9.8 *Alternating Series*

The Integral Test, P-Test and comparison tests each apply only to series with positive terms. We now examine some series with both positive and negative terms, focusing on series with terms that alternate between positive and negative values.

Examples of Alternating Series

An **alternating series** is a series with terms that alternate between positive and negative. Each of the following are alternating series:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$\sum_{k=1}^{\infty} (-1)^{k} \frac{k}{k+2} = -\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \cdots$$

$$\sum_{k=1}^{\infty} (-1)^{k} \frac{1}{\sqrt{2k+1}} = -\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{9}} - \frac{1}{\sqrt{11}} + \cdots$$

We call this first example the **alternating harmonic series**.

The margin graphs show values of several partial sums s_n for each of these series. As n increases, the partial sums s_n alternately get larger and smaller, a typical pattern for the partial sums of alternating series. This same pattern appears in tables of the partial sums:

$\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

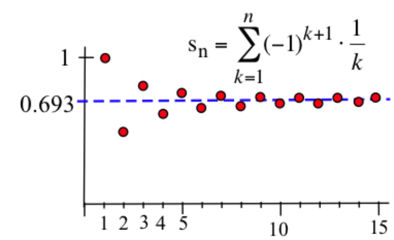
n	s_n
1	1.0
2	0.5
3	0.8333
4	0.5833
5	0.7833
6	0.6166
7	0.7595
8	0.6345
9	0.7456
10	0.6456

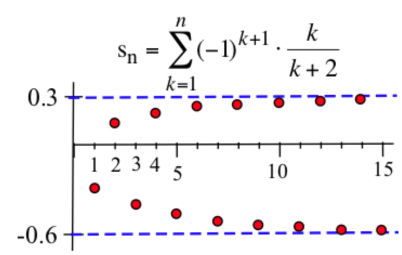
$\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k} \cdot k}{k+2}$

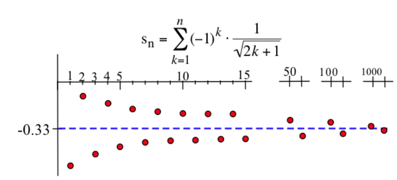
n	s_n
1	-0.3333
2	0.1667
3	-0.4333
4	0.2333
5	-0.4810
6	0.2690
7	-0.5087
8	0.2913
9	-0.5269
10	0.3064

$\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k}}{\sqrt{2k+1}}$

n	s_n
1	-0.5774
2	-0.1301
⋮	⋮
50	-0.2828
51	-0.3813
⋮	⋮
1000	-0.3211
1001	-0.3435







For the first and third series, the partial sums appear to be converging to a limit. For the second series, the partial sums appear to jump back and forth between values near 0.3 and values near -0.6: because $\displaystyle\lim_{k\to\infty} \frac{(-1)^k \cdot k}{k+2}$ does not exist (and, in particular, does not equal 0), the Test for Divergence tells us that this series diverges. The next result will help us determine that the other two series converge.

Alternating Series Test

The next result provides an easy way to determine that *some* alternating series converge: if the absolute values of the terms of an alternating series decrease monotonically to 0, then the series converges.

> **Alternating Series Test (AST)**
>
> If the numbers a_k satisfy the three conditions:
>
> - $a_k > 0$ for all k (each a_k is positive)
> - $a_k > a_{k+1}$ ($\{a_k\}$ is monotonically decreasing)
> - $\lim_{k\to\infty} a_k = 0$
>
> then the alternating series $\sum_{k=0}^{\infty} (-1)^k a_k$ converges.

Proof. To show that the alternating series converges, we need to show that the sequence of partial sums approaches a finite limit. We do so by showing that the subsequences of even partial sums $\{s_2, s_4, s_6, \ldots\}$ and odd partial sums $\{s_1, s_3, s_5, \ldots\}$ each approach the same value.

Because $\{a_k\}$ is a decreasing sequence, we know that $a_1 > a_2$, $a_3 > a_4$ and so forth, so that for the even partial sums:

$$s_2 = a_1 - a_2 > 0$$
$$s_4 = s_2 + (a_3 - a_4) > s_2$$
$$s_6 = s_4 + (a_5 - a_6) > s_4$$

In general, the sequence of even partial sums is positive and increasing:

$$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2}) > s_{2n} > 0$$

because $\{a_k\}$ is decreasing, so that $a_{2n+1} > a_{2n+2}$ for all n. Furthermore:

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots - a_{2n-2} + a_{2n-1} - a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} < a_1 \end{aligned}$$

so the sequence of even partial sums is bounded above by a_1. Because the sequence $\{s_2, s_4, s_6, \ldots\}$ of even partial sums is increasing and bounded above, the Monotone Convergence Theorem tells us the sequence of even partial sums converges to some finite limit: $\lim_{n\to\infty} s_{2n} = L$

We can write any odd partial sum as $s_{2n+1} = s_{2n} + a_{2n+1}$ so that:

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1} = L + 0 = L$$

Because the sequence of even partial sums and the sequence of odd partial sums both approach the same limit, L, we can conclude that the limit of the sequence consisting of all partial sums is L and that the alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$ converges (to L). □

Example 1. Show that each alternating series below satisfies the three conditions in the hypotheses of the Alternating Series Test, allowing you to conclude that each of them converges.

(a) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

(b) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{\sqrt{k}} = \frac{3}{1} - \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{3}} - \frac{3}{\sqrt{4}} + \cdots$

(c) $\sum_{k=2}^{\infty} (-1)^{k} \frac{7}{k \cdot \ln(k)} = \frac{7}{2\ln(2)} - \frac{7}{3\ln(3)} + \frac{7}{4\ln(4)} - \frac{7}{5\ln(5)} + \cdots$

Solution. (a) Here $a_k = \frac{1}{k}$, so $a_k > 0$ for all $k \geq 1$. The function $f(x) = x^{-1}$ satisfies $f(k) = a_k$ for $k \geq 1$, and $f'(x) = -x^{-2} < 0$, which tells us that $f(x)$ is decreasing, hence a_k is a decreasing sequence. Finally, $\lim_{k\to\infty} \frac{1}{k} = 0$, so all three conditions are satisfied.

In each part of this Example we use the third technique from Practice 4 in Section 9.2 to show that a sequence is monotonically decreasing. This is not the only possible method, but turns out to be convenient for the sequences under consideration here.

(b) Here $a_k = \frac{3}{\sqrt{k}}$, so $a_k > 0$ for all $k \geq 1$. The function $f(x) = 3x^{-\frac{1}{2}}$ satisfies $f(k) = a_k$ for $k \geq 1$, and $f'(x) = -\frac{3}{2}x^{-\frac{3}{2}} < 0$, which tells us that $f(x)$ is decreasing, hence a_k is a decreasing sequence. Finally, $\lim_{k\to\infty} \frac{3}{\sqrt{k}} = 0$, so all three conditions are satisfied.

(c) Here $a_k = \frac{1}{k \cdot \ln(k)}$, so $a_k > 0$ for all $k \geq 2$. If $f(x) = x \cdot \ln(x)$, $a_k = \frac{1}{f(k)}$ for $k \geq 2$, and $f'(x) = 1 + \ln(x) > 0$ for $x \geq 2$, which tells us that $f(x)$ is increasing, hence a_k is a decreasing sequence. Finally, $\lim_{k\to\infty} \frac{1}{k \cdot \ln(k)} = 0$, so all three conditions are satisfied. The Alternating Series Test therefore tells us that all three series converge. ◀

Practice 1. Show that each alternating series below satisfies the three conditions in the hypotheses of the Alternating Series Test, allowing you to conclude that each of them converges.

(a) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$

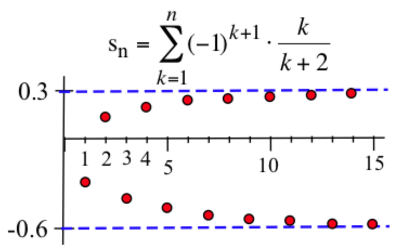
(b) $\sum_{k=2}^{\infty} (-1)^{k} \frac{3}{\ln(k)} = \frac{3}{\ln(2)} - \frac{3}{\ln(3)} + \frac{3}{4\ln(4)} - \frac{3}{\ln(5)} + \cdots$

Examples of Divergent Alternating Series

If $a_k > 0$ fails to hold for a series written in the form $\sum_{k=0}^{\infty} (-1)^k a_k$, then the series does not alternate and the AST fails to apply. Such a series may converge or diverge.

For examples, think of any convergent or divergent series will all positive terms.

If $\lim_{k\to\infty} a_k = 0$ fails to hold for *any* series (not just alternating series), then the Test for Divergence tells us that the series diverges.



Example 2. Does $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k+2}$ converge?

Solution. Here $a_k = \frac{k}{k+2} > 0$ but $\lim_{k\to\infty} \frac{k}{k+2} = 1 \neq 0$, so the Test For Divergence tell us the series diverges. (The margin figure shows some of the partial sums for this series. You should notice that the even and the odd partial sums are approaching two different values.) ◀

Practice 2. Does $\sum_{k=1}^{\infty} (-1)^{k+1} k$ converge? Does $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}}$ converge?

If $a_k > 0$ and $\lim_{k\to\infty} a_k = 0$ for an alternating series $\sum_{k=0}^{\infty} (-1)^k a_k$ but the terms $\{a_k\}$ are not monotonically decreasing, the AST does not apply and the series may converge or may diverge. For example:

$$\frac{3}{2} - \frac{1}{2} + \frac{3}{4} - \frac{1}{4} + \frac{3}{6} - \frac{1}{6} + \frac{3}{8} - \frac{1}{8} + \cdots$$

is an alternating series whose terms approach 0, but the even partial sums of this series are:

$$s_2 = \frac{3}{2} - \frac{1}{2} = 1$$
$$s_4 = \left(\frac{3}{2} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{1}{4}\right) = 1 + \frac{1}{2}$$
$$s_6 = \left(\frac{3}{2} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{1}{4}\right) + \left(\frac{3}{6} - \frac{1}{6}\right) = 1 + \frac{1}{2} + \frac{1}{3}$$

and, in general:

$$s_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

You should recognize these even partial sums as partial sums of the harmonic series, which diverges, so the (even) partial sums of our given series diverge, hence the sequence of all partial sums of that series diverges, so the given series diverges.

Can you think of an alternating series with terms that approach 0 non-monotonically that converges?

Approximating the Sum of an Alternating Series

If you know that a series converges and if you add up the first "many" terms, then you should expect that the resulting partial sum is reasonably "close" to the value S obtained by adding all the terms together. Generally, however, we do not know how close the partial sum is to S (because we don't know the value of S). The situation with many alternating series is much nicer.

In Section 9.6, we used a consequence of the Integral Test to help find upper and lower bounds for $\sum_{k=1}^{\infty} a_k$, but this only applied to series with positive terms for which we could find a continuous, positive, decreasing function $f(x)$ with $f(k) = a_k$.

Alternating Series Estimation Bound

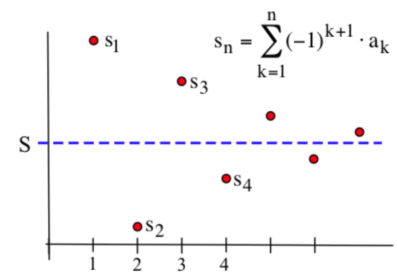
If $S = \sum_{k=0}^{\infty} (-1)^k a_k$ and the numbers a_k satisfy:

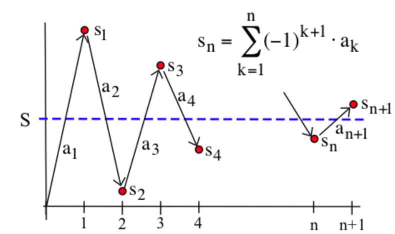
- $a_k > 0$ for all k (each a_k is positive)
- $a_k > a_{k+1}$ ($\{a_k\}$ is monotonically decreasing)
- $\lim_{k\to\infty} a_k = 0$

then $|S - s_n| < a_{n+1}$ for any partial sum $s_n = \sum_{k=0}^{n} (-1)^k a_k$.

This estimation bound only applies to *alternating* series. It is often tempting—but wrong—to use it with other series.

The geometric idea behind this estimation bound exploits the fact that when an alternating series satisfies the hypotheses of the AST, then the graph of the sequence $\{s_n\}$ of partial sums is "trumpet-shaped" or "funnel-shaped" (see margin). The partial sums alternately fall above and below the value S and "squeeze" in on the value S. Because the distance from s_n to S is less than the distance between the successive terms s_n and s_{n+1} (see second margin figure) and $|s_{n+1} - s_n| = a_{n+1}$.





Proof. The distance between the sum S and the n-th partial sum s_n is:

$$\begin{aligned}
|S - s_n| &= \left| \left[\sum_{k=0}^{\infty} (-1)^k a_k \right] - \left[\sum_{k=0}^{n} (-1)^k a_k \right] \right| = \left| \sum_{k=n+1}^{\infty} (-1)^k a_k \right| \\
&= \left| (-1)^{n+1} a_{n+1} + (-1)^{n+2} a_{n+2} + (-1)^{n+3} a_{n+3} + \cdots \right| \\
&= \left| (-1)^{n+1} \left[a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots \right] \right| \\
&= |a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots| \\
&= |(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + (a_{n+5} - a_{n+6}) + \cdots| \\
&= (a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + (a_{n+5} - a_{n+6}) + \cdots \\
&= a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \cdots < a_{n+1}
\end{aligned}$$

The last two equalities hold because $\{a_k\}$ is monotonically decreasing, so that $a_m > a_{m+1} \Rightarrow a_m - a_{m+1} > 0$ for any $m \geq 1$. ☐

We typically use this estimation bound in two different ways. Sometimes you know the value of n and you want to know how close s_n is to S. Other times you know how close you need s_n to be to S and want to find a value of n to ensure that level of closeness. The next two Examples illustrate these two different uses of the Alternating Series Estimation Bound.

Example 3. How close is $s_4 = \sum_{k=1}^{4} \frac{(-1)^{k+1}}{k^2}$ to $S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$?

Solution. The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ satisfies the hypotheses of the AST with $a_k = \frac{1}{k^2}$: for $k \geq 1$, $a_k > 0$ and $\{a_k\}$ is a decreasing sequence with limit 0. Computing the partial sum:

$$s_4 = \sum_{k=1}^{4} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} \approx 0.79861$$

so that:

$$|S - s_4| < a_5 = \frac{1}{25} = 0.04 \Rightarrow |S - 0.79861| < 0.04$$

hence $-0.04 < S - 0.79861 < 0.04 \Rightarrow 0.75861 < S < 0.83861$. ◀

Practice 3. Evaluate s_4 and s_9 for $S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$ and determine bounds for $|S - s_4|$ and $|S - s_9|$.

Example 4. Find a value N so that s_N will be within 0.001 of the exact value of $S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ and evaluate s_N.

Solution. The given series satisfies the hypotheses of the AST with $a_k = \frac{1}{k!}$ because $\frac{1}{k!}$ decreases monotonically to 0, so we know that $|S - s_N| < a_{N+1} = \frac{1}{(N+1)!}$. We need to find N so that $\frac{1}{(N+1)!} \leq 0.001 = \frac{1}{1000}$. With a little experimentation using a calculator, you can see that $6! = 720$, which doesn't quite work, but that $7! = 5040 > 1000$ so that $\frac{1}{7!} = \frac{1}{5040} \approx 0.000198 < 0.001$. With $N + 1 = 7 \Rightarrow N = 6$, $s_6 \approx 0.631944$ is the first partial sum *guaranteed* to be within 0.001 of S. In fact, s_6 is guaranteed to be within 0.000198 of S, so $0.631746 < S < 0.632142$. ◀

Practice 4. Find a value N so that s_N will be within 0.001 of the exact value of $S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 + 5}$ and evaluate s_N.

Wrap-Up

The Alternating Series Estimation Bound guarantees that s_n will be within a_{n+1} of S. In fact, s_n is often much closer to S.

Because the first finite number of terms do not affect the convergence or divergence of a series (they *do* affect its sum, S) you can use the Alternating Series Test and the Alternating Series Estimation Bound as long as the terms of a series "eventually" satisfy the required hypotheses. More precisely, "eventually" means that there is a value N so that for $n \geq N$ the series is an alternating series and the absolute value of the terms of that series decrease monotonically to 0.

If a series has some positive terms and some negative terms but those terms do **not** "eventually" alternate in sign, then you can **not** use the Alternating Series Test: it simply does not apply to such series. A result from the next section will allow us to show that *some* series of this type converge, but in general you will need more advanced methods to investigate the convergence and divergence of series with both positive and negative terms that do not (eventually) alternate.

9.8 Problems

Problems 1–6 give the values of the first four terms of a series. For each series, (a) calculate and graph the first four partial sums and (b) Determine whether or not it is an alternating series.

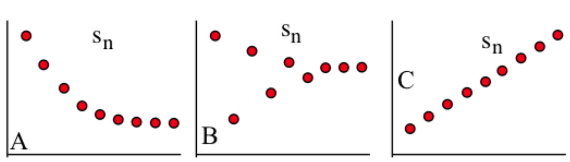
1. $1, -0.8, 0.6, -0.4$
2. $-1, 1.5, -0.7, 1$
3. $-1, 2, -3, 4$
4. $2, -1, -0.5, 0.3$
5. $-1, -0.6, 0.4, 0.2$
6. $2, -1, 0.5, -0.3$

Problems 7–12 give the values of the first five partial sums of a series. Which of the series are not alternating series. Why?

7. $2, 1, 3, 2, 4$
8. $2, 1, 1.8, 1.4, 1.6$
9. $2, 3, 2.1, 2.9, 2.8$
10. $-3, -1, -2.5, -1.5, -2$
11. $-1, 1, -0.8, -0.6, -0.4$
12. $-2.3, -1.6, -1.4, -1.8, -1.7$

Problems 13–16 shows the graphs of the partial sums of three series. Which is/are not the partial sums of alternating series? Why?

13.



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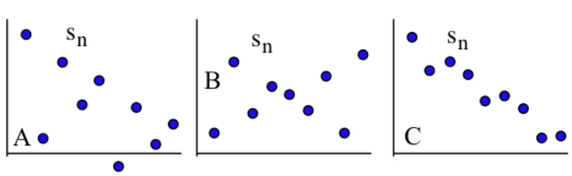

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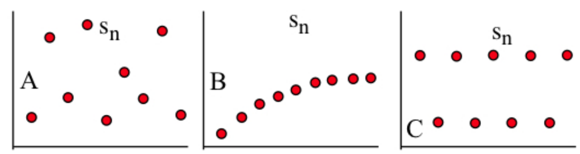
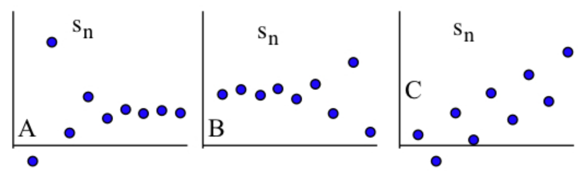


16.



In Problems 17–31, determine whether the given series converges or diverges.

17. $\sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{k+5}$
18. $\sum_{k=3}^{\infty} (-1)^k \cdot \frac{1}{\ln(k)}$
19. $\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{k}{k^2+3}$
20. $\sum_{k=5}^{\infty} (-0.99)^{k+1}$
21. $\sum_{k=1}^{\infty} (-1)^k \cdot \frac{k+3}{k+7}$
22. $\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \sin\left(\frac{1}{k}\right)$
23. $\sum_{k=4}^{\infty} \frac{\cos(k\pi)}{k}$
24. $\sum_{k=3}^{\infty} \frac{\sin(k\pi)}{k}$
25. $\sum_{k=2}^{\infty} (-1)^k \cdot \frac{\ln(k)}{k}$
26. $\sum_{k=3}^{\infty} (-1)^{k+1} \cdot \frac{\ln(k)}{\ln(k^3)}$
27. $\sum_{m=2}^{\infty} (-1)^m \cdot \frac{\ln(m^3)}{\ln(m^{10})}$
28. $\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{5}{\sqrt{k+7}}$
29. $\sum_{k=1}^{\infty} \frac{(-2)^{k+1}}{1+(-3)^k}$
30. $\sum_{m=1}^{\infty} \frac{(-2)^{m+1}}{1+3^m}$
31. $\sum_{k=1}^{\infty} \cos(k\pi)\sin(k\pi)$
32. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$

In Problems 33–42, (a) calculate s_4 for each series, (b) determine an upper bound for the distance between s_4 and the exact value S of the infinite series, then (c) use s_4 to find lower and upper bounds for the value of S.

33. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{1}{k^2}$

34. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{1}{k+6}$

35. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\ln(k+1)}$

36. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{2}{\sqrt{k+21}}$

37. $\displaystyle\sum_{k=1}^{\infty} (-0.8)^{k+1}$

38. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n$

39. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \sin\left(\frac{1}{k}\right)$

40. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{1}{k^4}$

41. $\displaystyle\sum_{k=1}^{\infty} (-1)^{k} \cdot \frac{1}{k^3}$

42. $\displaystyle\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k+\ln(k)}$

In Problems 43–52, find the number of terms N needed to guarantee that s_N is within the specified distance D of the exact value S of the sum of the series from the specified Problem.

43. Problem 34, $D = 0.01$

44. 35, $D = 0.01$

45. 36, $D = 0.01$

46. 37, $D = 0.003$

47. 38, $D = 0.002$

48. 39, $D = 0.06$

49. 40, $D = 0.001$

50. 41, $D = 0.0001$

51. 42, $D = 0.04$

52. 33, $D = 0.00005$

Alternating Power Series

Problems 53–63 ask you to use the series $S(x)$, $C(x)$ and $E(x)$ given below as infinite series depending on a continuous variable x:

$$S(x) = x - \frac{1}{3\cdot 2}x^3 + \frac{1}{5\cdot 4\cdot 3\cdot 2}x^5 + \cdots + \frac{(-1)^k}{(2k+1)!}x^{2k+1} + \cdots$$

$$C(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4\cdot 3\cdot 2}x^4 - \frac{1}{6\cdot 5\cdot 4\cdot 3\cdot 2}x^6 + \cdots + \frac{(-1)^k}{(2k)!}x^{2k} + \cdots$$

$$E(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \cdots + \frac{1}{k!}x^k + \cdots$$

You may recognize these series as infinite extensions of MacLaurin polynomials for $\sin(x)$, $\cos(x)$ and e^x from Section 8.7 (which we will study further in Chapter 10). In Problems 53–63, (a) substitute the given value for x in the series, (b) evaluate s_3, the sum of the first three terms of the series, and (c) determine an upper bound on the distance between s_3 and the sum of the entire infinite series.

53. $x = 0.3$ in $S(x)$

54. $x = 0.5$ in $S(x)$

55. $x = 0.1$ in $S(x)$

56. $x = 1$ in $S(x)$

57. $x = 1$ in $C(x)$

58. $x = -0.3$ in $S(x)$

59. $x = -0.2$ in $C(x)$

60. $x = 0.5$ in $C(x)$

61. $x = -1$ in $E(x)$

62. $x = -0.3$ in $C(x)$

63. $x = -0.2$ in $E(x)$

64. $x = -0.5$ in $E(x)$

9.8 Practice Answers

1. (a) For $k \geq 1$, $a_k = \frac{1}{k^2} > 0$. Taking $f(x) = \frac{1}{x^2}$, $f(k) = a_k$ for $k \geq 1$ and $f'(x) = -\frac{2}{x^3} < 0$ for $x \geq 1$, so $f(x)$ is decreasing when $x \geq 1$, telling us that $\{a_k\}$ is a decreasing sequence for $k \geq 1$. Finally, $\lim_{k\to\infty} \dfrac{1}{k^2} = 0$.

   (b) For $k \geq 2$, $a_k = \frac{3}{\ln(k)} > 0$. Because $\ln(x)$ is an increasing function for all x, $\{a_k\}$ is a decreasing sequence for $k \geq 2$. Finally, $\lim_{k\to\infty} \dfrac{3}{\ln(k)} = 0$.

2. Because $\lim_{k\to\infty} (-1)^{k+1}k \neq 0$, $\sum_{k=1}^{\infty} (-1)^{k+1}k$ diverges. For the second series, take $a_k = \frac{1}{\sqrt{2k+1}}$ so that $a_k > 0$ for $k \geq 1$. Furthermore, $\sqrt{2x+1}$ is an increasing function, so $\left\{\sqrt{2k+1}\right\}$ is an increasing sequence, hence $\{a_k\}$ is a decreasing sequence for $k \geq 1$. Finally, $\lim_{k\to\infty} \dfrac{1}{\sqrt{2k+1}} = 0$, so the series satisfies all three conditions of the Alternating Series Test, hence the series converges.

3. For $S = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k^3}$, $a_k = \dfrac{1}{k^3}$ and the partial sums are:

$$s_4 = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} = \frac{1549}{1728} \approx 0.896412$$
$$s_9 = s_4 + \frac{1}{125} - \frac{1}{216} + \frac{1}{343} - \frac{1}{512} + \frac{1}{729} \approx 0.9021165$$

   A bound for $|S - s_4|$ is a_5 and a bound for $|S - s_4|$ is a_{10} so:

$$|S - s_4| < a_5 = \frac{1}{5^3} = \frac{1}{125} = 0.008 \Rightarrow s_4 - 0.008 < S < s_4 + 0.008$$
$$\Rightarrow 0.888412 < S < 0.904412$$
$$|S - s_9| < a_{10} = \frac{1}{10^3} = \frac{1}{1000} = 0.001 \Rightarrow s_9 - 0.001 < S < s_9 + 0.001$$
$$\Rightarrow 0.9020165 < S < 0.9022165$$

4. We need to find an integer N so that $|a_{N+1}| < 0.001$:

$$\frac{1}{(N+1)^3 + 5} < \frac{1}{1000} \Rightarrow (N+1)^3 + 5 > 1000 \Rightarrow (N+1)^3 > 995$$
$$\Rightarrow N + 1 > \sqrt[3]{995} \approx 9.98 \Rightarrow N > 8.98$$

   so $N = 9$ should work:

$$s_9 = \frac{1}{6} - \frac{1}{13} + \frac{1}{32} - \frac{1}{69} + \frac{1}{130} - \frac{1}{221} + \frac{1}{348} - \frac{1}{517} - \frac{1}{734} \approx 0.111970$$