9.9 *Absolute Convergence and the Ratio Test*

The series we examined so far have generally behaved very regularly with regard to the signs of the terms: the signs of the terms were typically either all positive or they alternated between + and −. Yet the signs of terms in a series need not behave in such regular ways, and in this section we examine techniques for determining whether some of those series converge or diverge.

Two Examples

Consider a series with terms of magnitudes $\frac{1}{k}$ but unknown signs:

$$\sum_{k=1}^{\infty} \frac{\star}{k} = \frac{\star}{1} + \frac{\star}{2} + \frac{\star}{3} + \frac{\star}{4} + \cdots$$

where $\star$ represents $+1$ or -1 in each term. Does this series converge? Replacing $\star$ with 1 in *all* terms results in:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which diverges. Replacing $\star$ with -1 in *all* terms results in:

$$\sum_{k=1}^{\infty} \frac{-1}{k} = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$$

or -1 times the harmonic series, which also diverges. But replacing $\star$ alternately with $+1$ and -1 results in:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

or the alternating harmonic series, which converges (by the Alternating Series Test). The answer to the question above ("Does this series converge?") is: "It depends on the signs of the terms."

Now consider a series with terms of magnitudes $\frac{1}{k^2}$:

$$\sum_{k=1}^{\infty} \frac{\star}{k^2} = \frac{\star}{1} + \frac{\star}{4} + \frac{\star}{9} + \frac{\star}{16} + \cdots$$

where $\star$ represents $+1$ or -1 in each term. Does this series converge? Replacing $\star$ with 1 in *all* terms results in:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

or a p-series with $p = 2 > 1$, which converges (by the P-Test) to a value S. Replacing $\star$ with -1 in *all* terms results in:

$$\sum_{k=1}^{\infty} \frac{-1}{k^2} = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \cdots = -S$$

so this series also converges. Replacing $\star$ alternately with $+1$ and -1 results in:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

which converges (by the Alternating Series Test) to some value between $-S$ and S. Replacing $\star$ with $+1$ and -1 in any other way results in a series with partial sums all between $-S$ and S: although we know the partial sums are bounded, we don't (yet) know that they converge (they could bounce back and forth between distinct finite values).

The answer to the question above ("Does this series converge?") appears to be: "Yes, regardless of the signs of the terms." This is in fact true, but will require a careful proof.

Absolute Convergence and Conditional Convergence

If a series $\sum_{k=1}^{\infty} a_k$ converges no matter how the signs of each term are chosen, then $\sum_{k=1}^{\infty} |a_k|$ must converge (just choose all signs to be $+$).

> **Definition**:
> A series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ converges.

Any series of the form $\sum_{k=1}^{\infty} \frac{\star}{k^2}$ (where $\star$ represents $+1$ or -1) is absolutely convergent because $\sum_{k=1}^{\infty} \left|\frac{\star}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is convergent by the P-Test (with $p = 2 > 1$).

The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges but is **not** absolutely convergent because $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges. We call such a series **conditionally convergent**.

> **Definition**:
> A series $\sum_{k=1}^{\infty} a_k$ is **conditionally convergent** if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges.

Example 1. Determine whether these series are absolutely convergent, conditionally convergent or divergent.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ (b) $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ (c) $\sum_{m=1}^{\infty} (-1)^{k+1} \frac{m^2}{m+1}$

Solution. (a) $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{\sqrt{k}}\right| = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$, which diverges (by the P-Test with $p=\frac{1}{2}<1$) so $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}}$ does not converge absolutely, but this series does converge (by the Alternating Series Test), so it is conditionally convergent.

See Example 1(b) in Section 9.8.

(b) For all k, $|\sin(k)|<1$, so $\left|\dfrac{\sin(k)}{k^2}\right| < \dfrac{1}{k^2}$. Because $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges (by the P-Test with $p=2>1$), $\displaystyle\sum_{k=1}^{\infty}\left|\frac{\sin(k)}{k^2}\right|$ converges by the Basic Comparison Test, hence $\displaystyle\sum_{k=1}^{\infty}\frac{\sin(k)}{k^2}$ is absolutely convergent.

(c) $\displaystyle\lim_{m\to\infty}(-1)^{k+1}\frac{m^2}{m+1}\neq 0$, so $\displaystyle\sum_{m=1}^{\infty}(-1)^{k+1}\frac{m^2}{m+1}$ diverges (by the Test for Divergence). ◀

Practice 1. Determine whether each series is absolutely convergent, conditionally convergent or divergent.

$$\text{(a)} \sum_{k=2}^{\infty}(-1)^k\frac{5}{\ln(k)} \qquad \text{(b)} \sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2}$$

We now know that $\displaystyle\sum_{k=1}^{\infty}\frac{\star}{k^2}$ (where $\star$ represents $+1$ or -1) is absolutely convergent no matter how you choose the values of $\star$, and we know that $\displaystyle\sum_{k=1}^{\infty}\frac{\star}{k^2}$ converges for certain choices of $\star$, but does this series converge for *all* possible choices of $\star$? The following important result provides the answer to this question.

This theorem tells us that any absolutely convergent series is convergent.

Absolute Convergence Theorem

If $\displaystyle\sum_{k=1}^{\infty}|a_k|$ converges then $\displaystyle\sum_{k=1}^{\infty}a_k$ converges.

Proof. If $a_k \geq 0$ then $a_k = |a_k|$ and if $a_k < 0$ then $a_k = -|a_k|$ so for all k:

$$-|a_k| \leq a_k \leq |a_k| \quad \Rightarrow \quad 0 \leq |a_k| + a_k \leq 2|a_k|$$

Because $\displaystyle\sum_{k=1}^{\infty}2|a_k|$ converges, the BCT says $\displaystyle\sum_{k=1}^{\infty}\Big[|a_k|+a_k\Big]$ converges. Hence the difference between this series and $\displaystyle\sum_{k=1}^{\infty}|a_k|$ converges and:

$$\sum_{k=1}^{\infty}\Big[|a_k|+a_k\Big] - \sum_{k=1}^{\infty}|a_k| = \sum_{k=1}^{\infty}\Big[|a_k|+a_k-|a_k|\Big] = \sum_{k=1}^{\infty}a_k$$

must converge. □

The contrapositive form of the Absolute Convergence Theorem can be useful when showing that a series is *not* absolutely convergent.

Corollary

If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} |a_k|$ diverges.

This result tells us that no divergent series can be absolutely convergent.

The Ratio Test

If $|r| \geq 1$, then $\sum_{k=0}^{\infty} r^k$ diverges; if $|r| < 1$, $\sum_{k=0}^{\infty} |r|^k$ converges, so $\sum_{k=0}^{\infty} r^k$ converges absolutely. The following test extends these results about geometric series to a more general class of infinite series that behave like geometric series in a certain way: when the absolute value of the ratio of successive terms (eventually) exceeds 1, the series diverges; when this ratio is (eventually) less than 1, the series converges absolutely.

Ratio Test

If $L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$ then:

- $L < 1 \;\Rightarrow\; \sum_{k=1}^{\infty} a_k$ converges absolutely
- $L > 1 \;\Rightarrow\; \sum_{k=1}^{\infty} a_k$ diverges
- $L = 1 \;\Rightarrow\; \sum_{k=1}^{\infty} a_k$ may converge or may diverge

This is the last major convergence test we'll study. You'll use it often in Chapter 10 when you want to determine the interval on which a power series converges.

The Ratio Test can also be inconclusive if the limit in the hypothesis does not exist.

Proof. If $L < 1$, let r be any number so that $L < r < 1$. Because $L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$, the ratio $\left| \frac{a_{k+1}}{a_k} \right|$ must get close to L (and be less than r). More precisely, there is an N so that $k \geq N$ guarantees:

$$\left| \frac{a_{k+1}}{a_k} \right| < r \quad \Rightarrow \quad |a_{k+1}| < r \cdot |a_k|$$

Applying this result repeatedly, we know that:

$$|a_{N+1}| < r\,|a_N| \Rightarrow |a_{N+2}| < r\,|a_{N+1}| < r^2\,|a_N| \Rightarrow |a_{N+3}| < r^3\,|a_N|$$

and so forth, so that $|a_{N+j}| < r^j\,|a_N|$ for all $j \geq 0$. Hence:

$$\begin{aligned}\sum_{k=1}^{\infty} |a_k| &= \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_k| \leq \sum_{k=1}^{N-1} |a_k| + \sum_{j=0}^{\infty} r^j\,|a_N| \\ &= \sum_{k=1}^{N-1} |a_k| + |a_N| \sum_{j=0}^{\infty} r^j = \sum_{k=1}^{N-1} |a_k| + \frac{|a_N|}{1-r}\end{aligned}$$

We have shown that $\sum_{k=1}^{\infty} |a_k|$ is less than a finite sum plus the sum of a geometric series with ratio r where $|r| < 1$. The Basic Comparison Test tells us that $\sum_{k=1}^{\infty} |a_k|$ converges, hence $\sum_{k=1}^{\infty} a_k$ converges absolutely.

If $L > 1$, the ratio $\left|\frac{a_{k+1}}{a_k}\right|$ exceeds 1 for all $k \geq N$ for some integer N. Therefore $|a_{k+1}| > |a_k|$ when $k \geq N$ so that:

$$|a_N| < |a_{N+1}| < |a_{N+2}| < \cdots < |a_k|$$

for any $k > N$. This means that $\lim_{k\to\infty} |a_k| \geq |a_N| > 0$ so $\lim_{k\to\infty} a_k \neq 0$ and the Test for Divergence tells us that $\sum_{k=1}^{\infty} a_k$ diverges.

The following Practice problem provides an example of a convergent series with $L = 1$ and a divergent series with $L = 1$. □

Practice 2. Show that $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$ for both $\sum_{k=1}^{\infty} \frac{1}{k}$ (the harmonic series, which diverges) and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ (a convergent p-series with $p = 2$).

A powerful aspect of the Ratio Test is that it is very "mechanical": you simply calculate a particular limit and the value of this limit (often) tells you whether the series converges or diverges.

If the terms of a series involve factorials or k-th powers, the Ratio Test will often be the best test to use, so it will typically be the first test you try.

Example 2. Use the Ratio Test to determine whether each series converges absolutely:

$$\text{(a)} \sum_{k=1}^{\infty} \frac{2^k \cdot k}{5^k} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{n^2}{n!}$$

Solution. (a) If $a_k = \frac{2^k \cdot k}{5^k}$ then $a_{k+1} = \frac{2^{k+1} \cdot (k+1)}{5^{k+1}}$ so that:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{\frac{2^{k+1}\cdot(k+1)}{5^{k+1}}}{\frac{2^k \cdot k}{5^k}} = \frac{2^{k+1}}{2^k} \cdot \frac{5^k}{5^{k+1}} \cdot \frac{k+1}{k} = \frac{2}{5} \cdot \frac{k+1}{k} \to \frac{2}{5} = L$$

Because $L < 1$, the Ratio Test says $\sum_{k=1}^{\infty} \frac{2^k \cdot k}{5^k}$ converges absolutely.

(b) If $a_n = \frac{n^2}{n!}$ then $a_{n+1} = \frac{(n+1)^2}{(n+1)!}$ so that:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} = \frac{(n+1)^2}{n^2} \cdot \frac{n!}{(n+1)!}$$

$$= \left(\frac{n+1}{n}\right)^2 \cdot \frac{n!}{(n+1)\cdot n!} = \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{n+1} \to 1 \cdot 0 = L$$

Because $L < 1$, the Ratio Test says $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ converges absolutely. ◄

Practice 3. Use the Ratio Test to determine whether each series converges absolutely:

$$\text{(a)} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{e^k}{k!} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{n^5}{3^n}$$

The Ratio Test is often very useful for determining values of a variable that guarantee the absolute convergence (hence convergence) of a power series. You will use this method often in Chapter 10.

Example 3. On what interval does $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k}$ converge absolutely?

Solution. If $a_k = \frac{(x-3)^k}{k}$ then $a_{k+1} = \frac{(x-3)^{k+1}}{k+1}$ so that:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{(x-3)^{k+1}}{k+1}}{\frac{(x-3)^k}{k}}\right| = \left|\frac{(x-3)^{k+1}}{(x-3)^k} \cdot \frac{k}{k+1}\right| = |x-3| \cdot \frac{k}{k+1} \to |x-3|$$

We need $L = |x-3| < 1$, which means that:

$$|x-3| < 1 \quad \Rightarrow \quad -1 < x-3 < 1 \quad \Rightarrow \quad 2 < x < 4$$

If $x < 2$ or $x > 4$, $L = |x-3| > 1$ so the series diverges. If $x = 4$, $x - 3 = 1$ and the series becomes the harmonic series, which diverges; if $x = 2$, $x - 3 = -1$ and the series becomes the alternating harmonic series, which converges conditionally. So the series converges when $2 \le x < 4$ but converges absolutely only when $2 < x < 4$. ◀

Practice 4. On what interval does $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$ converge absolutely?

9.9 Problems

In Problems 1–30, determine whether the given series converges absolutely, converges conditionally or diverges, giving reasons for your conclusion.

1. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+2}$
2. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$
3. $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{5}{n^3}$
4. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{1+\ln(n)}$
5. $\sum_{k=0}^{\infty} (-0.5)^k$
6. $\sum_{k=0}^{\infty} (-0.5)^{-k}$
7. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$
8. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3+k^2}$
9. $\sum_{n=2}^{\infty} (-1)^n \cdot \frac{\ln(n)}{n}$
10. $\sum_{n=2}^{\infty} (-1)^n \cdot \frac{\ln(n)}{n^2}$
11. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k+\ln(k)}$
12. $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{5}{\sqrt{n+7}}$
13. $\sum_{k=1}^{\infty} (-1)^k \cdot \sin\left(\frac{1}{k}\right)$
14. $\sum_{k=1}^{\infty} (-1)^k \cdot \sin\left(\frac{1}{k^2}\right)$
15. $\sum_{k=1}^{\infty} (-1)^k \sqrt{k} \sin\left(\frac{1}{k^2}\right)$
16. $\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k}$
17. $\sum_{m=2}^{\infty} (-1)^m \cdot \frac{\ln(m)}{\ln(m^3)}$
18. $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{n^2+7}{n^2+10}$

19. $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{n^2+7}{n^3+10}$

20. $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{n^2+7}{n^4+10}$

21. $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(n^2+7)^2}{n^2+10}$

22. $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$

23. $\sum_{k=1}^{\infty} \frac{\sin(k\pi)}{k}$

24. $\sum_{k=1}^{\infty} (-k)^{-k}$

25. $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1+\sqrt{3n}}{n+2}$

26. $\sum_{k=1}^{\infty} \frac{(-2)^k}{k^2}$

27. $\sum_{k=1}^{\infty} \frac{(-3)^k}{k^3}$

28. $\sum_{n=2}^{\infty} (-1)^n \left(\frac{\ln(n)}{\ln(n^5)}\right)^2$

29. $\sum_{k=1}^{\infty} \frac{(-2)^k}{k \cdot 3^k}$

30. $\sum_{k=1}^{\infty} \frac{3^k}{k^3}$

The Ratio Test often arises with series that involve factorials. To help you prepare for these situations, Problems 31–39 ask you to simplify the factorial expression on the left side of the equality to arrive at the expression on the right side. Recall that: $0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 24$ and, in general, $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$.

31. $\frac{n!}{(n+1)!} = \frac{1}{n+1}$

32. $\frac{n!}{(n+3)!} = \frac{1}{(n+1)(n+2)(n+3)}$

33. $\frac{(n-1)!}{(n+1)!} = \frac{1}{n(n+1)}$

34. $\frac{(2n)!}{(2n+1)!} = \frac{1}{2n+1}$

35. $\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}$

36. $\frac{(n+1)!}{(n+2)!} = \frac{1}{n+2}$

37. $\frac{2n!}{(2n)!} = \frac{2}{(n+1)(n+2)\cdots(2n)}$

38. $\frac{(2n)!}{(2(n+1)!} = \frac{1}{(2n+1)(2n+2)}$

39. $\frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{2} \cdot \frac{n}{3} \cdots \frac{n}{n-1} \cdot \frac{n}{n}$

40. For $n > 0$, which is larger, $7!$ or $\frac{(n+7)!}{n!}$?

In Problems 41–58, apply the Ratio Test to the series. If the Ratio Test is inconclusive, use some other method to determine whether the series converges absolutely, converges conditionally or diverges.

41. $\sum_{k=1}^{\infty} \frac{1}{k}$

42. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

43. $\sum_{k=1}^{\infty} \frac{1}{k^3}$

44. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

45. $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$

46. $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$

47. $\sum_{n=0}^{\infty} 1^n$

48. $\sum_{n=0}^{\infty} (-2)^n$

49. $\sum_{k=0}^{\infty} \frac{1}{k!}$

50. $\sum_{k=0}^{\infty} \frac{5}{k!}$

51. $\sum_{k=0}^{\infty} \frac{2^k}{k!}$

52. $\sum_{k=0}^{\infty} \frac{5^k}{k!}$

53. $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{3k}$

54. $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{2k}$

55. $\sum_{k=5}^{\infty} (0.9)^{2k+1}$

56. $\sum_{k=100}^{\infty} (1.1)^{\frac{k}{2}}$

57. $\sum_{k=100}^{\infty} (-1.1)^k$

58. $\sum_{k=5}^{\infty} (-0.8)^{2k+1}$

In 59–76, apply the Ratio Test to determine the values of x for which the series converges absolutely.

59. $\sum_{k=0}^{\infty} (x-5)^k$

60. $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k}$

61. $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$

62. $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2}$

63. $\sum_{k=0}^{\infty} \frac{(x-2)^k}{k!}$

64. $\sum_{n=0}^{\infty} \frac{(x-10)^n}{n!}$

65. $\sum_{k=1}^{\infty} \frac{(2x-12)^k}{k^2}$

66. $\sum_{k=1}^{\infty} \frac{(4x-12)^k}{k^2}$

67. $\sum_{k=1}^{\infty} \frac{(6x-12)^k}{k!}$

68. $\sum_{k=0}^{\infty} (x-3)^{2k}$

69. $\sum_{k=1}^{\infty} \frac{(x+1)^{2k}}{k}$

70. $\sum_{k=1}^{\infty} \frac{(x+1)^{2k+1}}{k^2}$

71. $\sum_{n=1}^{\infty} \frac{(x-5)^{3n+1}}{n^2}$

72. $\sum_{n=0}^{\infty} \frac{(x+4)^{2n+1}}{n!}$

73. $\sum_{n=0}^{\infty} \frac{(x+3)^{2n+1}}{(n+1)!}$

74. $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

75. $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

76. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

77. $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$

78. $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$

The Root Test

A geometric series $\sum_{k=0}^{\infty} r^k$ diverges if $|r| \geq 1$ and converges (absolutely) if $|r| < 1$. In a geometric series:

$$a_k = r^k \quad \Rightarrow \quad |a_k| = \left|r^k\right| \quad \Rightarrow \quad \sqrt[k]{|a_k|} = |r|$$

The following test extends this result about a geometric series to a more general class of infinite series that behave like geometric series in a certain way: when the k-th root of the absolute value of the k-th term (eventually) exceeds 1, the series diverges; when this k-th root is (eventually) less than 1, the series converges absolutely.

Root Test

If $L = \lim_{k \to \infty} \sqrt[k]{|a_k|}$ then:

- $L < 1 \;\Rightarrow\; \sum_{k=1}^{\infty} a_k$ converges absolutely
- $L > 1 \;\Rightarrow\; \sum_{k=1}^{\infty} a_k$ diverges
- $L = 1 \;\Rightarrow\; \sum_{k=1}^{\infty} a_k$ may converge or may diverge

The Root Test comes in handy less often than the Ratio Test, but can be useful on occasion. If the Ratio Test is conclusive, the Root Test will be also (but may be more difficult to apply); there are some series, however, for which the Ratio Test is inconclusive but the Root Test is not.

The Root Test can also be inconclusive if the limit in the hypothesis does not exist.

Proof. If $L < 1$, choose any number r with $L < r < 1$. Then there is an integer N so that for $k \geq N$:

$$\sqrt[k]{|a_k|} < r \quad \Rightarrow \quad |a_k| < r^k$$

Therefore:

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} |a_k| \leq \sum_{k=1}^{N-1} |a_k| + \sum_{k=N}^{\infty} r^k$$

This last quantity is the sum of a finite sum and the tail end of a convergent geometric series (because $0 \leq r < 1$). The Basic Comparison Test tells us that $\sum_{k=1}^{\infty} |a_k|$ converges, hence $\sum_{k=1}^{\infty} a_k$ converges absolutely.

If $L > 1$, then there is an integer N so that for $k \geq N$:

$$\sqrt[k]{|a_k|} > 1 \quad \Rightarrow \quad |a_k| > 1 \quad \Rightarrow \quad \lim_{k \to \infty} |a_k| \neq 0 \quad \Rightarrow \quad \lim_{k \to \infty} a_k \neq 0$$

The Test for Divergence then tell us that $\sum_{k=1}^{\infty} a_k$ diverges.

The series $\sum_{k=1}^{\infty} 1$ diverges while $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges; applying the Root Test to each results in $L = 1$. □

Example 6 in Section 3.7 shows:

$$\lim_{k \to \infty} \left(\frac{1}{k}\right)^{\frac{1}{k}} = \lim_{x \to 0^+} x^x = 1$$

In Problems 79–94, apply the Root Test to the series. If the Root Test is inconclusive, use some other method to determine whether the series converges absolutely, converges conditionally or diverges.

79. $\sum_{k=0}^{\infty} \left(\frac{2}{7}\right)^k$

80. $\sum_{k=0}^{\infty} \left(\frac{8}{7}\right)^k$

81. $\sum_{k=1}^{\infty} \frac{1}{k^3}$

82. $\sum_{k=1}^{\infty} \frac{1}{k}$

83. $\sum_{k=1}^{\infty} \frac{1}{k^k}$

84. $\sum_{k=1}^{\infty} \frac{3^k}{k^k}$

85. $\sum_{k=5}^{\infty} \left(\frac{1}{2} - \frac{2}{k}\right)^k$

86. $\sum_{k=1}^{\infty} \left(\frac{1}{2} + \frac{2}{k}\right)^k$

87. $\sum_{k=1}^{\infty} \left(\frac{2+k}{k}\right)^k$

88. $\sum_{k=1}^{\infty} \left(\frac{k}{2+k}\right)^k$

89. $\sum_{k=1}^{\infty} \cos^k(k\pi)$

90. $\sum_{k=1}^{\infty} \sin^k\left(\frac{(6k+1)\pi}{6}\right)$

91. $\sum_{k=0}^{\infty} \left(\frac{2k^3+1}{3k^3+2}\right)^k$

92. $\sum_{k=0}^{\infty} \left(\frac{3k^2+1}{2k^2+3}\right)^k$

93. $\sum_{k=1}^{\infty} \frac{(2k)^k}{k^{2k}}$

94. $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right)^{k^2}$

Rearrangements

See Problems 103–104 for a proof of this statement.

Absolutely convergent series share an important property with finite sums: no matter what order you add the numbers, the sum will always be the same.

Conditionally convergent series do not possess this property: the order in which you add the terms *does* matter. If you reorder the terms of a conditionally convergent series, the sum after the rearrangement may be different than the sum before the rearrangement. A rather amazing fact is that you can rearrange the terms of a conditionally convergent series to obtain *any* sum you want!

We illustrate this strange result by showing that you can rearrange the alternating harmonic series:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

which converges conditionally to a value $S \approx 0.69$, so that the sum of the rearranged series is 2.

First, note that the sum of the positive terms of the alternating harmonic series:

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

diverges (by the Limit Comparison Test with the harmonic series), so the partial sums of this series of all-positive terms must exceed any positive number you pick (if n becomes large enough).

Similarly, the sum of the negative terms of the harmonic series:

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \cdots = -\frac{1}{2} \cdot \sum_{m=1}^{\infty} \frac{1}{m}$$

diverges (because it's a nonzero multiple of the harmonic series), so the partial sums of this series of all-negative terms must eventually be lower than any negative number you pick (if m becomes large enough).

Begin the rearrangement process by adding up just enough positive terms so that the sum of those terms exceeds 2:

$$s_7 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} \approx 1.955133755$$
$$s_8 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} \approx 2.021800422$$

This procedure will still work if you "overshoot" the target by adding on more positive terms than necessary, but not doing so makes the process more efficient.

so the first eight positive terms will do the trick. Now add on enough negative terms to bring the total below 2:

$$s_9 = 1 + \frac{1}{3} + \cdots + \frac{1}{15} - \frac{1}{2} \approx 1.521800422$$

(a single negative term does the job). Then add on more positive terms until the total once again exceeds 2:

$$s_{21} \approx 1.97967321 \Rightarrow s_{22} = s_{21} + \frac{1}{41} \approx 2.004063454$$

Now add enough negative terms to bring the total back down below 2:

$$s_{23} = s_{22} - \frac{1}{4} \approx 1.754063454$$

We will always have enough terms at our disposal to exceed the target number: if not, the partial sums of the positive terms would be bounded above and the sum of the positive terms would be finite (and likewise for the negative terms).

As you continue to repeat this process, you will "eventually" use all of the terms of the original conditionally convergent series, while the partial sums of the new "rearranged" series become — and remain — arbitrarily close to the target number, 2.

This method can be used to rearrange the terms of the alternating harmonic series — or any conditionally convergent series — to get a sum of 0.3, 3, 30 or any positive target number you want.

How do you think the strategy needs to be modified to rearrange a conditionally convergent series to add up to a negative target number?

In Problems 95–100, use the strategy outlined above to find the first 15 terms of a rearrangement of the given conditionally convergent series so that the rearranged series converges to the given target T.

95. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, $T = 0.3$

96. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, $T = 1$

97. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$, $T = 1$

98. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, $T = 0.7$

99. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$, $T = 0.4$

100. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, $T = -1$

101. Show that the Ratio Test applied to the series:
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$
is inconclusive. Then apply the Root Test.

102. Show that the Ratio Test applied to the series:
$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \cdots$$
is inconclusive. Then apply the Root Test.

103. Given an absolutely convergent series $\displaystyle\sum_{k=1}^{\infty} a_k$, let $\{b_n\}$ be any rearrangement of $\{a_k\}$ (so that these sequences both have the same terms, just listed in a different order). Show that the partial sums of $\displaystyle\sum_{k=1}^{\infty} a_k$ are bounded by some number $C > 0$. Then show that the partial sums of $\displaystyle\sum_{n=1}^{\infty} b_n$ are also increasing and bounded by the same number C. Conclude that $\displaystyle\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

104. Refer to Problem 103. Let $a = \sum_{k=1}^{\infty} a_k$, $A = \sum_{k=1}^{\infty} |a_k|$ $b = \sum_{n=1}^{\infty} b_n$, $s_K = \sum_{k=1}^{K} a_k$ and $S_N = \sum_{n=1}^{N} a_n$.

(a) Given $\epsilon > 0$, show that there is some integer K so that $|a - s_K| < \dfrac{\epsilon}{2}$ and that this inequality holds for any larger value of K.

(b) Show that by choosing a (possibly larger) value of K:

$$\left| A - \sum_{k=1}^{K} |a_k| \right| < \frac{\epsilon}{2}$$

holds for all larger values of K.

(c) Choose N to be large enough so that all of the terms $a_1, a_2, \ldots, a_K$ appear among the terms $b_1, b_2, \ldots, b_N$ and show this holds true for even larger values of N.

(d) Show that $|S_N - s_K| \leq \left| A - \sum_{k=1}^{K} |a_k| \right|$.

(e) Use the triangle inequality to conclude that $|a - S_N| < \epsilon$, showing that the partial sums of $\sum_{n=1}^{\infty} b_n$ have limit $\sum_{k=1}^{\infty} a_k$.

9.9 Practice Answers

1. (a) $\sum_{k=1}^{\infty} \left| (-1)^k \frac{5}{\ln(k)} \right| = \sum_{k=1}^{\infty} \frac{5}{\ln(k)}$ diverges (use the BCT with $\sum_{k=1}^{\infty} \frac{5}{k}$ to show this) so $\sum_{k=1}^{\infty} (-1)^k \frac{5}{\ln(k)}$ does not converge absolutely, but it does converge (by the AST—you should check this), so it converges conditionally.

Note that: $\cos(k\pi) = \pm 1$

(b) $\sum_{k=1}^{\infty} \left| \frac{\cos(k\pi)}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ and this series converges (by the P-Test, with $p = 2 > 1$), so $\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k^2}$ converges absolutely.

2. For the harmonic series, $a_k = \dfrac{1}{k} \Rightarrow a_{k+1} = \dfrac{1}{k+1}$ so:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

For the other series, $a_k = \dfrac{1}{k^2} \Rightarrow a_{k+1} = \dfrac{1}{(k+1)^2}$ so:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = \lim_{k\to\infty} \left[\frac{k}{k+1} \right]^2 = 1$$

3. (a) Applying the Ratio Test:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+2} \frac{e^{k+1}}{(k+1)!}}{(-1)^{k+1} \frac{e^k}{k!}} \right| = \lim_{k\to\infty} e \cdot \frac{k!}{(k+1)!}$$
$$= \lim_{k\to\infty} \frac{e}{k+1} = 0$$

Because $0 < 1$, the series converges absolutely.

(b) Applying the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^5}{3^{n+1}}}{\frac{n^5}{3^n}}\right| = \lim_{n\to\infty}\frac{1}{3}\cdot\frac{(n+1)^5}{n^5}$$
$$= \lim_{n\to\infty}\frac{1}{3}\cdot\left[\frac{n+1}{n}\right]^5 = \frac{1}{3}\cdot 1 = \frac{1}{3}$$

Because $\frac{1}{3} < 1$, the series converges absolutely.

4. If $a_k = \dfrac{(x-5)^k}{k^2}$ then $a_{k+1} = \dfrac{(x-5)^{k+1}}{(k+1)^2}$ so that:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{(x-5)^{k+1}}{(k+1)^2}}{\frac{(x-5)^k}{k^2}}\right| = \left|\frac{(x-5)^{k+1}}{(x-5)^k}\cdot\frac{k^2}{(k+1)^2}\right| = |x-5|\cdot\left[\frac{k}{k+1}\right]^2$$

which has limit $L = |x-5|$. We need $L < 1$, which means that:

$$|x-5| < 1 \quad\Rightarrow\quad -1 < x-5 < 1 \quad\Rightarrow\quad 4 < x < 6$$

If $x < 4$ or $x > 6$, $L = |x-5| > 1$ so the series diverges. If $x = 4$, $x - 5 = -1$ and the series becomes $\sum_{k=1}^{\infty}\frac{(-1)^2}{k^2}$, which converges absolutely by the P-Test; if $x = 6$, $x - 5 = 1$ and the series becomes $\sum_{k=1}^{\infty}\frac{1}{k^2}$, which also converges absolutely by the P-Test. The series converges absolutely on $[4,6]$ and diverges outside of this interval.