85. $\sqrt[k]{\left(\frac{1}{2}-\frac{2}{k}\right)^k} = \frac{1}{2} - \frac{2}{k} \to \frac{1}{2} < 1$; AC.

87. $\sqrt[k]{\left(\frac{2+k}{k}\right)^k} = \frac{2+k}{k} \to 1$, so Root Test inconclusive; diverges by Test for Divergence.

89. $\sqrt[k]{|\cos(k\pi)|^k} = 1$, so Root Test is inconclusive; diverges by Test for Divergence.

91. $L = \frac{2}{3} < 1$; absolutely convergent.

93. $\sqrt[k]{\frac{(2k)^k}{k^{2k}}} = \frac{2k}{k^2} = \frac{2}{k} \to 0 < 1$; AC.

95. $\frac{1}{2} - 1 + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{16} - \frac{1}{3} + \frac{1}{18} + \frac{1}{20} + \frac{1}{22} + \frac{1}{24} + \frac{1}{26}$

97. $\frac{1}{\sqrt{2}} + \frac{1}{2} - 1 + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{10}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{14}} - \frac{1}{\sqrt{5}} + \frac{1}{4} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{20}} - \frac{1}{3}$

99. $\frac{1}{\sqrt{2}} - 1 + \frac{1}{2} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{10}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{14}} - \frac{1}{3} + \frac{1}{4} + \frac{1}{\sqrt{18}} - \frac{1}{\sqrt{11}}$

101. On your own.

103. On your own.

Section 10.1

1. This is a geometric series with ratio x, so it converges precisely when $|x| < 1$; the interval of convergence is $(-1, 1)$. (Graph it yourself.)

3. Applying the Ratio Test:
$$\left|\frac{3^{k+1} \cdot x^{k+1}}{3^k \cdot x^k}\right| = |3x|$$
for all values of x, so the series converges when $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$ and diverges when $|x| > \frac{1}{3}$. At $x = \frac{1}{3}$ the series becomes $\sum_{k=1}^{\infty} 1$, which diverges by the Test for Divergence; at $x = -\frac{1}{3}$, the series becomes $\sum_{k=1}^{\infty} (-1)^k$, which also diverges by the Test for Divergence. The interval of convergence is therefore $(-\frac{1}{3}, \frac{1}{3})$. (The graph is left to you.)

5. Applying the Ratio Test:
$$\left|\frac{\frac{x^{k+1}}{k+1}}{\frac{x^k}{k}}\right| = \frac{k}{k+1} \cdot |x| \longrightarrow |x|$$
so the series converges when $|x| < 1$ and diverges when $|x| > 1$. At $x = 1$ the series becomes the harmonic series, which diverges; at $x = -1$, the series becomes the alternating harmonic series, which converges conditionally (by the Alternating Series Test). The interval of convergence is therefore $[-1, 1)$. (The graph is left to you.)

7. Applying the Ratio Test:
$$\left|\frac{(k+1) \cdot x^{k+1}}{k \cdot x^k}\right| = \frac{k+1}{k} \cdot |x| \longrightarrow |x|$$
so the series converges when $|x| < 1$ and diverges when $|x| > 1$. At $x = 1$ the series becomes $\sum_{k=1}^{\infty} k$, which diverges by the Test for Divergence; at $x = -1$, the series becomes $\sum_{k=1}^{\infty} k \cdot (-1)^k$, which also diverges by the Test for Divergence. The interval of convergence is therefore $(-1, 1)$.

9. Applying the Ratio Test:
$$\left|\frac{(k+1) \cdot x^{2k+3}}{k \cdot x^{2k+1}}\right| = \frac{k+1}{k} \cdot x^2 \longrightarrow x^2$$
so the series converges when $x^2 < 1 \Rightarrow |x| < 1$ and diverges when $|x| > 1$. At $x = 1$ the series becomes $\sum_{k=1}^{\infty} k$, which diverges by the Test for Divergence; at $x = -1$, the series becomes $\sum_{k=1}^{\infty} -k$, which also diverges by the Test for Divergence. The interval of convergence is therefore $(-1, 1)$.

11. Applying the Ratio Test:
$$\left|\frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}}\right| = \frac{k! \cdot |x|}{(k+1)!} = \frac{k! \cdot |x|}{(k+1) \cdot k!} = \frac{|x|}{k+1} \longrightarrow 0$$
for any x, so the interval of convergence is therefore $(-\infty, \infty)$.

13. Applying the Ratio Test:
$$\left|\frac{(k+1) \cdot \frac{x^{2k+2}}{4^{2k+2}}}{k \cdot \frac{x^{2k}}{4^{2k}}}\right| = \frac{(k+1) \cdot x^2}{16k} \longrightarrow \frac{x^2}{16}$$
so the series converges when $\frac{x^2}{16} < 1 \Rightarrow x^2 < 16 \Rightarrow |x| < 4$ and diverges when $|x| > 4$. At $x = \pm 4$ the series becomes $\sum_{k=1}^{\infty} k$, which diverges, so the interval of convergence is $(-4, 4)$.

15. Applying the Ratio Test:

$$\left|\frac{x^{k+1}}{2^{k+1}} \cdot \frac{2^k}{x^k}\right| = \frac{|x|}{2}$$

for all x, so the series converges when $\frac{|x|}{2} < 1 \Rightarrow |x| < 2$ and diverges when $|x| > 2$. At $x = 2$ the series becomes $\sum_{k=1}^{\infty} 1$, which diverges; at $x = -2$, it becomes $\sum_{k=1}^{\infty} (-1)^k$, which also diverges. The interval of convergence is $(-2, 2)$.

17. $R = \frac{1}{2}(1 - (-1)) = 1$

19. $R = 1$ 21. $R = 1$ 23. $R = 4$

25. $\sum_{k=0}^{\infty} \frac{x^k}{5^k}$ is one possibility.

27. $\sum_{k=0}^{\infty} \frac{x^k}{2^k \cdot k^2}$ is one possibility.

29. $R = \frac{1}{2}(5 - (-5)) = 5$ 31. $R = 2$

33. This is a geometric series with ratio x, so it converges precisely when $|x| < 1$, hence its interval of convergence is $(-1, 1)$. On that interval:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

35. This is a geometric series with ratio $2x$, so it converges precisely when $|2x| < 1$, hence its interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$. On that interval:

$$\sum_{k=0}^{\infty} (2x)^k = \frac{1}{1-2x}$$

37. This is a geometric series with ratio x, so its interval of convergence is $(-1, 1)$. On that interval:

$$\sum_{k=1}^{\infty} x^k = \left[\sum_{k=0}^{\infty} x^k\right] - 1 = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

39. This is a geometric series with ratio x^3, so it converges precisely when $|x^3| < 1 \Rightarrow |x| < 1$, hence its interval of convergence is $(-1, 1)$, where:

$$\sum_{k=0}^{\infty} (x^3)^k = \frac{1}{1-x^3}$$

41. This is a geometric series with ratio $4x$, so it converges precisely when $|4x| < 1$, hence its interval of convergence is $\left(-\frac{1}{4}, \frac{1}{4}\right)$. On that interval:

$$\sum_{k=0}^{\infty} (4x)^k = \frac{1}{1-4x}$$

Section 10.2

1. This is a geometric series with ratio $x + 2$, so it converges precisely when:

$$|x+2| < 1 \Rightarrow -1 < x+2 < 1 \Rightarrow -3 < x < -1$$

The interval of convergence is $(-3, -1)$, so $R = \frac{1}{2}(-1 - (-3)) = 1$. (The graph is left to you.)

3. This is a geometric series with ratio $x + 5$, so it converges when:

$$|x+5| < 1 \Rightarrow -1 < x+5 < 1 \Rightarrow -6 < x < -4$$

and diverges everywhere else. The interval of convergence is $(-6, -4)$, so $R = 1$.

5. Applying the Ratio Test:

$$\left|\frac{\frac{(x-2)^{k+1}}{k+1}}{\frac{(x-1)^k}{k}}\right| = \frac{k}{k+1} \cdot |x-2| \longrightarrow |x-2|$$

so the series converges when:

$$|x-2| < 1 \Rightarrow -1 < x-2 < 1 \Rightarrow 1 < x < 3$$

and diverges when $x < 1$ or $x > 3$. At $x = 3$ the series becomes the harmonic series, which diverges; at $x = 1$, the series becomes the alternating harmonic series, which converges conditionally. The interval of convergence is therefore $[1, 3)$, hence $R = 1$.

7. Applying the Ratio Test:

$$\left|\frac{\frac{(x-7)^{2k+3}}{(k+1)^2}}{\frac{(x-7)^{2k+1}}{k^2}}\right| = \left(\frac{k}{k+1}\right)^2 \cdot (x-7)^2 \longrightarrow (x-7)^2$$

so the series converges when:

$$(x-7)^2 < 1 \Rightarrow |x-7| < 1 \Rightarrow -1 < x-7 < 1 \Rightarrow 6 < x < 8$$

and diverges when $x < 6$ or $x > 8$. At $x = 8$, the series becomes $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which converges (by the P-test, with $p = 2$); at $x = 6$, the series becomes

$\sum_{k=1}^{\infty} \frac{-1}{k^2}$, which likewise converges. The interval of convergence is $[6,8]$, hence $R = 1$.

9. This is a geometric series with ratio $2x-6$, so it converges precisely when:

$$|2x-6| < 1 \Rightarrow -1 < 2x-6 < 1 \Rightarrow 5 < 2x < 7 \Rightarrow 2.5 < x < 3.5$$

The interval of convergence is $(2.5, 3.5)$; $R = 0.5$.

11. Applying the Ratio Test:

$$\left| \frac{\frac{(x-5)^{k+1}}{(k+1)!}}{\frac{(x-5)^k}{k!}} \right| = \frac{k! \cdot |x-5|}{(k+1)!} = \frac{|x-5|}{k+1} \longrightarrow 0$$

for any x, so the interval of convergence is $(-\infty, \infty)$ and $R = \infty$.

13. Applying the Ratio Test:

$$\left| \frac{(k+1)! \cdot (x-7)^{k+1}}{k! \cdot x^k} \right| = (k+1) \cdot |x-7|$$

which has a limit of ∞ as $k \to \infty$ for all x except $x = 7$ (in which case the limit is 0, so the series converges). The interval of convergence is therefore the single point$\{7\}$ and $R = 0$.

15. The center of the interval is $x = 5$ but the power series is centered at $x = 4$.

17. The interval of convergence must be centered at $x = 7$ so the only candidates for it are: $(5,9)$, $[1,13]$, $(-1,15]$, $[3,11)$, $[0,14)$ and $\{7\}$.

19. The interval of convergence must be centered at $x = 1$ so the possibilities are: $(0,2)$, $(-5,7)$, $[0,2]$, $(-3,5]$, $(-9,11]$, $[0,2)$ and $\{1\}$.

21. $R = \frac{1}{2}(6-0) = 3$

23. $R = \frac{1}{2}(8-2) = 3$

25. $\sum_{k=0}^{\infty} \frac{(x-3)^k}{3^k}$ is one possibility.

27. $\sum_{k=0}^{\infty} \frac{(5-x)^k}{k \cdot 3^k}$ is one possibility.

29. This is a geometric series with ratio $x-3$, so it converges precisely when:

$$|x-3| < 1 \Rightarrow -1 < x-3 < 1 \Rightarrow 2 < x < 4$$

On that interval:

$$\sum_{k=0}^{\infty} (x-3)^k = \frac{1}{1-(x-3)} = \frac{1}{4-x}$$

31. This is a geometric series with ratio $\frac{x-6}{5}$, so it converges precisely when:

$$\left| \frac{x-6}{5} \right| < 1 \Rightarrow |x-6| < 5 \Rightarrow -5 < x-6 < 5 \Rightarrow 1 < x < 11$$

On that interval:

$$\sum_{k=0}^{\infty} (\frac{x-6}{5})^k = \frac{1}{1-\frac{x-6}{5}} = \frac{5}{11-x}$$

33. This is a geometric series with ratio $\frac{1}{2}\sin(x)$. Because $\left|\frac{1}{2}\sin(x)\right| \le \frac{1}{2} < 1$ for all values of x, the interval of convergence is $(-\infty, \infty)$ and:

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\sin(x)\right)^k = \frac{1}{1-\frac{1}{2}\sin(x)} = \frac{2}{2-\sin(x)}$$

35. This is a geometric series with ratio $x-a$, so it converges precisely when:

$$|x-a| < 1 \Rightarrow -1 < x-a < 1 \Rightarrow a-1 < x < a+1$$

37. Applying the Ratio Test:

$$\left| \frac{\frac{(x-a)^{k+1}}{k+1}}{\frac{(x-a)^k}{k}} \right| = \frac{k}{k+1} \cdot |x-a| \longrightarrow |x-a|$$

so the series converges when:

$$|x-a| < 1 \Rightarrow -1 < x-a < 1 \Rightarrow a-1 < x < a+1$$

and diverges when $x < a-1$ or $x > a+1$. At $x = a+1$ the series becomes the harmonic series, which diverges; at $x = a-1$, the series becomes the alternating harmonic series, which converges conditionally. The interval of convergence is therefore $[a-1, a+1)$.

39. This is a geometric series with ratio ax, so it converges precisely when:

$$|ax| < 1 \Rightarrow -1 < ax < 1 \Rightarrow -\frac{1}{a} < x < \frac{1}{a}$$

41. This is a geometric series with ratio $ax-b$, so it converges precisely when:

$$|ax-b| < 1 \Rightarrow -1 < ax-b < 1 \Rightarrow -\frac{b-1}{a} < x < \frac{b+1}{a}$$

Section 10.3

1. Starting with the geometric series:

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k = 1 + u + u^2 + u^3 + \cdots$$

and using the substitution $u = x^4$ yields:

$$\frac{1}{1-x^4} = \sum_{k=0}^{\infty} x^{4k} = 1 + x^4 + x^8 + x^{12} + \cdots$$

3. Substitute $u = -x^4$ in the geometric series:

$$\frac{1}{1+x^4} = \sum_{k=0}^{\infty} (-1)^k \cdot x^{4k} = 1 - x^4 + x^8 - x^{12} + \cdots$$

5. Rewrite the function as:

$$\frac{1}{5+x} = \frac{\frac{1}{5}}{1+\frac{x}{5}} = \frac{\frac{1}{5}}{1-(-\frac{x}{5})}$$

and put $u = -\frac{x}{5}$ in the geometric series:

$$\frac{1}{5+x} = \frac{1}{5}\sum_{k=0}^{\infty} (-\frac{x}{5})^k = \frac{1}{5} - \frac{x}{25} + \frac{x^2}{125} - \frac{x^3}{625} + \cdots$$

7. Substitute $u = -x^3$ in the geometric series:

$$\frac{1}{1+x^3} = \sum_{k=0}^{\infty} (-1)^k \cdot x^{3k} = 1 - x^3 + x^6 - x^9 + \cdots$$

and multiply the result by x^2:

$$\frac{x^2}{1+x^3} = \sum_{k=0}^{\infty} (-1)^k \cdot x^{3k+2} = x^2 - x^5 + x^8 - x^{11} + \cdots$$

9. Into the first result from Example 3:

$$\ln(1-u) = -\sum_{k=1}^{\infty} \frac{u^k}{k} = -u - \frac{1}{2}u^2 - \frac{1}{3}u^3 - \cdots$$

substitute $u = -x^2$ to get:

$$\begin{aligned}\ln(1+x^2) &= \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{x^{2k}}{k} \\ &= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \cdots\end{aligned}$$

11. Substitute $u = x^2$ to get into the second result from Example 3 and multiply by x:

$$\begin{aligned} \arctan(u) &= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{u^{2k+1}}{2k+1} = u - \frac{1}{3}u^3 + \frac{1}{5}u^5 - \frac{1}{7}u^7 + \cdots \\ \Rightarrow \quad \arctan\left(x^2\right) &= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{4k+2}}{2k+1} = x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} - \frac{1}{7}x^{14} + \cdots \\ \Rightarrow \quad x \cdot \arctan\left(x^2\right) &= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{4k+3}}{2k+1} = x^3 - \frac{1}{3}x^7 + \frac{1}{5}x^{11} - \frac{1}{7}x^{15} + \cdots \end{aligned}$$

13. Substitute $u = x^2$ into the result from Example 2:

$$\begin{aligned} \frac{1}{(1-u)^2} &= \sum_{k=1}^{\infty} k \cdot u^{k-1} = 1 + 2u + 3u^2 + 4u^3 + 5u^4 + \cdots \\ \Rightarrow \quad \frac{1}{(1-x^2)^2} &= \sum_{k=1}^{\infty} k \cdot x^{2k-2} = 1 + 2x^2 + 3x^4 + 4x^6 + 5x^8 + \cdots \end{aligned}$$

15. Differentiate the result from Example 2 and then divide by 2:

$$\begin{aligned} (1-x)^{-2} &= \sum_{k=1}^{\infty} k \cdot x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \\ \Rightarrow \quad 2(1-x)^{-3} &= \sum_{k=1}^{\infty} k(k-1) \cdot x^{k-2} = 2 + 6x + 12x^2 + 20x^3 + \cdots \\ \Rightarrow \quad \frac{1}{(1-x)^3} &= \sum_{k=1}^{\infty} \frac{k(k-1)}{2} \cdot x^{k-2} = 1 + 3x + 6x^2 + 10x^3 + \cdots \end{aligned}$$

17. Replace x with $-x^2$ in the result from Problem 15:

$$\frac{1}{(1+x^2)^3} = \sum_{k=1}^{\infty} (-1)^k \frac{k(k-1)}{2} \cdot x^{2k-4} = 1 - 3x^2 + 6x^4 - 10x^6 + \cdots$$

19. Integrate the first result from Practice 1 between $x = 0$ and $x = \frac{1}{2}$:

$$\int_0^{\frac{1}{2}} \frac{1}{1-x^3}\, dx = \int_0^{\frac{1}{2}} \left[\sum_{k=0}^{\infty} x^{3k} \right] dx = \int_0^{\frac{1}{2}} \left[1 + x^3 + x^6 + \cdots \right] dx$$

$$= \sum_{k=0}^{\infty} \left[\frac{1}{3k+1} x^{3k+1} \right]_0^{\frac{1}{2}} = \left[x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + \cdots \right]_0^{\frac{1}{2}} = \frac{1}{2} + \frac{1}{64} + \frac{1}{896} + \cdots \approx 0.5167$$

21. Integrate the result from Practice 2 between $x = 0$ and $x = \frac{3}{5}$:

$$\int_0^{\frac{3}{5}} \ln(1+x)\, dx = \int_0^{\frac{3}{5}} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \cdot x^{k+1} \right] dx = \int_0^{\frac{3}{5}} \left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \right] dx$$

$$= \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+2)(k+1)} \cdot x^{k+2} \right]_0^{\frac{3}{5}} = \left[\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \cdots \right]_0^{\frac{3}{5}} = \frac{9}{50} - \frac{9}{250} + \frac{27}{2500} - \cdots$$

or about 0.1548 (adding up the first three terms of the numerical sum).

23. Multiply the second result from Example 3 by x^2 and integrate between $x = 0$ and $x = \frac{1}{2}$:

$$\int_0^{\frac{1}{2}} x^2 \cdot \arctan(x)\, dx = \int_0^{\frac{1}{2}} \left[\sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+3}}{2k+1} \right] dx = \int_0^{\frac{1}{2}} \left[x^3 - \frac{1}{3}x^5 + \frac{1}{5}x^7 - \cdots \right] dx$$

$$= \left[\sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+4}}{(2k+4)(2k+1)} \right]_0^{\frac{1}{2}} = \left[\frac{1}{4}x^4 - \frac{1}{18}x^6 + \frac{1}{40}x^8 - \cdots \right]_0^{\frac{1}{2}}$$

$$= \frac{1}{64} - \frac{1}{1152} + \frac{1}{10240} - \cdots \approx 0.01485$$

25. Integrate the result from Example 2 between $x = 0$ and $x = 0.3$:

$$\int_0^{0.3} \frac{1}{(1-x)^2}\, dx = \int_0^{0.3} \left[\sum_{k=1}^{\infty} k \cdot x^{k-1} \right] dx = \int_0^{0.3} \left[1 + 2x + 3x^2 + \cdots \right] dx$$

$$= \left[\sum_{k=1}^{\infty} \cdot x^k \right]_0^{0.3} = \left[x + x^2 + x^3 + \cdots \right]_0^{0.3} = 0.3 + 0.09 + 0.027 + \cdots \approx 0.417$$

27. If $x \neq 0$, divide the second result from Example 3 by x to get:

$$\frac{\arctan(x)}{x} = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k}}{2k+1} = 1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 + \cdots$$

As $x \to 0$, the last expression approaches 1, so $\lim\limits_{x \to 0} \dfrac{\arctan(x)}{x} = 1$.

29. If $x \neq 0$, divide the result from Practice 2 by $2x$:

$$\frac{\ln(1+x)}{2x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+2} \cdot x^k = \frac{1}{2} - \frac{1}{4}x + \frac{1}{6}x^2 - \frac{1}{8}x^3 + \cdots \longrightarrow \frac{1}{2} \text{ (as } x \to 0)$$

31. If $x \neq 0$, divide the power series for $\arctan(x)$ obtained in the solution to Example 4 by x^2 to get:

$$\frac{\arctan(x^2)}{x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{4k} = 1 - \frac{1}{3}x^4 + \frac{1}{5}x^8 - \frac{1}{7}x^{12} + \cdots \longrightarrow 1 \text{ (as } x \to 0)$$

33. If $x \neq 0$, replace x with $-x^2$ in the power series for $\ln(1+x)$ and divide by $3x$ to get:

$$\frac{\ln(1-x^2)}{3x} = -\frac{1}{3x}\sum_{k=1}^{\infty} \frac{x^{2k}}{k} = -\sum_{k=1}^{\infty} \frac{x^{2k-1}}{3k} = -\frac{1}{3}x - \frac{1}{6}x^3 - \frac{1}{9}x^5 - \cdots \longrightarrow 0 \text{ (as } x \to 0)$$

(Check that you get the same result in Problems 27–34 from applying L'Hôpital's Rule.)

35. $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k \cdot x^k$ (a geometric series with ratio $-x$ precisely when $|-x| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$, so the interval of convergence is $(-1, 1)$.

37. From Example 3, we know that:

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots = -\sum_{k=0}^{\infty} \frac{1}{k+1} \cdot x^{k+1}$$

Applying the Ratio Test to this series:

$$\left| \frac{-\frac{x^{k+1}}{k+1}}{-\frac{x^k}{k}} \right| = \frac{k}{k+1} \cdot |x| \longrightarrow |x|$$

as $k \to \infty$, so the series converge when $|x| < 1$ and diverges when $|x| > 1$. At $x = 1$, the series becomes a multiple of the harmonic series, which diverges; at $x = -1$, the series becomes a multiple of the alternating harmonic series, which converges conditionally. So the interval of convergence is $[-1, 1)$.

39. From Example 3, we know that:

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

Applying the Ratio Test to this series:

$$\left| \frac{\frac{(-1)^{k+1}x^{2k+3}}{2k+3}}{\frac{(-1)^k x^{2k+1}}{2k+1}} \right| = \frac{2k+1}{2k+3} \cdot x^2 \longrightarrow 1$$

as $k \to \infty$, so the series convereges when $x^2 < 1 \Rightarrow |x| < 1$ and diverges when $x^2 > 1 \Rightarrow |x| > 1$. At $x = \pm 1$, the series converges conditionally (by the Alternating Series Test—check this) so the interval of convergence is $[-1, 1]$.

41. From Example 2 we know that:

$$(1-x)^{-2} = \sum_{k=1}^{\infty} k \cdot x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

Applying the Ratio Test to this series:

$$\left| \frac{(k+1) \cdot x^k}{k \cdot x^{k-1}} \right| = \frac{k+1}{k} \cdot |x| \longrightarrow 1$$

as $k \to \infty$, so the series converges when $|x| < 1$ and diverges when $|x| > 1$. At $x = 1$, the series becomes $\sum_{k=1}^{\infty} k$, which diverges; at $x = -1$, the series becomes $\sum_{k=1}^{\infty} k \cdot (-1)^{k-1}$, which also diverges. The interval of convergence is $(-1, 1)$.

Section 10.4

1. With $f(x)=\ln(1+x)$, $f(0)=\ln(1)=0$ and:
$$f'(x)=(1+x)^{-1}\Rightarrow f'(0)=1$$
$$f''(x)=-(1+x)^{-2}\Rightarrow f''(0)=-1$$
$$f'''(x)=2(1+x)^{-3}\Rightarrow f'''(0)=2$$
$$f^{(4)}(x)=-6(1+x)^{-4}\Rightarrow f^{(4)}(0)=-6$$
and so on, so the first few terms of the MacLaurin series for $f(x)$ are:
$$0+1\cdot x+\frac{-1}{2!}x^2+\frac{2}{3!}x^3+\frac{-6}{4!}x^4+\cdots$$
$$=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$$

3. $f(x)=\arctan(x)\Rightarrow f(0)=\arctan(0)=0$ and:
$$f'(x)=\frac{1}{1+x^2}\Rightarrow f'(0)=1$$
$$f''(x)=-\frac{2x}{(1+x^2)^2}\Rightarrow f''(0)=0$$
$$f'''(x)=\frac{6x^2-2}{(1+x^2)^3}\Rightarrow f'''(0)=-2$$
$$f^{(4)}(x)=\frac{24\left(x-x^3\right)}{(1+x^2)^4}\Rightarrow f^{(4)}(0)=0$$
and so on, so the first few terms of the MacLaurin series for $f(x)$ are:
$$0+1\cdot x+\frac{0}{2!}\cdot x^2+\frac{-2}{3!}x^3+\frac{0}{4!}\cdot x^4+\cdots$$
$$=x-\frac{1}{3}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$$

5. With $f(x)=\cos(x)$, $f(0)=\cos(0)=1$ and:
$$f'(x)=-\sin(x)\Rightarrow f'(0)=0$$
$$f''(x)=-\cos(x)\Rightarrow f''(0)=-1$$
$$f'''(x)=\sin(x)\Rightarrow f'''(0)=0$$
$$f^{(4)}(x)=\cos(x)\Rightarrow f^{(4)}(0)=1$$
From here the derivatives repeat the same pattern, so $f^{(5)}(0)=0$, $f^{(6)}(0)=-1$ and the first few terms of the MacLaurin series for $f(x)$ are:
$$1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots$$

7. With $f(x)=\sec(x)$, $f(0)=\sec(0)=1$ and:
$$f'(x)=\sec(x)\tan(x)\Rightarrow f'(0)=0$$
$$f''(x)=\sec^3(x)+\sec(x)\tan^2(x)\Rightarrow f''(0)=1$$
$$f'''(x)=5\sec^3(x)\tan(x)+\sec(x)\tan^3(x)$$
so that $f'''(0)=0$, while $f^{(4)}(x)=5\sec^5(x)+18\sec^3(x)\tan^2(x)+\sec(x)\tan^4(x)\Rightarrow f^{(4)}(0)=5$; the first terms of the MacLaurin series are:
$$1-x^2+\frac{5}{4!}x^4+\cdots\ =\ 1-x^2+\frac{5}{24}x^4+\cdots$$

9. With $f(x)=\ln(x)$, $f(1)=\ln(1)=0$ and:
$$f'(x)=x^{-1}\Rightarrow f'(1)=1$$
$$f''(x)=-x^{-2}\Rightarrow f''(1)=-1$$
$$f'''(x)=2x^{-3}\Rightarrow f'''(1)=2$$
$$f^{(4)}(x)=-6x^{-4}\Rightarrow f^{(4)}(1)=-6$$
and so on, so the first few terms of the Taylor series are:
$$(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\cdots$$

11. With $f(x)=\sin(x)$, $f\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1$ and:
$$f'(x)=\cos(x)\Rightarrow f'\left(\frac{\pi}{2}\right)=0$$
$$f''(x)=-\sin(x)\Rightarrow f''\left(\frac{\pi}{2}\right)=-1$$
$$f'''(x)=\cos(x)\Rightarrow f'''\left(\frac{\pi}{2}\right)=0$$
$$f^{(4)}(x)=\sin(x)\Rightarrow f^{(4)}\left(\frac{\pi}{2}\right)=1$$
From here the derivatives repeat the same pattern, so $f^{(5)}\left(\frac{\pi}{2}\right)=0$, $f^{(6)}\left(\frac{\pi}{2}\right)=-1$ and the first few terms of the Taylor series:
$$1-\frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4-\frac{1}{6!}\left(\frac{x-\pi}{2}\right)^6+\cdots$$

13. With $f(x)=\sqrt(x)$, $f(9)=\sqrt(9)=3$ and:
$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}\Rightarrow f'(9)=\frac{1}{6}$$
$$f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}\Rightarrow f''(9)=-\frac{1}{108}$$
$$f'''(x)=\frac{3}{8}x^{-\frac{5}{2}}\Rightarrow f'''(9)=\frac{1}{648}$$

and so on, so the first few terms of the Taylor series are:

$3+\frac{1}{6}(x-9)-\frac{1}{108}(x-9)^2+\frac{1}{648}(x-9)^3-\cdots$

15. Using $P_4(x)=1-\frac{1}{2}x^2+\frac{1}{24}x^4\approx\cos(x)$:

x	$\cos(x)$	$P_4(x)$
0.1	0.995004165	0.995004167
0.2	0.98006657	0.98006666
0.5	0.87758	0.87604
1.0	0.54030	0.54167
2.0	−0.4161	−0.3333

17. Using $P_4(x)=x-\frac{1}{3}x^3+\frac{1}{5}x^5\approx\arctan(x)$:

x	$\arctan(x)$	$P_4(x)$
0.1	0.09966865	0.09966867
0.2	0.197396	0.197397
0.5	0.4636	0.4646
1.0	0.7854	0.8667
2.0	1.1071	5.7333

19. With $\sin(u)=u-\frac{1}{6}u^3+\frac{1}{120}u^5-\cdots$, put $u=x^2$ so that $\sin\left(x^2\right)=x^2-\frac{1}{6}x^6+\frac{1}{120}x^{10}-\cdots$ and:

$\int\sin\left(x^2\right)\,dx=C+\frac{1}{3}x^3-\frac{1}{42}x^7+\frac{1}{1320}x^{11}-\cdots$

21. With $\sin(u)=u-\frac{1}{6}u^3+\frac{1}{120}u^5-\cdots$, put $u=x^3$ so that $\sin\left(x^3\right)=x^3-\frac{1}{6}x^9+\frac{1}{120}x^{15}-\cdots$ and:

$\int\sin\left(x^3\right)\,dx=C+\frac{1}{4}x^4-\frac{1}{60}x^{10}+\frac{1}{1920}x^{16}-\cdots$

23. With $e^u=1+u+\frac{1}{2}u^2+\cdots$, put $u=-x^2$ so that $e^{-x^2}=1-x^2+\frac{1}{2}x^4+\cdots$ and:

$$\int e^{-x^2}\,dx=C+x-\frac{1}{3}x^3+\frac{1}{10}x^5-\cdots$$

25. With $e^u=1+u+\frac{1}{2}u^2+\cdots$, put $u=-x^3$ so that $e^{-x^3}=1-x^3+\frac{1}{2}x^6+\cdots$ and:

$$\int e^{-x^3}\,dx=C+x-\frac{1}{4}x^4+\frac{1}{14}x^7-\cdots$$

27. Multiply $\sin(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots$ by x to get $x\cdot\sin(x)=x^2-\frac{1}{6}x^4+\frac{1}{120}x^6-\cdots$ so:

$$\int x\cdot\sin(x)\,dx=C+\frac{1}{3}x^3-\frac{1}{30}x^5+\frac{1}{840}x^7-\cdots$$

29. Multiply $\sin(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots$ by x^2 to get $x\cdot\sin(x)=x^3-\frac{1}{6}x^5+\frac{1}{120}x^7-\cdots$ so:

$$\int x^2\cdot\sin(x)\,dx=C+\frac{1}{4}x^4-\frac{1}{36}x^6+\frac{1}{960}x^8-\cdots$$

31. Subtract $\cos(x)=1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots$ from 1 and divide by x^2 to get:

$$\frac{1-\cos(x)}{x^2}=\frac{\frac{1}{2}x^2-\frac{1}{24}x^4+\cdots}{x^2}=\frac{1}{2}-\frac{1}{24}x^2+\cdots$$

which has limit $\frac{1}{2}$ as $x\to0$.

33. Subtract $e^x=1+x+\frac{1}{2}x^2+\cdots$ from 1 and divide by x to get:

$$\frac{1-e^x}{x}=\frac{x+\frac{1}{2}x^2+\cdots}{x}=1+\frac{1}{2}x+\cdots$$

which has limit 1 as $x\to0$.

35. Dividing $\sin(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots$ by x:

$$\frac{x-\frac{1}{6}x^3+\frac{1}{120}x^5-\cdots}{x}=1-\frac{1}{6}x^2+\frac{1}{120}x^4\cdots$$

yields a limit of 1 as $x\to0$.

37. Subtract $\sin(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7\cdots$ from $x-\frac{1}{6}x^3$ and divide by x^5 to get:

$$\frac{-\frac{1}{120}x^5+\frac{1}{5040}x^7-\cdots}{x^5}=\frac{1}{120}-\frac{1}{5040}x^2+\cdots$$

which has a limit of $\frac{1}{120}$ as $x\to0$.

39. Starting with $e^u = 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots$, put $u = x$, then $u = -x$ to get:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right] - \frac{1}{2}\left[1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \cdots\right]$$
$$= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots$$

41. $\mathbf{D}\left(\sinh(x)\right) = \mathbf{D}\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots\right) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots = \cosh(x)$

43. $e^{i\left(\frac{\pi}{2}\right)} = \cos\left(\frac{\pi}{2}\right) + i \cdot \sin\left(\frac{\pi}{2}\right) = 0 + i \cdot 1 = i$, while $e^{\pi i} = \cos(\pi) + i \cdot \sin(\pi) = -1 + i \cdot 0 = -1$.

45. $\binom{3}{0} = 1$ by definition, while $\binom{3}{1} = \frac{3}{1!} = 3$, $\binom{3}{2} = \frac{3 \cdot 2}{2!} = 3$ and $\binom{3}{3} = \frac{3 \cdot 2 \cdot 1}{3!} = 1$; these agree with the numbers 1, 3, 3, 1 from Pascal's triangle and with the coefficients of $(1+x)^3 = 1 + 3x + 3x^2 + x^3$.

47. The MacLaurin series for $(1+x)^{\frac{5}{2}}$ is:

$$1 + \frac{5}{2}x + \left(\frac{5}{2}\right)\left(\frac{3}{2}\right) \cdot \frac{1}{2!}x^2 + \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \cdot \frac{1}{3!}x^3 + \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \cdot \frac{1}{4!}x^4 + \cdots$$

49. Using the Binomial Series Theorem, the MacLaurin series for $\frac{1}{\sqrt{1+u}} = (1+u)^{-\frac{1}{2}}$ is:

$$1 - \frac{1}{2}u + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \cdot \frac{1}{2!}u^2 - \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) \cdot \frac{1}{3!}u^3 + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right) \cdot \frac{1}{4!}u^4 + \cdots$$

Putting $u = -x^2$ gives $\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots$, so integrating term-by-term (and using the fact that $\arcsin(0) = 0$) yields:

$$\arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots$$

51. With $f(x) = (1+x)^m \Rightarrow f(0) = 1$, $f'(x) = m(1+x)^{m-1} \Rightarrow f'(0) = m$, $f''(x) = m(m-1)(1+x)^{m-2} \Rightarrow f''(0) = m(m-1)$, $f'''(x) = m(m-1)(m-2)(1+x)^{m-3} \Rightarrow f'''(0) = m(m-1)(m-2)$ and $f^{(4)}(x) = m(m-1)(m-2)(m-3)(1+x)^{m-4} \Rightarrow f^{(4)}(0) = m(m-1)(m-2)(m-3)$ so the MacLaurin series is:

$$(1+x)^m = 1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \frac{m(m-1)(m-2)(m-3)}{4!}x^4 + \cdots$$

Section 10.5

1. $P_0(x) = 0$, $P_1(x) = P_2(x) = x$, $P_3(x) = P_4(x) = x - \frac{1}{6}x^3$; use technology to create graphs.

3. $f(x) = \ln(x) \Rightarrow f'(x) = x^{-1} \Rightarrow f''(x) = -x^{-2} \Rightarrow f'''(x) = 2x^{-3} \Rightarrow f^{(4)}(x) = -6x^{-3}$, so $P_0(x) = 0$, $P_1(x) = x$, $P_2(x) = x - \frac{1}{2}x^2$, $P_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$ and $P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

5. $P_0(x) = 1$, $P_1(x) = 1 + (x-1) = P_2(x) = P_3(x) = P_4(x)$

7. $f(x) = (1+x)^{-\frac{1}{2}} \Rightarrow f'(x) = -\frac{1}{2}(1+x)^{-\frac{3}{2}} \Rightarrow f''(x) = \frac{3}{4}(1+x)^{-\frac{5}{2}} \Rightarrow f'''(x) = -\frac{15}{8}(1+x)^{-\frac{7}{2}} \Rightarrow f^{(4)}(x) = \frac{105}{16}(1+x)^{-\frac{9}{2}}$, so $P_0(x) = 1$, $P_1(x) = 1 - \frac{1}{2}x$, $P_2(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2$, $P_3(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$, $P_4(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4$

9. $f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x) \Rightarrow f'''(x) = -\cos(x) \Rightarrow f^{(4)}(x) = \sin(x)$ so $P_0(x) = 1 = P_1(x)$, $P_2(x) = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 = P_3(x)$ and $P_4(x) = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4$

11. $R_5(x) = \frac{\sin(z)}{6!}(x-0)^6$

$|R_5(x)| \le \frac{1}{720}\left(\frac{\pi}{2}\right)^6 = \frac{\pi^6}{46080} \approx 0.021$

13. $R_5(x) = \frac{\cos(z)}{6!}(x-0)^6$

$|R_5(x)| \le \frac{1}{720}(\pi)^6 = \frac{\pi^6}{720} \approx 1.335$

15. $R_{10}(x) = \frac{-\cos(z)}{10!}(x-0)^{10}$

$|R_{10}(x)| \le \frac{1}{362880} \cdot 2^6 = \frac{1}{56700} \approx 0.0000176$

17. $R_6(x) = \frac{e^z}{6!}(x-0)^6$

$|R_6(x)| \le \frac{e^2}{720} \cdot 2^6 \le \frac{2.72^2 \cdot 64}{720} \approx 0.658$

19. $|R_n(x)| \le \frac{1}{(n+1)!} < \frac{1}{1000} \Rightarrow n \ge 6$

21. $|R_n(x)| \le \frac{1.6^{n+1}}{(n+1)!} < \frac{1}{100000} \Rightarrow n \ge 10$

23. $|R_n(x)| \le \frac{2.72^2 \cdot 2^{n+1}}{(n+1)!} < \frac{1}{1000} \Rightarrow n \ge 10$

25. Any derivative of $f(x) = \cos(x)$ is either $\pm\sin(x)$ or $\pm\cos(x)$, so $\left|f^{(n+1)}(z)\right| \le 1$ for any z. Hence:

$$|R_n(x)| = \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}|x-0|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!}$$

As noted in the solution to Example 5, this expression approaches 0 as $n \to \infty$ (for any x).

27. (a) Using the definition of the derivative:

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-h^{-2}}-0}{h}$$

(b) For $h > 0$, let $y = \frac{1}{h} \Rightarrow h = \frac{1}{y}$ so that:

$$f'(0) = \lim_{y\to\infty} \frac{e^{-y^2}}{\frac{1}{y}} = \lim_{y\to\infty} \frac{y}{e^{y^2}}$$

By L'Hôpital's Rule:

$$f'(0) = \lim_{y\to\infty} \frac{1}{2y \cdot e^{y^2}} = 0$$

The process for $h < 0$ is quite similar.

29. (a) $\sum_{k=1}^{5} \frac{(-1)^{k+1}\cdot 4}{2k-1} = \frac{1052}{315} \approx 3.33968$

(b) $\left|\frac{(-1)^{50+1}\cdot 4}{2\cdot 50-1}\right| = \frac{4}{99} \approx 0.0404$

(c) $\left|\frac{(-1)^{k+1}\cdot 4}{2k-1}\right| < \frac{1}{10000} \Rightarrow k \ge 20001$

31. (a) $4\arctan\left(\frac{1}{5}\right) \approx \frac{4}{5} - \frac{4}{375} + \frac{4}{15625} \approx 0.789589$ and $\arctan\left(\frac{1}{239}\right) \approx 0.004184$ so:

$$\pi \approx 4\,[0.197397 - 0.004184] \approx 3.14162$$

(b) We are using smaller values of x in the arctan series, and the powers of these smaller values of x approach 0 more quickly than the values of x used in Methods I and II.