac{3}{4}$$

as a telescoping series and show that its sum is $\frac{3}{4}$.

Limit Comparison Test

Consider the three infinite series:

$$\sum_{k=2}^{\infty} \frac{1}{k^2}, \quad \sum_{k=2}^{\infty} \frac{1}{k^2+1} \quad \text{and} \quad \sum_{k=2}^{\infty} \frac{1}{k^2-1}$$

All of them converge. Showing that the first series converges is easy: use the P-Test with $p = 2 > 1$. Because $0 < \frac{1}{k^2+1} < \frac{1}{k^2}$, it's relatively easy to show the second series converges (using the BCT and comparing it to the first series). The third series looks quite similar to the first two, but each of the three methods we used in the preceding discussion to show it converges was rather complicated. There must be a better way!

Limit Comparison Test (LCT)

If $a_k > 0$ and $b_k > 0$ for all k and $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ where $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge or both diverge.

Proof. If the hypotheses hold and $\sum_{k=1}^{\infty} b_k$ converges, then, because $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ and $0 < L < \infty$, there is some integer N so that:

$$k \geq N \Rightarrow \frac{a_k}{b_k} \leq L+1 \Rightarrow a_k \leq (L+1)b_k$$

and the Basic Comparison Test then tells us that $\sum_{k=1}^{\infty} a_k$ converges. If the hypotheses hold and $\sum_{k=1}^{\infty} b_k$ diverges, there is some integer N so that:

$$k \geq N \Rightarrow \frac{a_k}{b_k} \geq \frac{L}{2} > 0 \Rightarrow a_k \geq \frac{L}{2} \cdot b_k$$

The Basic Comparison Test then tells us that $\sum_{k=1}^{\infty} a_k$ diverges. □

Example 3. Use the LCT to show that $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges.

Solution. For $k \geq 2$, let $a_k = \frac{1}{k^2-1}$ and $b_k = \frac{1}{k^2}$ so $a_k > 0$, $b_k > 0$ and:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{k^2-1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1$$

Because $0 < 1 < \infty$ and $\sum_{k=2}^{\infty} \frac{1}{k^2}$ (a p-series, with $p = 2$) converges, the LCT tells us that $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges. ◀

Practice 2. Use the Limit Comparison Test to show whether the series $\sum_{k=1}^{\infty} \frac{k^2+5k}{k^3+k^2+7}$ and $\sum_{k=3}^{\infty} \frac{5}{\sqrt{k^4-11}}$ converge or diverge.

The Limit Comparison Test allows us to "ignore" some parts of the terms of a series that cause algebraic difficulties when using the BCT, but which have no effect on the convergence of the series.

Using "Dominant Terms"

To use the Limit Comparison Test with a new series, we need to find another, simpler series that we already know converges or diverges to compare with our new series. When the new series involves a rational expression, one effective method to find an appropriate simpler series is ignore everything but the largest power of the variable (the dominant term) in the numerator and denominator of the new series. The Limit Comparison Test will then allow us to conclude that the new series converges if and only if the "dominant term" series converges.

Example 4. For each of series below, form a new series using only the dominant terms from the numerator and the denominator. Does the "dominant term" series converge?

(a) $\sum_{k=3}^{\infty} \frac{5k^2-3k+2}{17+2k^4}$ (b) $\sum_{k=1}^{\infty} \frac{1+4k}{\sqrt{k^3+5k}}$ (c) $\sum_{k=1}^{\infty} \frac{k^{23}+1}{5k^{10}+k^{26}+3}$

Solution. (a) The dominant terms of the numerator and denominator are $5k^2$ and $2k^4$, respectively, so the "dominant term" series is $\sum_{k=3}^{\infty} \frac{5k^2}{2k^4} = \frac{5}{2} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2}$ (a p-series with $p = 2$), which converges.

(b) The dominant terms are $4k$ and $\sqrt{k^3} = k^{\frac{3}{2}}$, so the "dominant term" series is $\sum_{k=1}^{\infty} \frac{4k}{k^{\frac{3}{2}}} = 4 \cdot \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ (a p-series with $p = \frac{1}{2}$), which diverges.

(c) The dominant terms are k^{23} and k^{26}, so the "dominant term" series is $\sum_{k=1}^{\infty} \frac{k^{23}}{k^{26}} = \sum_{k=1}^{\infty} \frac{1}{k^3}$ (a p-series with $p = 3$), which converges.

Using the Limit Comparison Test to compare each of the given series with their corresponding "dominant term" series, we can conclude that the first and third converge and that the second diverges. ◀

Practice 3. For each of series below, form a new series using only the dominant terms from the numerator and the denominator. Does the "dominant term" series converge? Does the given series converge?

(a) $\sum_{k=1}^{\infty} \frac{3k^4 - 5k + 2}{1 + 17k^2 + 9k^5}$ (b) $\sum_{k=1}^{\infty} \frac{\sqrt{1+9k}}{k^2 + 5k - 2}$ (c) $\sum_{k=1}^{\infty} \frac{k^{25} + 1}{5k^{10} + k^{26} + 3}$

Experienced calculus students commonly use "dominant terms" to make quick and accurate judgments about the convergence or divergence of a series. With practice, so can you.

9.7 Problems

In Problems 1–12, use the Basic Comparison Test to determine whether the series converges or diverges.

1. $\sum_{k=1}^{\infty} \frac{\cos^2(k)}{k^2}$
2. $\sum_{k=1}^{\infty} \frac{3}{k^3 + 7}$
3. $\sum_{n=3}^{\infty} \frac{5}{n - 1}$
4. $\sum_{k=1}^{\infty} \frac{2 + \sin(k)}{k^3}$
5. $\sum_{m=1}^{\infty} \frac{3 + \cos(m)}{m}$
6. $\sum_{k=1}^{\infty} \frac{\arctan(k)}{k^{\frac{3}{2}}}$
7. $\sum_{k=2}^{\infty} \frac{\ln(k)}{k}$
8. $\sum_{k=2}^{\infty} \frac{k - 1}{k \cdot 1.5^k}$
9. $\sum_{k=1}^{\infty} \frac{k + 9}{k \cdot 2^k}$
10. $\sum_{n=2}^{\infty} \frac{n^3 + 7}{n^4 - 1}$
11. $\sum_{k=1}^{\infty} \frac{1}{k!}$
12. $\sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + \cdots + n}$

In Problems 13–22 use the Limit Comparison Test (or the Test for Divergence) to determine whether the given series converges or diverges.

13. $\sum_{k=3}^{\infty} \frac{k + 1}{k^2 + 4}$
14. $\sum_{k=1}^{\infty} \frac{7}{\sqrt{k^3 + 3}}$
15. $\sum_{w=1}^{\infty} \frac{5}{w + 1}$
16. $\sum_{n=1}^{\infty} \frac{7n^3 - 4n + 3}{3n^4 + 7n^3 + 9}$
17. $\sum_{k=1}^{\infty} \frac{k^3}{(1 + k^2)^3}$
18. $\sum_{k=1}^{\infty} \left(\frac{\arctan(k)}{k}\right)^2$
19. $\sum_{n=1}^{\infty} \frac{5 - \frac{1}{n}}{n}$
20. $\sum_{w=1}^{\infty} \left(1 + \frac{1}{w}\right)^w$
21. $\sum_{k=2}^{\infty} \left(\frac{1 - \frac{1}{k}}{k}\right)^3$
22. $\sum_{k=2}^{\infty} \sqrt{\frac{k^3 - 4}{k^5 + 1}}$

In 23–32, use a "dominant term" series to determine whether the given series converges or diverges.

23. $\sum_{n=3}^{\infty} \frac{n + 100}{n^3 + 4}$
24. $\sum_{n=3}^{\infty} \frac{n + 100}{n^2 - 4}$
25. $\sum_{k=1}^{\infty} \frac{7k}{\sqrt{k^3 + 5}}$
26. $\sum_{k=1}^{\infty} \frac{5}{k + 1}$
27. $\sum_{k=2}^{\infty} \frac{k^3 - 4k + 3}{2k^4 + 7k^6 + 9}$
28. $\sum_{n=1}^{\infty} \frac{5n^3 + 7n^2 + 9}{(1 + n^3)^2}$
29. $\sum_{k=1}^{\infty} \left(\frac{\arctan(3k)}{2k}\right)^2$
30. $\sum_{n=1}^{\infty} \left(\frac{3 - \frac{1}{n}}{n}\right)^2$
31. $\sum_{k=1}^{\infty} \frac{\sqrt{k^3 + 4k^2}}{k^2 + 3k - 2}$
32. $\sum_{k=2}^{\infty} \frac{\arcsin\left(1 - \frac{1}{k^2}\right)}{k}$

In 33–78, use any method from this or previous sections to determine whether the series converges or diverges. Include reasoning for your answers.

33. $\sum_{n=2}^{\infty} \frac{n^2+10}{n^3-3}$

34. $\sum_{k=1}^{\infty} \frac{3k}{\sqrt{k^5+7}}$

35. $\sum_{k=1}^{\infty} \frac{3}{2k+1}$

36. $\sum_{n=2}^{\infty} \frac{n^2-n+1}{3n^4+2n^2+1}$

37. $\sum_{n=1}^{\infty} \frac{2n^3+n^2+6}{(3+n^2)^2}$

38. $\sum_{k=1}^{\infty} \left(\frac{\arctan(k)}{3}\right)^k$

39. $\sum_{k=3}^{\infty} \sqrt{\frac{1-\frac{2}{k}}{k}}$

40. $\sum_{n=3}^{\infty} \frac{\sqrt{n^2+4n}}{(n-2)^3}$

41. $\sum_{k=1}^{\infty} \frac{(k+5)^2}{k^2 \cdot 3^k}$

42. $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2+4}$

43. $\sum_{k=1}^{\infty} \frac{k+2}{\sqrt{k^2+1}}$

44. $\sum_{k=1}^{\infty} \frac{\sin(k\pi)}{k+1}$

45. $\sum_{k=0}^{\infty} \frac{3}{e^k+k}$

46. $\sum_{n=0}^{\infty} \frac{(2+3n)^2+9}{(1+n^3)^2}$

47. $\sum_{n=1}^{\infty} \left(\frac{\tan(3)}{2+n}\right)^2$

48. $\sum_{n=1}^{\infty} n \cdot \sin\left(\frac{1}{n}\right)$

49. $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$

50. $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$

51. $\sum_{n=1}^{\infty} \sin^3\left(\frac{1}{n}\right)$

52. $\sum_{n=1}^{\infty} \cos^2\left(\frac{1}{n}\right)$

53. $\sum_{n=1}^{\infty} \cos^3\left(\frac{1}{n}\right)$

54. $\sum_{n=1}^{\infty} \tan^2\left(\frac{1}{n}\right)$

55. $\sum_{k=1}^{\infty} \left(1-\frac{2}{k}\right)^k$

56. $\sum_{k=1}^{\infty} \left(1+\frac{2}{k}\right)^k$

57. $\sum_{k=1}^{\infty} \frac{5}{3^k}$

58. $\sum_{n=1}^{\infty} \frac{5+\cos(n^3)}{n^2}$

59. $\sum_{n=1}^{\infty} \frac{2}{3+\sin(n^3)}$

60. $\sum_{k=1}^{\infty} \frac{5}{\left(\frac{1}{3}\right)^k}$

61. $\sum_{k=0}^{\infty} e^{-k}$

62. $\sum_{k=0}^{\infty} \left(\frac{\pi}{e}\right)^k$

63. $\sum_{k=0}^{\infty} \left(\frac{\pi^2}{e^3}\right)^k$

64. $\sum_{k=1}^{\infty} \cos\left(\frac{1}{k^3}\right)$

65. $\sum_{n=1}^{\infty} \frac{5+\cos(n^2)}{n^3}$

66. $\sum_{k=1}^{\infty} \frac{1}{k \cdot [3+\ln(k)]}$

67. $\sum_{m=1}^{\infty} \frac{1}{m \cdot [3+\ln(m)]^2}$

68. $\sum_{n=1}^{\infty} \frac{4}{n \cdot \arctan(n)}$

69. $\sum_{n=1}^{\infty} \frac{4\arctan(n)}{n}$

70. $\sum_{k=2}^{\infty} \frac{\ln(k)}{k^3}$

71. $\sum_{k=2}^{\infty} \frac{\ln(k)}{k^2}$

72. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$

73. $\sum_{n=1}^{\infty} \frac{1+n}{1+n^2}$

74. $\sum_{k=2}^{\infty} \left[\sin(k)-\sin(k+1)\right]$

75. $\sum_{k=1}^{\infty} \sqrt{\frac{k^3+5}{k^5+3}}$

76. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

77. $\sum_{n=1}^{\infty} n^{\frac{1}{n}}$

78. $\sum_{k=1}^{\infty} \sqrt[3]{\frac{k^3+7}{k^8+5}}$

9.7 Practice Answers

1. Considering the first series, for any integer $k \geq 3$:

$$k-2 < k \quad \Rightarrow \quad \sqrt{k-2} < \sqrt{k} \quad \Rightarrow \quad \frac{1}{\sqrt{k}} < \frac{1}{\sqrt{k-2}}$$

Because $\sum_{k=3}^{\infty} \frac{1}{\sqrt{k}}$ (a p-series with $p = \frac{1}{2} \leq 1$) diverges and each term of $\sum_{k=3}^{\infty} \frac{1}{\sqrt{k-2}}$ is bigger, the BCT says $\sum_{k=3}^{\infty} \frac{1}{\sqrt{k}}$ also diverges.

Considering the second series, for any k:

$$0 < 7 \quad \Rightarrow \quad 2^k < 2^k + 7 \quad \Rightarrow \quad \frac{1}{2^k} > \frac{1}{2^k + 7}$$

Because $\sum_{k=3}^{\infty} \left(\frac{1}{2}\right)^k$ (a geometric series with ratio $|r| = \frac{1}{2} < 1$) converges and each term of $\sum_{k=3}^{\infty} \frac{1}{2^k + 7}$ is even smaller, the BCT tells us that $\sum_{k=3}^{\infty} \frac{1}{2^k + 7}$ also converges.

2. For large values of k, the terms of the series $\sum_{k=1}^{\infty} \frac{k^2 + 5}{k^3 + k^2 + 7}$ behave like $\frac{k^2}{k^3} = \frac{1}{k}$ so we will compare the given series to $\sum_{k=1}^{\infty} \frac{1}{k}$, which we know diverges (it's the harmonic series). Computing the limit of the ratio of the corresponding terms of these series:

$$\lim_{k\to\infty} \frac{\frac{k^2+5}{k^3+k^2+7}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^3 + 5k}{k^3 + k^2 + 7} = \lim_{k\to\infty} \frac{1 + \frac{5}{k^2}}{1 + \frac{1}{k} + \frac{7}{k^3}} = 1$$

Because $0 < 1 < \infty$, the LCT says $\sum_{k=1}^{\infty} \frac{k^2 + 5}{k^3 + k^2 + 7}$ also diverges.

For large values of k, the terms of the series $\sum_{k=3}^{\infty} \frac{5}{\sqrt{k^4 - 11}}$ behave like $\frac{5}{\sqrt{k^4}} = \frac{5}{k^2}$ so we will compare the given series to $\sum_{k=3}^{\infty} \frac{5}{k^2}$, which we know converges (it's a constant multiple of a p-series with $p = 2 > 1$). The limit of the ratio of the corresponding terms of these series is:

$$\lim_{k\to\infty} \frac{\frac{5}{\sqrt{k^4-11}}}{\frac{5}{k^2}} = \lim_{k\to\infty} \frac{k^2}{\sqrt{k^4 - 11}} = \lim_{k\to\infty} \frac{1}{\sqrt{1 - \frac{11}{k^4}}} = 1$$

Because $0 < 1 < \infty$, the LCT says $\sum_{k=3}^{\infty} \frac{5}{\sqrt{k^4 - 11}}$ also converges.

3. (a) $\sum_{k=1}^{\infty} \frac{3k^4}{9k^5} = \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (it's a multiple of the harmonic series), so $\sum_{k=1}^{\infty} \frac{3k^4 - 5k + 2}{1 + 17k^2 + 9k^5}$ also diverges (by the LCT).

(b) $\sum_{k=1}^{\infty} \frac{\sqrt{9k}}{k^2} = 3 \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ converges (it's a multiple of a p-series with $p = \frac{3}{2} > 1$), so $\sum_{k=1}^{\infty} \frac{\sqrt{1 + 9k}}{k^2 + 5k - 2}$ also converges (by the LCT).

(c) $\sum_{k=1}^{\infty} \frac{k^{25}}{k^{26}} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so $\sum_{k=1}^{\infty} \frac{k^{25} + 1}{5k^{10} + k^{26} + 3}$ diverges (LCT).

Why does $\sum_{k=1}^{\infty} \frac{1}{k}$ diverge?