Section 8.1

1. Set $u = x^2 + 7 \Rightarrow du = 2x\,dx$ so that:
$$\int 6x\left(x^2+7\right)^2 dx = \int 3u^2\,du = u^3 + C = \left(x^2+7\right)^3 + C$$

3. Set $u = t^2 - 3 \Rightarrow du = 2t\,dt$ so $t = 2 \Rightarrow u = 1$, $t = 4 \Rightarrow u = 13$ and:
$$\int_2^4 \frac{6t}{\sqrt{t^2-3}}\,dt = \int_1^{13} 3u^{-\frac{1}{2}}\,du = \left[6u^{\frac{1}{2}}\right]_1^{13} = 6\sqrt{13} - 6 \approx 15.63$$

5. Set $u = x^2 + 3 \Rightarrow du = 2x\,dx$ so that:
$$\int \frac{12x}{x^2+3}\,dx = \int \frac{6}{u}\,du = \ln(|u|) + C = 6\ln\left(x^2+3\right) + C$$

7. Set $u = 3y + 2 \Rightarrow du = 3\,dy \Rightarrow \frac{1}{3}\,du = dy$ so that:
$$\int \sin(3y+2)\,dy = \int \frac{1}{3}\sin(u)\,du = -\frac{1}{3}\cos(u) + C = -\frac{1}{3}\cos(3y+2) + C$$

9. Set $u = e^x + 3 \Rightarrow du = e^x\,dx$ so that $x = -1 \Rightarrow u = e^{-1} + 3$, $x = 0 \Rightarrow 4$ and:
$$\int_{-1}^0 e^x \sec^2(e^x+3)\,dx = \int_{e^{-1}+3}^4 \sec^2(u)\,du = \left[\tan(u)\right]_{e^{-1}+3}^4 = \tan(4) - \tan\left(e^{-1}+3\right)$$
or approximately 0.9276.

11. Set $u = \ln(x) \Rightarrow du = \dfrac{1}{x}\,dx$ so that:
$$\int \frac{\ln(x)}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\left[\ln(x)\right]^2 + C$$

13. Set $u = \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$ so that:
$$\int \cos(\theta)e^{\sin(\theta)}\,d\theta = \int e^u\,du = e^u + C = e^{\sin(\theta)} + C$$

15. Set $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$ so $x = 1 \Rightarrow u = 3$, $x = 3 \Rightarrow 9$ and:
$$\int_1^3 \frac{5}{1+9x^2}\,dx = \frac{5}{3}\int_1^9 \frac{1}{1+u^2}\,du = \frac{5}{3}\Big[\arctan(u)\Big]_3^9 = \frac{5}{3}\left[\arctan(9) - \arctan(3)\right] \approx 0.3518$$

17. Set $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2}\,dx$ so that $x = 1 \Rightarrow u = 1$, $x = 2 \Rightarrow \frac{1}{2}$ and:
$$\int_1^2 \frac{1}{x^2}\cdot\cos\left(\frac{1}{x}\right)dx = -\int_1^{\frac{1}{2}} \cos(u)\,du = \int_{\frac{1}{2}}^1 \cos(u)\,du = \Big[\sin(u)\Big]_{\frac{1}{2}}^1 = \sin(1) - \sin\left(\frac{1}{2}\right) \approx 0.3620$$

19. Set $u = 5 + \sin^2(\theta) \Rightarrow du = 2\sin(\theta)\cos(\theta)\,d\theta$ so:
$$\int \frac{6\sin(\theta)\cos(\theta)}{5+\sin^2(\theta)}\,d\theta = \int \frac{3}{u}\,du = 3\ln(|u|) + C = 3\ln\left(5+\sin^2(\theta)\right) + C$$

21. Set $u = 2x + 5 \Rightarrow du = 2\,dx$ so that:
$$\int \frac{10}{2x+5}\,dx = \int \frac{5}{u}\,du = 5\ln(|u|) + C = 5\ln(|2x+5|) + C$$

23. Set $u = 5x^2 + 3 \Rightarrow du = 10x\,dx$ so $x = 1 \Rightarrow u = 8$, $x = 3 \Rightarrow u = 48$ and:
$$\int_1^3 \frac{20x}{5x^2+3}\,dx = \int_8^{48} \frac{2}{u}\,du = \Big[2\ln(|u|)\Big]_8^{48} = 2\ln(48) - 2\ln(8) = \ln(36) \approx 3.5835$$

25. Set $u = x + 3 \Rightarrow du = dx$ so $x = 0 \Rightarrow u = 3$, $x = 1 \Rightarrow u = 4$ and:
$$\int_0^1 \frac{7}{(x+3)^2+4}\,dx = 7\int_3^4 \frac{1}{u^2+2^2}\,du = 7\left[\frac{1}{2}\arctan(u)\right]_3^4 = \frac{7}{2}\left[\arctan(2) - \arctan\left(\frac{3}{2}\right)\right] \approx 0.4352$$

27. Set $u = e^t \Rightarrow du = e^t\,dt$ so:
$$\int \frac{e^t}{1+e^{2t}}\,dt = \int \frac{1}{1+u^2}\,du = \arctan(u) = \arctan\left(e^t\right) + C$$

29. Set $u = 1 + \ln(x) \Rightarrow du = \dfrac{1}{x}\,dx$ so $x = 1 \Rightarrow u = 1$, $x = 3 \Rightarrow u = 1 + \ln(3)$ and:
$$\int_1^3 \frac{3}{x\left[1+\ln(x)\right]}\,dx = \int_1^{1+\ln(3)} \frac{3}{u}\,du = 3\Big[\ln(u)\Big]_1^{1+\ln(3)} = 3\ln(1+\ln(3)) \approx 2.2238$$

31. Set $u = 1 - x^2 \Rightarrow du = -2x\,dx$ so $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$ and:

$$\int_0^1 2x\sqrt{1-x^2}\,dx = -\int_1^0 \sqrt{u}\,du = \int_0^1 u^{\frac{1}{2}}\,du = \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$$

33. Set $u = 1 + \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$ so:

$$\int \cos(\theta)\,[1+\sin(\theta)]^3\,d\theta = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\,[1+\sin(\theta)]^4 + C$$

35. Set $u = \ln(x) \Rightarrow du = \frac{1}{x}\,dx$ so $x = 1 \Rightarrow u = 0$, $x = e \Rightarrow u = 1$ and:

$$\int_1^e \frac{\sqrt{\ln(x)}}{x}\,dx = \int_0^1 \sqrt{u}\,du = \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}$$

37. Set $u = 5 + \tan(\theta) \Rightarrow du = \sec^2(\theta)\,d\theta$ so:

$$\int \frac{\sec^2(\theta)}{5+\tan(\theta)}\,d\theta = \int \frac{1}{u}\,du = \ln\left(|5+\tan(\theta)|\right) C$$

39. $\ln\left(|\sec(y-5)|\right) + C$

41. $\frac{1}{5}\left[e^{5u}\right]_0^1 = \frac{1}{5}\left[e^5 - 1\right] \approx 29.483$

43. $\frac{1}{6}\ln\left(\left|\sec\left(2+3t^2\right) + \tan\left(2+3t^2\right)\right|\right) + C$

45. 0

47. Set $u = 3x^2 \Rightarrow du = 6x\,dx \Rightarrow \frac{1}{6}\,du = x\,dx$ so $x = 1 \Rightarrow u = 3$, $x = \infty \Rightarrow u = \infty$ and:

$$\begin{aligned}\int_1^\infty \frac{x}{1+9x^4}\,dx &= \frac{1}{6}\int_3^\infty \frac{1}{1+u^2}\,du \\ &= \lim_{M\to\infty} \frac{1}{6}\int_1^M \frac{1}{1+u^2}\,du \\ &= \lim_{M\to\infty} \frac{1}{6}\Big[\arctan(u)\Big]_3^M \\ &= \lim_{M\to\infty} \frac{1}{6}\Big[\arctan(M) - \arctan(3)\Big] \\ &= \frac{1}{6}\left[\frac{\pi}{2} - \arctan(3)\right] \approx 0.536\end{aligned}$$

49. $\int \frac{7}{(x+2)^2+1^2}\,dx = 7\arctan(x+2) + C$

51. $\int \frac{2}{(x-3)^2+7^2}\,dx = \frac{2}{7}\arctan\left(\frac{x-3}{7}\right) + C$

53. $\int \frac{3}{(x+5)^2+2^2}\,dx = \frac{3}{2}\arctan\left(\frac{x+5}{2}\right) + C$

55. $$\int \frac{2x+4}{x^2+4x+5}\,dx + \int \frac{7}{(x+2)^2+1^2}\,dx = \ln\left(x^2+4x+5\right) + 7\arctan(x+2) + C$$

57. $$\int \frac{2(2x-6)}{x^2-6x+10}\,dx + \int \frac{19}{(x-3)^2+1^2}\,dx = 2\ln\left(x^2-6x+10\right) + 19\arctan(x-3) + C$$

59. $$\int \frac{3(2x-4)}{x^2-4x+13}\,dx + \int \frac{17}{(x-2)^2+3^2}\,dx = 3\ln\left(x^2-4x+13\right) + \frac{17}{3}\arctan\left(\frac{x-2}{3}\right) + C$$

61. $\int \frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2} + C$

63. $$\int \frac{x+3}{(x-3)^2}\,dx = \int \frac{x-3}{(x-3)^2}\,dx + \int \frac{6}{(x-3)^2}\,dx = \ln\left(|x-3|\right) - \frac{6}{x-3} + C$$

65. Rewrite $\int_3^\infty \frac{1}{(x-3)^2}\,dx$ as:

$$\lim_{a\to 3^+}\int_b^4 (x-3)^{-2}\,dx + \lim_{M\to\infty}\int_4^M (x-3)^{-2}\,dx$$

The first limit diverges, so the integral diverges.

Section 8.2

1. Setting $u = \ln(x)$ leaves $dv = 12x\,dx$ so we have $du = \frac{1}{x}\,dx$, $v = 6x^2$ and therefore:

$$\int 12x\ln(x)\,dx = 6x^2\ln(x) - \int 6x^2\cdot\frac{1}{x}\,dx = 6x^2\ln(x) - \int 6x\,dx = 6x^2\ln(x) - 3x^2 + C$$

3. Setting $dv = x^4\,dx$ leaves $u = \ln(x)$ so $du = \frac{1}{x}\,dx$, $v = \frac{1}{5}x^5$ and:

$$\int x^4\ln(x)\,dx = \frac{1}{5}x^5\ln(x) - \int \frac{1}{5}x^5\cdot\frac{1}{x}\,dx = \frac{1}{5}x^5\ln(x) - \frac{1}{5}\int x^4\,dx = \frac{1}{5}x^5\ln(x) - \frac{1}{25}x^5 + C$$

5. Setting $dv = x\,dx$ leaves $u = \arctan(x)$ so $du = \frac{1}{1+x^2}\,dx$, $v = \frac{1}{2}x^2$ and:

$$\begin{aligned}\int x\arctan(x)\,dx &= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \\ &= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\int\left[1 - \frac{1}{1+x^2}\right]dx \\ &= \frac{1}{2}x^2\arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + C\end{aligned}$$

7. Setting $u = x$ leaves $dv = e^{-3x}\,dx$ so $du = dx$, $v = -\frac{1}{3}e^{-3x}$ and:
$$\int xe^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \left(-\frac{1}{3}\right)\int e^{-3x}\,dx$$
$$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$$

9. Setting $u = x$ leaves $dv = \sec(x)\tan(x)\,dx$ so $du = dx$, $v = \sec(x)$ and:
$$\int x\sec(x)\tan(x)\,dx = x\sec(x) - \int \sec(x)\,dx$$
$$= x\sec(x) - \ln\left(|\sec(x)+\tan(x)|\right) + C$$
or approximately -2.887.

11. Setting $u = 7x$ leaves $dv = \cos(3x)\,dx$ so that $du = 7\,dx$, $v = \frac{1}{3}\sin(3x)$ and:
$$\int 7x\cos(3x)\,dx = \frac{7}{3}x\sin(3x) - \frac{7}{3}\int \sin(3x)\,dx$$
$$= \frac{7}{3}x\sin(3x) + \frac{7}{9}\cos(3x) + C$$
and the definite integral is then:
$$\left[\frac{7}{3}x\sin(x) + \frac{7}{9}\cos(x)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{7}{9} - \frac{7\pi}{6}$$

13. Don't use integration by parts on this one! Use the substitution $w = 3x^2 \Rightarrow dw = 6x\,dx$:
$$\int 12x\cos(3x^2)\,dx = \int 2\cos(w)\,dw = 2\sin(w) + C$$
$$= 2\sin(3x^2) + C$$

15. Setting $u = \ln(2x+5)$ leaves $dv = dx$ so that $du = \frac{2}{2x+5}\,dx$, $v = x$ and:
$$\int \ln(2x+5)\,dx = x\ln(2x+5) - \int \frac{2x}{2x+5}\,dx$$
$$= x\ln(2x+5) - \int \left[1 - \frac{5}{2x+5}\right]\,dx$$
$$= x\ln(2x+5) - x + \frac{5}{2}\ln(2x+5) + C$$
and the definite integral is then:
$$\left[x\ln(2x+5) - x + \frac{5}{2}\ln(2x+5)\right]_1^3$$
$$= \left[3\ln(11) - 3 + \frac{5}{2}\ln(11)\right] - \left[\ln(7) - 1 + \frac{5}{2}\ln(7)\right]$$
$$= \frac{11}{2}\ln(11) - \frac{7}{2}\ln(7) - 2 \approx 4.38$$

17. Setting $u = (\ln(x))^2$ leaves $dv = dx$ so that $du = 2\ln(x)\cdot\frac{1}{x}\,dx$, $v = x$ and:
$$\int (\ln(x))^2\,dx = x(\ln(x))^2 - 2\int \ln(x)\,dx$$
$$= x(\ln(x))^2 - 2\left[x\ln(x) - x\right] + C$$
$$= x(\ln(x))^2 - 2x\ln(x) + 2x + C$$
and the definite integral is then:
$$\left[x(\ln(x))^2 - 2x\ln(x) + 2x\right]_1^e$$
$$= [e - 2e + 2e] - [0 - 0 + 2] = e - 2 \approx 0.728$$

19. Setting $u = \arcsin(x)$ leaves $dv = dx$ so that $du = \frac{1}{\sqrt{1-x^2}}\,dx$, $v = x$ and:
$$\int \arcsin(x)\,dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$
For this new integral use the substitution $w = 1 - x^2 \Rightarrow dw = -2x\,dx \Rightarrow -\frac{1}{2}\,dw = x\,dx$ so that:
$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int w^{-\frac{1}{2}}\,dw = -w^{\frac{1}{2}} + K$$
$$= -\sqrt{1-x^2} + K$$
and the original integral is then:
$$\int \arcsin(x)\,dx = x\arcsin(x) + \sqrt{1-x^2} + C$$

21. Setting $u = \arctan(3x)$ leaves $dv = x\,dx$ so that $du = \frac{3}{1+9x^2}\,dx$, $v = \frac{1}{2}x^2$ and:
$$\int x\arctan(3x)\,dx = \frac{1}{2}x^2\arctan(3x) - \frac{3}{2}\int \frac{x^2}{1+9x^2}\,dx$$
$$= \frac{1}{2}x^2\arctan(3x) - \frac{3}{2}\cdot\frac{1}{9}\int \left[1 - \frac{1}{1+9x^2}\right]\,dx$$
$$= \frac{1}{2}x^2\arctan(3x) - \frac{1}{6}\left[x - \frac{1}{3}\arctan(3x)\right] + C$$
$$= \frac{1}{2}x^2\arctan(3x) - \frac{1}{6}x + \frac{1}{18}\arctan(3x) + C$$

23. Don't use integration by parts on this one! Use the substitution $w = \ln(x) \Rightarrow dw = \frac{1}{x}\,dx$ so that:
$$\int \frac{\ln(x)}{x}\,dx = \int w\,dw = \frac{1}{2}w^2 + C = \frac{1}{2}(\ln(x))^2 + C$$
and the definite integral is: $\frac{1}{2}(\ln(2))^2 \approx 0.240$.

25. On your own.

27. Write $\sec^n(x) = \sec^{n-2}(x) \cdot \sec^2(x)$ and set $u = \sec^{n-2}(x)$, leaving $dv = \sec^2(x)\,dx$, so that:

$$du = (n-2)\sec^{n-3}(x) \cdot \sec(x)\tan(x)\,dx = (n-2)\sec^{n-2}(x)\tan(x)\,dx \quad \text{and} \quad v = \tan(x)$$

Integration by parts then says:

$$\int \sec^{n-2}(x)\,dx = \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\,dx$$

Now use the identity $\tan^2(x) = \sec^2(x) - 1$ to write this new integral as:

$$\int \sec^{n-2}(x)\left[\sec^2(x) - 1\right]\,dx = \int \sec^n(x)\,dx - \int \sec^{n-2}(x)\,dx$$

and combining these results yields:

$$\int \sec^{n-2}(x)\,dx = \sec^{n-2}(x)\tan(x) - (n-2)\left[\int \sec^n(x)\,dx - \int \sec^{n-2}(x)\,dx\right]$$

Moving the first of the two integrals on the right side to the left side:

$$(n-1)\int \sec^{n-2}(x)\,dx = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)\,dx$$

and solving for the original integral yields:

$$\int \sec^{n-2}(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

29. $\displaystyle\int \sin^3(x)\,dx = \frac{1}{3}\left[-\sin^2(x)\cos(x) + 2\int \sin(x)\,dx\right] = -\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x) + C$

31. $\displaystyle\int \sin^5(x)\,dx = \frac{1}{5}\left[-\sin^4(x)\cos(x) + 4\int \sin^3(x)\,dx\right]$ and using the result of Problem 29 yields:

$$\int \sin^5(x)\,dx = -\frac{1}{5}\sin^4(x)\cos(x) - \frac{4}{15}\sin^2(x)\cos(x) - \frac{8}{15}\cos(x) + C$$

33. $\displaystyle\int \cos^4(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)\,dx = \frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\left[\frac{1}{2}\cos(x)\sin(x) + x\right] + C$

35. $\displaystyle\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\left(|\sec(x) + \tan(x)|\right) + C$

37. $\displaystyle\int \sec^5(x)\,dx = \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{4}\int \sec^3(x)\,dx$ and using the result of Problem 33 yields:

$$\int \sec^5(x)\,dx = \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x) + \frac{3}{8}\ln\left(|\sec(x) + \tan(x)|\right) + C$$

39. $\displaystyle\int \cos^3(u)\,du = \frac{1}{3}\cos^2(u)\sin(u) + \frac{2}{3}\int \cos(u)\,du == \frac{1}{3}\cos^2(u)\sin(u) + \frac{2}{3}\sin(u) + C$ so that:

$$\begin{aligned}\int \cos^3(2x+3)\,dx &= \frac{1}{2}\left[\frac{1}{3}\cos^2(2x+3)\sin(2x+3) + \frac{2}{3}\sin(2x+3)\right] + C \\ &= \frac{1}{6}\cos^2(2x+3)\sin(2x+3) + \frac{1}{3}\sin(2x+3) + C\end{aligned}$$

41. Set $u = x^n$ and $dv = e^{ax}\,dx$ so $du = n \cdot x^{n-1}\,dx$, $v = \frac{1}{a}e^{ax}$ and $\displaystyle\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$

43. Set $u = (\ln(x))^n$, leaving $dv = x\,dx$, so that $du = n\,(\ln(x))^{n-1} \cdot \frac{1}{x}\,dx$, $v = \frac{1}{2}x^2$ and:
$$\int x\,(\ln(x))^n\;dx = \frac{1}{2}x^2\,(\ln(x))^n - \frac{n}{2}\int x\,(\ln(x))^{n-1}\;dx$$

45. (a) Set $u = x \Rightarrow du = dx$ and $dv = (2x+5)^{19}\;dx \Rightarrow v = \frac{1}{40}\,(2x+5)^{20}$:
$$\int x\,(2x+5)^{19}\;dx$$
$$= \frac{x}{40}\,(2x+5)^{20} - \frac{1}{40}\int (2x+5)^{20}\;dx$$
$$= \frac{x}{40}\,(2x+5)^{20} - \frac{1}{40}\cdot\frac{1}{42}\,(2x+5)^{21} + C$$
(b) Set $w = 2x+5 \Rightarrow dw = 2\,dx \Rightarrow \frac{1}{2}\,dw = dx$ and note that $x = \dfrac{1}{2}(w-5)$:
$$\int x\,(2x+5)^{19}\;dx = \int \frac{1}{2}(w-5)\cdot w^{19}\cdot\frac{1}{2}\,dw$$
$$= \frac{1}{2}\int\left[w^{20} - 5w^{19}\right]\;dw$$
$$= \frac{1}{4}\left[\frac{1}{21}w^{21} - \frac{5}{20}w^{20}\right] + K$$
$$= \frac{1}{84}\,(2x+5)^{21} - \frac{1}{16}\,(2x+5)^{20} + K$$
These answers look different, but you can verify that the derivative of each answer is $x(2x+5)^{19}$.

47. Use the result of Problem 43 (twice) to get:
$$\frac{1}{2}x^2\left[(\ln(x))^2 - \ln(x) + \frac{1}{2}\right] + C$$

49. Apply integration by parts twice to get a reappearing integral and, eventually:
$$\frac{1}{2}\left[-e^{-1}\cos(1) - e^{-1}\sin(1) + 1\right] \approx 0.24584$$

51. Substitute $y = \ln(x) \Rightarrow x = e^y \Rightarrow dx = e^y\,dy$ to turn the integral into $\displaystyle\int \sin(y)\cdot e^y\,dy$ and then apply integration by parts twice to get a reappearing integral and, eventually:
$$\frac{1}{2}x\left[\sin\left(\ln(x)\right) - \cos\left(\ln(x)\right)\right] + C$$

53. Substitute $y = \sqrt{x} \Rightarrow x = y^2 \Rightarrow dx = 2y\,dy$, then apply integration by parts to get:
$$2\left[\sqrt{x}\sin\left(\sqrt{x}\right) + \cos\left(\sqrt{x}\right)\right] + C$$

55. Integration by parts (twice) results in a reappearing integral and, eventually:
$$\frac{1}{10}e^{3x}\left[3\sin(x) - \cos(x)\right] + C$$

57. Integration by parts yields $-(x+1)e^{-x} + C$ so:
$$\int_0^M xe^{-x}\,dx = \Big[-(x+1)e^{-x}\Big]_0^M = -\frac{M+1}{e^M} + 1$$
and $\displaystyle\int_0^\infty xe^{-x}\,dx = \lim_{M\to\infty}\left[1 - \frac{M+1}{e^M}\right] = 0.$

59. After integrating by parts (twice):
$$\lim_{M\to 0}\frac{1}{2}\left[-e^{-x}\cos(x) - e^{-x}\sin(x)\right]_0^M = \frac{1}{2}$$

61. The substitution $w = x+1$ results in:
$$\int (w-1)\sqrt{w}\,dw = \frac{2}{5}w^{\frac{5}{2}} - \frac{2}{3}w^{\frac{3}{2}} + C$$
and then resubstitute $w = x+1$.

63. Use the substitution $w = x^2$ to get $\frac{1}{2}\sin(x^2) + C$.

65. Use integration by parts twice, starting with $u = x^2$ and $dv = \cos(x)\,dx$ to get:
$$x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

67. Use the substitution $y = x^2 + 1$ so so that:
$$\int \frac{1}{2}(y-1)y^{\frac{1}{3}}\,dy = \frac{3}{14}\left(x^2+1\right)^{\frac{7}{3}} - \frac{3}{8}(x^2+1)^{\frac{4}{3}} + C$$

69. Integration by parts on the right side yields $y = -x\cos(x) + \sin(x) + C$ and using the initial condition tells us $0 = y(0) = 0 + C \Rightarrow C = 0$.

71. Separate variables to get $\int e^y\,dy = \int xe^{-x}\,dx$ so, using integration by parts on the right side:
$$e^y = -(x+1)e^{-x} + C \Rightarrow e^1 = -1 + C$$
resulting in $y = e + 1 - (x+1)e^{-x}$.

73. (a) When $0 \le x \le 1$, $x\sin(x) \le \sin(x)$, so you should expect the second integral to be larger. (b) Using integration by parts on the first integral yields $\sin(1) - \cos(1) \approx 0.3012$, while the value of the second integral is $1 - \cos(1) \approx 0.4597$.

75. (a) Make an informed prediction. (b) From Problem 17, we know that the first volume is $\pi(e-2) \approx 2.257$, while using integration by parts on the second integral yields $2\pi \approx 6.283$.

77. Using the tube method, the volume is given by $\int_0^\pi 2\pi x \cdot \sin(x)\,dx$ and, using integration by parts with $u = 2\pi x \Rightarrow du = 2\pi\,dx$ so that $dv = \sin(x)\,dx \Rightarrow v = -\cos(x)$, this yields:

$$\Big[-2\pi x\cos(x)\Big]_0^\pi - \int_0^\pi [-\cos(x)]\cdot 2\pi\,dx$$
$$= 2\pi^2 + 2\pi\int_0^\pi \cos(x)\,dx$$
$$= 2\pi^2 + 2\pi\Big[\sin(x)\Big]_0^\pi = 2\pi^2$$

79. Using the tube method, the volume is given by:

$$\int_0^\pi 2\pi x \cdot x\sin(x)\,dx = 2\pi\int_0^\pi x^2\sin(x)\,dx$$

Using integration by parts twice yields:

$$\Big[-x^2\cos(x) + 2\pi x\sin(x) + 2\cos(x)\Big]_0^\pi$$

which evaluates to $\pi^2 - 4 \approx 5.8670$.

81. The area of the region is given by:

$$\int_0^\infty xe^{-x}\,dx = \lim_{M\to\infty}\int_0^M xe^{-x}\,dx$$

Using integration by parts yields:

$$\lim_{M\to\infty}\Big[-xe^{-x} - e^{-x}\Big]_0^M = \lim_{M\to\infty}\left[-\frac{M}{e^M} - \frac{1}{e^M} + 1\right]$$

which equals 1, so the area is finite.

83. Using the disk method, the volume is given by:

$$\int_0^\infty \pi\left[xe^{-x}\right]^2\,dx = \lim_{M\to\infty}\int_0^M x^2e^{-2x}\,dx$$

Using integration by parts (twice) yields:

$$\lim_{M\to\infty}\left[-\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^M$$
$$= \lim_{M\to\infty}\left[-\frac{M^2}{2e^{2M}} - \frac{M}{2e^{2M}} - \frac{1}{4e^M} + \frac{1}{4}\right]$$

which equals $\frac{1}{4}$, so the volume is finite.

85. Using the tube method, the volume is given by:

$$\int_0^\infty 2\pi x\cdot xe^{-x}\,dx = \lim_{M\to\infty} 2\pi\int_0^M x^2e^{-x}\,dx$$

Using integration by parts (twice) yields:

$$\lim_{M\to\infty}\Big[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big]_0^M$$

which becomes:

$$\lim_{M\to\infty}\left[-\frac{M^2}{e^M} - \frac{2M}{e^M} - \frac{2}{e^M} + 2\right] = 2$$

so the volume is finite.

Section 8.3

1. Decompose the integrand as:

$$\frac{7x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Multiplying by $x(x+1)$ yields:

$$7x + 2 = A(x+1) + Bx$$

If $x = 0$ this tells us that $2 = A$; if $x = -1$, $-5 = -B \Rightarrow B = 5$, so the integral becomes:

$$\int\left[\frac{2}{x} + \frac{5}{x+1}\right]dx = 2\ln(|x|) + 5\ln(|x+1|) + C$$

3. Factor the denominator and decompose as:

$$\frac{11x+25}{(x+1)(x+8)} = \frac{A}{x+1} + \frac{B}{x+8}$$

Multiplying by $(x+1)(x+8)$ yields:

$$11x + 25 = A(x+8) + B(x+1)$$

If $x = -1$, $14 = 7A \Rightarrow A = 2$; if $x = -8$, $-63 = -7B \Rightarrow B = 9$, so:

$$\int\left[\frac{2}{x+1} + \frac{9}{x+8}\right]dx = \ln\left((x+1)^2\cdot|x+8|^9\right) + C$$

5. First use polynomial division, then factor the denominator to rewrite the integral as:

$$\int\left[2 + \frac{5}{x}\right]dx = 2x + 5\ln(|x|) + C$$

7. Decompose the integrand as:

$$\frac{6x^2+9x-15}{x(x+5)(x-1)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-1}$$

Multiplying by $x(x+5)(x-1)$:

$$6x^2 + 9x - 15$$
$$= A(x+5)(x-1) + Bx(x-1) + Cx(x+5)$$

If $x = 0$, $-15 = -5A \Rightarrow A = 3$; if $x = -5$, then $90 = 30B \Rightarrow B = 3$; and if $x = 1$, $0 = 6C \Rightarrow C = 0$. So the integral becomes:

$$\int\left[\frac{3}{x} + \frac{3}{x+5}\right]dx = 3\ln(|x|) + 3\ln(|x+5|) + C$$

9. Factor the denominator and decompose:

$$\frac{8x^2-x+3}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

Multiplying by the denominator $x(x^2+1)$ yields:

$$8x^2-x+3=[A+B]x^2+Cx+A$$

so $A=3$, $C=-1$, $A+B=8\Rightarrow B=5$ and:

$$\int\left[\frac{3}{x}+\frac{5x-1}{x^2+1}\right]dx$$
$$=\int\left[\frac{3}{x}+\frac{5}{2}\cdot\frac{2x}{x^2+1}-\frac{1}{x^2+1}\right]dx$$
$$=3\ln(|x|)+\frac{5}{2}\ln\left(x^2+1\right)-\arctan(x)+C$$

11. Decompose the integrand as:

$$\frac{11x^2+23x+6}{x^2(x+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}$$

and multiply both sides by $x^2(x+2)$ to get:

$$11x^2+23x+6=Ax(x+2)+B(x+2)+Cx^2$$
$$=[A+C]x^2+[2A+B]x+2B$$

so $2B=6\Rightarrow B=3$, $2A+B=23\Rightarrow 2A=20\Rightarrow A=10$, $A+C=11\Rightarrow C=1$ and:

$$\int\left[\frac{10}{x}+\frac{3}{x^2}+\frac{1}{x+2}\right]dx$$
$$=10\ln(|x|)-\frac{3}{x}+\ln(|x+2|)+C$$

13. Decompose the integrand as:

$$\frac{3x+13}{(x+2)(x-5)}=\frac{A}{x+2}+\frac{B}{x-5}$$

and multiply by the denominator $(x+2)(x-5)$:

$$3x+13=A(x-5)+B(x+2)$$

With $x=-2$, $7=-7A\Rightarrow A=-1$; with $x=5$, $28=7B\Rightarrow B=4$ so the integral becomes:

$$\int\left[\frac{-1}{x+2}+\frac{4}{x-5}\right]dx=\ln\left(\frac{(x-5)^4}{|x+2|}\right)+C$$

15. The integrand decomposes as:

$$\frac{2}{(x-1)(x+1)}=\frac{1}{x-1}-\frac{1}{x+1}$$

so the integral becomes:

$$\int_2^5\left[\frac{1}{x-1}-\frac{1}{x+1}\right]dx=\left[\ln\left(\left|\frac{x-1}{x+1}\right|\right)\right]_2^5$$

which evaluates to $\ln\left(\frac{2}{3}\right)-\ln\left(\frac{1}{3}\right)=\ln(2)$.

17. Use polynomial division to rewrite the integrand:

$$\int\left[2+\frac{5x-1}{(x-1)(x+1)}\right]dx$$
$$=\int\left[2+\frac{2}{x-1}+\frac{3}{x+1}\right]dx$$
$$=2x+2\ln(|x-1|)+3\ln(|x+1|)+C$$

19. Use polynomial division to rewrite the integrand:

$$\int\left[3+\frac{x+9}{(x+1)(x+5)}\right]dx$$
$$=\int\left[3+\frac{2}{x+1}-\frac{1}{x+5}\right]dx$$
$$=3x+2\ln(|x+1|)-\ln(|x+5|)+C$$

21. Factor the denominator and decompose:

$$\frac{3x^2-1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

and then multiply by the denominator $x(x^2+1)$:

$$3x^2-1=A(x^2+1)+(Bx+C)x$$
$$=[A+B]x^2+Cx+A$$

so $A=-1$, $C=0$ and $A+B=3\Rightarrow B=4$ and:

$$\int\left[\frac{-1}{x}+\frac{4x}{x^2+1}\right]dx=-\ln(|x|)+2\ln\left(x^2+1\right)+C$$

23. Use polynomial division to rewrite the integrand:

$$\int\left[x+\frac{6(x+5)}{(x-2)(x+5)}\right]dx=\frac{1}{2}x^2+6\ln(|x-2|)+C$$

25. Factor the denominator and decompose:

$$\frac{12x^2+19x-6}{x^2(x+3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+3}$$

then multiply by the denominator $x^2(x+3)$:

$$12x^2+19x-6=Ax(x+3)+B(x+3)+Cx^2$$

When $x=0$, $-6=3B\Rightarrow B=-2$; when $x=-3$, $45=9C\Rightarrow C=5$; and when $x=1$, $25=4A-2(4)+5(1)^2\Rightarrow 4A=28\Rightarrow A=7$ so:

$$\int\left[\frac{7}{x}-\frac{2}{x^2}+\frac{5}{x+3}\right]dx$$
$$=7\ln(|x|)+\frac{2}{x}+5\ln(|x+3|)+C$$
$$=\frac{2}{x}+\ln\left(|x|^7\cdot|x+3|^5\right)+C$$

27. Factor the denominator and decompose:

$$\frac{7x^2+3x+7}{x^2(x+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

then multiply by the denominator $x(x^2+1)$:

$$7x^2+3x+7 = [A+B]x^2 + Cx + A$$

so $A = 7$, $C = 3$, $A + B = 7 \Rightarrow B = 0$ and:

$$\int \left[\frac{7}{x} + \frac{3}{x^2+1}\right] dx = 7\ln(|x|) + 3\arctan(x) + C$$

29. Using the result of Problem 15:

$$\int_2^\infty \frac{2}{x^2-1}\,dx = \lim_{M\to\infty}\left[\ln\left(\left|\frac{x-1}{x+1}\right|\right)\right]_2^M$$
$$= \lim_{M\to\infty}\left[\ln\left(\left|\frac{M-1}{M+1}\right|\right) - \ln\left(\frac{1}{3}\right)\right] = \ln(3)$$

31. Decompose the integrand as:

$$\frac{6x^2+5x+61}{(x-1)(x^2+4x+13)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+13}$$

and multiply by the common denominator to get:

$$6x^2+5x+61 = A(x^2+4x+13) + (Bx+C)(x-1)$$
$$= [A+B]x^2 + [4A-B+C]x + [13A-C]$$

so $A + B = 6$ and $4A - B + C = 5 \Rightarrow 5A + C = 11$ while $13A - C = 61$, so $18A = 72 \Rightarrow A = 4 \Rightarrow B = 2 \Rightarrow C = -9$ and the integral becomes:

$$\int \left[\frac{4}{x-1} + \frac{2x-9}{x^2+4x+13}\right] dx$$
$$= \int \left[\frac{4}{x-1} + \frac{2x+4}{x^2+4x+13} - \frac{13}{(x+2)^2+3^2}\right] dx$$
$$= 4\ln(|x-1|) + \ln(x^2+4x+13)$$
$$- \frac{13}{3}\arctan\left(\frac{x+2}{3}\right) + C$$

33. (a) $\displaystyle\int \frac{1}{(x+1)^2+1}\,dx = \arctan(x+1) + C$

(b) $\displaystyle\int \frac{1}{(x+1)^2}\,dx = \frac{-1}{x+1} + C$

(c) $\displaystyle\int \frac{1}{x(x+2)}\,dx = \frac{1}{2}\int\left[\frac{1}{x} - \frac{1}{x+2}\right] dx$

$$= \frac{1}{2}\ln\left(\left|\frac{x}{x+2}\right|\right) + C$$

35. Using the decomposition from Problem 1:

$$f(x) = 2x^{-1} + 5(x+1)^{-1}$$
$$f'(x) = -2x^{-2} - 5(x+1)^{-2}$$
$$f''(x) = 4x^{-3} + 10(x+1)^{-3}$$

37. Using the decomposition from Problem 3:

$$g(x) = 2(x+1)^{-1} + 9(x+8)^{-1}$$
$$g'(x) = -2(x+1)^{-2} - 9(x+8)^{-2}$$
$$g''(x) = 4(x+1)^{-3} + 18(x+8)^{-3}$$

39. Using results from Problem 5:

$$h(x) = 2 + 3x^{-1}$$
$$h'(x) = -3x^{-2}$$
$$h''(x) = 6x^{-3}$$

41. Using $u = \sin(\theta) \Rightarrow du = \cos(\theta)\,d\theta$:

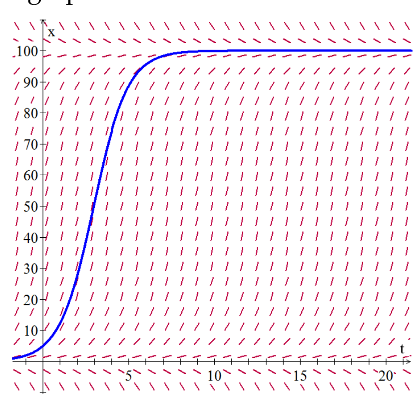
$$\int \frac{1}{1-u^2}\,du = \int \frac{1}{(1-u)(1+u)}\,du$$
$$= \frac{1}{2}\int\left[\frac{1}{1-u} + \frac{1}{1+u}\right] du$$
$$= -\frac{1}{2}\ln(|1-u|) + \frac{1}{2}\ln(|1+u|) + C$$
$$= \ln\left(\sqrt{\left|\frac{1+u}{1-u}\right|}\right) + C$$

Replacing u with $\sin(\theta)$ yields:

$$\frac{1+u}{1-u} = \frac{1+\sin(\theta)}{1-\sin(\theta)} \cdot \frac{1+\sin(\theta)}{1+\sin(\theta)} = \frac{(1+\sin(\theta))^2}{1-\sin^2(\theta)}$$
$$= \left[\frac{1+\sin(\theta)}{\cos(\theta)}\right]^2 = [\sec(\theta)+\tan(\theta)]^2$$

Combining results: $\ln(|\sec(\theta)+\tan(\theta)|) + C$.

43. (a) Here is the direction field for the ODE, along with a graph of the solution to the IVP:



(b) To solve $\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)$, first separate the variables:

$$\int \frac{1}{x\left(1-\frac{x}{100}\right)}\,dx = \int t\,dt$$

Then decompose the integrand on the left:

$$\frac{1}{x\left(1-\frac{x}{100}\right)} = \frac{1}{x} + \frac{\frac{1}{100}}{1-\frac{x}{100}} = \frac{1}{x} + \frac{1}{100-x}$$

Integrating both sides of the integral equation:

$$\ln\left(\left|\frac{x}{100-x}\right|\right) = t + C$$

Using the initial condition:

$$\ln\left(\left|\frac{5}{95}\right|\right) = t + C \;\Rightarrow\; C = -ln(19)$$

Solving for $x(t)$:

$$\frac{x}{100-x} = \frac{1}{19}e^t \;\Rightarrow\; 19x = (100-x)e^t$$

finally results in:

$$x(t) = \frac{100e^t}{e^t+19} = \frac{100}{1+19e^{-t}}$$

(c) See graph for (a).

(d) Using an intermediate equation from (b):

$$\ln\left(\left|\frac{20}{100-20}\right|\right) = t - \ln(19) \Rightarrow t = \ln\left(\frac{19}{4}\right)$$

or about 1.56. Similarly, $x = 50 \Rightarrow t = \ln(19) \approx 2.94$ and $x = 90 \Rightarrow t = \ln(171) \approx 5.14$; $x = 100$ is impossible.

(e) $x(t) \to 100$ (the carrying capacity)

(f) The bacteria begin to grow exponentially, but soon the growth rate slows and the number of bacteria approaches the carrying capacity.

(g) $\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)$ is biggest when $x = 50$, which by part (d) occurs when $t = \ln(19)$.

(h) $x = 50$

(i) The number of bacteria would decrease, approaching the carrying capacity.

45. (a) Following the same steps as in Problem 43 yields:

$$x(t) = \frac{M}{1 + e^{-t}\left(\frac{M}{x_0} - 1\right]}$$

(b) $x(t) \to M$

(c) When $x = \dfrac{M}{2} \Rightarrow t = \ln\left(\left|\dfrac{M-x_0}{x_0}\right|\right)$.

(d) $x = \frac{M}{2}$

47. (a) Separate variables and use partial fractions to get the integral equation:

$$\int \left[\frac{\frac{1}{b-a}}{a-x} + \frac{\frac{1}{a-b}}{b-x}\right] dx = \int 1\,t$$

Integrate both sides to get:

$$\ln\left(\left|\frac{b-x}{a-x}\right|\right) = (b-a)t + K$$

Use $x(0) = 0$ to get $K = \ln\left(\frac{b}{a}\right)$ so that:

$$\frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)t}$$

and solve for x to get:

$$x(t) = \frac{b\left[e^{(b-a)t} - 1\right]}{\frac{b}{a}e^{(b-a)t} - 1}$$

(b) Separate variables to get:

$$\int \frac{1}{(c-x)^2}\,dx = \int 1\,dt$$

and integrate to get:

$$\frac{1}{c-x} = t + C$$

The initial condition tells us $\frac{1}{c} = C$ so:

$$\frac{1}{x-c} = t + \frac{1}{c} \;\Rightarrow\; x(t) = c + \frac{1}{t + \frac{1}{c}}$$

Section 8.4

1. $x = 7\sin(\theta)$

3. $x = 9\tan(\theta)$

5. $x = \sqrt{7}\sec(\theta)$

7. $x = 10\sin(\theta)$

9. $x = 3\sin(\theta) \Rightarrow dx = 3\cos(\theta)\,d\theta$ and:

$$\frac{1}{\sqrt{9-(3\sin(\theta))^2}} = \frac{1}{\sqrt{9\left(1-\sin^2(\theta)\right)}} = \frac{1}{3\cos(\theta)}$$

11. $x = 3\sec(\theta) \Rightarrow dx = 3\sec(\theta)\tan(\theta)\,d\theta$ and:

$$\frac{1}{\sqrt{(3\sec(\theta))^2-9}} = \frac{1}{\sqrt{9\left(\sec^2(\theta)-1\right)}} = \frac{1}{3\tan(\theta)}$$

13. $x = \sqrt{2}\tan(\theta) \Rightarrow dx = \sqrt{2}\sec^2(\theta)\,d\theta$ and:

$$\frac{1}{\sqrt{2+\left(\sqrt{2}\tan(\theta)\right)^2}} = \frac{1}{\sqrt{2}\sec(\theta)} = \frac{1}{\sqrt{2}}\cos(\theta)$$

15. (a) $\theta = \arcsin\left(\frac{x}{3}\right)$
    (b) $f(\theta) = \cos(\theta)\tan(\theta)$ becomes:

$$\cos\left(\arcsin\left(\frac{x}{3}\right)\right)\tan\left(\arcsin\left(\frac{x}{3}\right)\right)$$

    (c) $\dfrac{\sqrt{9-x^2}}{3} \cdot \dfrac{x}{\sqrt{9-x^2}} = \dfrac{x}{3}$

17. (a) $\theta = \operatorname{arcsec}\left(\frac{x}{3}\right)$
    (b) $f(\theta) = \sqrt{1+\sin^2(\theta)}$ becomes:

$$\sqrt{1+\sin^2\left(\operatorname{arcsec}\left(\frac{x}{3}\right)\right)} = \sqrt{1+\left[\frac{\sqrt{x^2-9}}{x}\right]^2}$$

    (c) $\sqrt{1+\dfrac{x^2-9}{x^2}} = \sqrt{2-\dfrac{9}{x^2}}$

19. (a) $\theta = \arctan\left(\frac{x}{5}\right)$
    (b) $f(\theta) = \dfrac{\cos^2(\theta)}{1+\cot(\theta)}$ becomes:

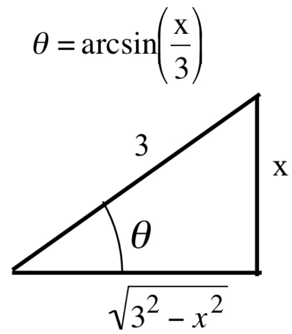
$$\frac{\left[\arctan\left(\frac{x}{5}\right)\cos\left(\right)\right]^2}{1+\cot\left(\arctan\left(\frac{x}{5}\right)\right)} = \frac{\frac{25}{25+x^2}}{1+\frac{5}{x}}$$

    (c) $\dfrac{25x}{(25+x^2)(x+5)}$

21. Using $x = 3\sin(\theta) \Rightarrow dx = 3\cos(\theta)\,d\theta$:

$$\int \frac{1}{x\sqrt{9-x^2}}\,dx = \int \frac{3\cos(\theta)}{3\sin(\theta)\cdot 3\cos(\theta)}\,d\theta$$
$$= \frac{1}{3}\int \csc(\theta)\,d\theta = -\frac{1}{3}\ln\left(|\csc(\theta)+\cot(\theta)|\right) + C$$
$$= -\frac{1}{3}\ln\left(\left|\frac{3}{x}+\frac{\sqrt{9-x^2}}{x}\right|\right) + C$$

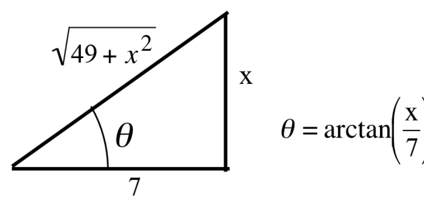
Remember to draw a triangle!



23. Using $x = 7\tan(\theta) \Rightarrow dx = 7\sec^2(\theta)\,d\theta$:

$$\int \frac{1}{\sqrt{49+x^2}}\,dx = \int \frac{7\sec^2(\theta)}{7\sec(\theta)}\,d\theta = \int \sec(\theta)\,d\theta$$
$$= \ln\left(|\sec(\theta)+\tan(\theta)|\right) + C$$
$$= \ln\left(\frac{\sqrt{49+x^2}}{7}+\frac{x}{7}\right) + C$$
$$= \ln\left(x+\sqrt{49+x^2}\right) + K$$

Remember to draw a triangle!



25. Using $x = 6\sin(\theta) \Rightarrow dx = 6\cos(\theta)\,d\theta$:

$$\int \sqrt{36-x^2}\,dx = \int \sqrt{36-36\sin^2(\theta)}\cdot 6\cos(\theta)\,d\theta$$
$$= 36\int \cos^2(\theta)\,d\theta = 18\int \left[1+\cos(2\theta)\right]\,d\theta$$
$$= 18\theta + 9\sin(2\theta) + C = 18\theta + 18\sin(\theta)\cos(\theta) + C$$
$$= 18\arcsin\left(\frac{x}{6}\right) + 18\cdot\frac{x}{6}\cdot\frac{\sqrt{36-x^2}}{6}$$
$$= 18\arcsin\left(\frac{x}{6}\right) + \frac{x}{2}\sqrt{36-x^2} + C$$

Remember to draw a triangle!

27. This is very similar to Problem 23:

$$\int \frac{1}{36+x^2}\,dx = \ln\left(x+\sqrt{36+x^2}\right)+K$$

29. Using $x = 7\sin(\theta) \Rightarrow dx = 7\cos(\theta)\,d\theta$:

$$\int \frac{x^2}{\sqrt{49-x^2}}\,dx = \int \frac{49\sin^2(\theta)}{\sqrt{49-49\sin^2(\theta)}}\cdot 7\cos(\theta)\,d\theta$$
$$= 49\int \sin^2(\theta)\,d\theta = \frac{49}{2}\int [1-\cos(2\theta)]\,d\theta$$
$$= \frac{49}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right]+C$$
$$= 49\,[\theta+\sin(\theta)\cos(\theta)]+C$$
$$= \frac{49}{2}\arcsin\left(\frac{x}{7}\right) - \frac{49}{2}\cdot\frac{x}{7}\cdot\frac{\sqrt{49-x^2}}{7}$$
$$= \frac{49}{2}\arcsin\left(\frac{x}{7}\right) + \frac{x}{2}\sqrt{49-x^2}+C$$

Remember to draw a triangle!

31. This does not require a trig substitution! Use $u = 25-x^2 \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}\,du = dx$:

$$\int \frac{x}{\sqrt{25-x^2}}\,dx = -\sqrt{25-x^2}+C$$

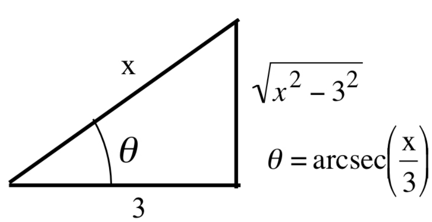
33. This does not require a trig substitution! Use $u = x^2+49 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\,du = dx$ to get:

$$\int \frac{x}{x^2+49}\,dx = \frac{1}{2}\ln\left(x^2+49\right)+C$$

35. Using $x = 3\sec(\theta) \Rightarrow dx = 3\sec(\theta)\tan(\theta)\,d\theta$:

$$\int \frac{1}{(x^2-9)^{\frac{3}{2}}}\,dx = \int \frac{3\sec(\theta)\tan(\theta)}{\left[\sqrt{9\sec^2(\theta)-9}\right]^3}\,d\theta$$
$$= \int \frac{3\sec(\theta)\tan(\theta)}{27\tan^3(\theta)}\,d\theta = \frac{1}{9}\int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta$$
$$= \frac{1}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta = -\frac{1}{9}\cdot\frac{1}{\sin(\theta)}+C$$
$$= -\frac{1}{9}\cdot\frac{x}{\sqrt{x^2-9}}+C = \frac{-x}{9\sqrt{x^2-9}}+C$$

Remember to draw a triangle!



37. This does not require a trig substitution! If $x > 5$:

$$\int \frac{5}{2x\sqrt{x^2-25}}\,dx = \frac{1}{2}\operatorname{arcsec}\left(\frac{x}{5}\right)+C$$

39. This does not require a trig substitution! Decompose the integrand using partial fractions:

$$\frac{1}{(5-x)(5+x)} = \frac{\frac{1}{10}}{5-x} + \frac{\frac{1}{10}}{5+x}$$

and then integrate:

$$\int \frac{1}{25-x^2}\,dx = \frac{1}{10}\left[-\ln(|5-x|)+\ln(|5+x|)+C\right.$$
$$= \frac{1}{10}\ln\left(\left|\frac{5+x}{5-x}\right|\right)+C$$

41. This resembles Problem 23 with a replacing 7:

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \ln\left(x+\sqrt{a^2+x^2}\right)+C$$

43. Using $x = a\tan(\theta) \Rightarrow dx = a\sec^2(\theta)\,d\theta$:

$$\int \frac{1}{x^2\sqrt{a^2+x^2}}\,dx = \int \frac{a\sec^2(\theta)}{a^2\tan^2(\theta)\cdot a\sec(\theta)}\,d\theta$$
$$= \frac{1}{a^2}\int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta$$
$$= \frac{1}{a^2}\int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta = \frac{1}{a}\cdot\frac{-1}{\sin(\theta)}+C$$
$$= -\frac{1}{a^2}\cdot\frac{\sqrt{a^2+x^2}}{x}+C = \frac{-\sqrt{a^2+x^2}}{a^2x}+C$$

45. Set $u = x+1 \Rightarrow du = dx$ so the integral becomes:

$$\int \frac{1}{\sqrt{u^2+3^2}}\,du = \ln\left(\left|u+\sqrt{u^2+9}\right|\right)+C$$

(using the same pattern as Problem 23) and then replace u with $x+1$ to get:

$$\ln\left(\left|x+1+\sqrt{(x+1)^2+9}\right|\right)+C$$

47. Complete the square in the denominator:

$$x^2+10x+29 = x^2+10x+25+4 = (x+5)^2+2^2$$

to get an integrand that does not require trig substitution:

$$\int \frac{1}{(x+1)^2+2^2}\,dx = \frac{1}{2}\arctan\left(\frac{x+5}{2}\right)+C$$

49. Complete the square in the denominator: $x^2 + 4x + 3 = x^2 + 4x + 4 - 1 = (x+2)^2 - 1^2$. Then substitute $x + 2 = \sec(\theta) \Rightarrow dx = \sec(\theta)\tan(\theta)\,d\theta$ (or substitute $u = x + 2$ and then do the trig substitution) to get:

$$\int \frac{1}{\sqrt{(x+2)^2 - 1}}\,dx = \int \frac{\sec(\theta)\tan(\theta)}{\tan(\theta)}\,d\theta = \int \sec(\theta)\,d\theta = \ln\left(|\sec(\theta) + \tan(\theta)|\right) + C$$
$$= \ln\left(\left|x + 2 + \sqrt{x^2 + 4x + 3}\right|\right) + C$$

51. (a) Using $x = \tan(\theta) \Rightarrow dx = \sec^2(\theta)\,d\theta$:

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{\sec^2(\theta)}{\left[\sec^2(\theta)\right]^2}\,d\theta = \int \cos^2(\theta)\,d\theta = \int \left[\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right]\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$
$$= \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C = \frac{1}{2}\arctan(x) + \frac{1}{2}\cdot\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}} + C$$
$$= \frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)} + C$$

(b) First write the denominator as $1 = 1 + x^2 - x^2$ so that:

$$\int \frac{(1+x^2) - x^2}{(1+x^2)^2}\,dx = \int \frac{1}{1+x^2}\,dx - \int \frac{x^2}{1+x^2}\,dx = \arctan(x) - \int x \cdot \frac{x}{(1+x^2)^2}\,dx$$

For the second integral, use $u = x \Rightarrow du = dx$ and $dv = \dfrac{x}{(1+x^2)^2}\,dx \Rightarrow v = -\dfrac{1}{2}\cdot\dfrac{1}{1+x^2}$ so that:

$$\int x \cdot \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}\cdot\frac{x}{1+x^2} + \frac{1}{2}\int \frac{1}{1+x^2}\,dx = \frac{-x}{2(1+x^2)} + \frac{1}{2}\arctan(x) + K$$

Putting this all together yields the same result as part (a).

(c) Trig substitution requires less cleverness.

53. Substitute $u = x^2 + 25 \Rightarrow du = 2x\,dx$ so that the integral does not require trig substitution:

$$\int \frac{4\cdot 2x}{(x^2+25)^2}\,dx = 4\int \frac{1}{u^2}\,du = -\frac{4}{u} + C = -\frac{4}{x^2+25} + C$$

Section 8.5

1. $\displaystyle\int \sin^2(3x)\,dx = \frac{x}{2} - \frac{\sin(6x)}{12} + C = \frac{x}{2} - \frac{\sin(3x)\cos(3x)}{6} + C$

3. With $u = e^x \Rightarrow du = e^x\,dx$ the integral becomes $\sin(u)\cos(u)\,du = \dfrac{1}{2}\sin^2(u) + C = \dfrac{1}{2}\sin^2\left(e^x\right) + C$. With $w = \cos\left(e^x\right) \Rightarrow dw = -e^x\sin\left(e^x\right)\,dx$, the integral becomes $-\displaystyle\int w\,dw = -\frac{1}{2}w^2 + K = -\frac{1}{2}\cos^2\left(e^x\right) + K$.

5. Substituting $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$ and using the result of Example 1 yields:

$$\frac{1}{3}\left[\frac{3}{8}(3x) - \frac{1}{4}\sin(6x) + \frac{1}{32}\sin(12x)\right]_0^{\pi} = \frac{3\pi}{8}$$

7. First split off one factor of $\cos(5x)$ to write the integrand as $\cos^2(5x)\cdot\cdot(5x) = \left[1 - \sin^2(5x)\right]\cos(5x)$ and then substitute $u = \sin(5x) \Rightarrow du = 5\cos(5x)\,dx \Rightarrow \frac{1}{5}\,du = \cos(5x)\,dx$ and note that $x = 0 \Rightarrow u = 0$ and $x = \pi \Rightarrow u = 0$ so the integral becomes:

$$\int_{x=0}^{x=\pi} \cos^2(5x)\cdot\cos(5x)\,dx = \int_{x=0}^{x=\pi}\left[1 - \sin^2(5x)\right]\cos(5x)\,dx = \frac{1}{5}\int_{u=0}^{u=0}\left[1 - u^2\right]\,du = 0$$

9. Substitute $u = \sin(7x) \Rightarrow du = 7\cos(7x)\,dx$:

$$\int \sin(7x)\cos(7x)\,dx = \frac{1}{7}\int u\,du = \frac{1}{14}u^2 + C = \frac{1}{14}\sin^2(7x) + C$$

Substituting $w = \cos(7x)$ yields an equivalent (but different-looking) result.

11. Substitute $u = \cos(7x) \Rightarrow du = -7\sin(7x)\,dx$:

$$\int \sin(7x)\cos^3(7x)\,dx = -\frac{1}{7}\int u^3\,du = -\frac{1}{28}u^4 + C = -\frac{1}{28}\cos^4(7x) + C$$

13. One option involves writing $\sin^2(3x)\cos^2(3x)$ as:

$$\sin^2(3x)\left[1 - \sin^2(3x)\right] = \sin^2(3x) - \sin^4(3x)$$

and then using the formula for $\int \sin^2(au)\,du$ and the result of Example 1. Or you can write:

$$\sin^2(3x)\cos^2(3x) = [\sin(3x)\cos(3x)]^2 = \left[\frac{1}{2}\sin(2\cdot 3x)\right]^2 = \frac{1}{4}\sin^2(6x)$$

and then use the formula for $\int \sin^2(au)\,du$:

$$\frac{1}{4}\int \sin^2(6x)\,dx = \frac{1}{4}\left[\frac{x}{2} - \frac{\sin(2\cdot 6x)}{4\cdot 6}\right] + C = \frac{x}{8} - \frac{1}{96}\sin(12x) + C$$

15. Split off one factor of $\sin(x)$, writing:

$$\sin^5(x)\cos^2(x) = \sin(x)\left[\sin^2(x)\right]^2\cos^2(x) = \sin(x)\left[1 - \cos^2(x)\right]^2\cos^2(x)$$

then substitute $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$:

$$\int \sin^5(x)\cos^2(x)\,dx = -\int \left[1 - u^2\right]^2 u^2\,du = -\int \left[1 - 2u^2 + u^4\right]u^2\,du = \int \left[-u^2 + 2u^4 - u^6\right]du$$
$$= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C = -\frac{1}{3}\cos^3(x) + \frac{2}{5}\cos^5(x) - \frac{1}{7}\cos^7(x) + C$$

17. First split $\sec^4(4x) = \sec^2(4x)\cdot\sec^2(4x)$ and write this as $\left[1 + \tan^2(4x)\right]\sec^2(4x)$, then substitute $u = \tan(4x) \Rightarrow du = 4\sec^2(4x)]\,dx$ so that:

$$\int \left[1 + \tan^2(4x)\right]\sec^2(4x)\,dx = \frac{1}{4}\int \left[1 + u^2\right]du = \frac{1}{4}u + \frac{1}{12}u^3 + C = \frac{1}{4}\tan(4x) + \frac{1}{12}\tan^3(4x) + C$$

19. Substitute $u = 4x \Rightarrow du = 4\,dx \Rightarrow \frac{1}{4}\,du = dx$ and apply the reduction formula repeatedly:

$$\int \tan^5(4x)\,dx = \frac{1}{4}\int \tan^5(u)\,du = \frac{1}{4}\left[\frac{1}{4}\tan^4(u) - \int \tan^3(x)\,dx\right]$$
$$= \frac{1}{4}\left[\frac{1}{4}\tan^4(u) - \left(\frac{1}{2}\tan^2(x) - \int \tan(x)\,dx\right)\right]$$
$$= \frac{1}{16}\tan^4(4x) - \frac{1}{8}\tan^2(4x) + \frac{1}{4}\ln\left(|\sec(4x)|\right) + C$$

21. Substitute $u = \tan(5x) \Rightarrow du = 5\sec^2(5x)\,dx \Rightarrow \frac{1}{5}\,du = \sec^2(5x)\,dx$:

$$\frac{1}{5}\int u\,du = \frac{1}{10}u^2 + C = \frac{1}{10}\tan^2(5x) + C$$

23. Split $\sec^3(5x)\tan(5x) = \sec^2(5x)\cdot\sec(5x)\tan(5x)$ so $u = \sec(5x) \Rightarrow du = 5\sec(5x)\tan(5x)\,dx$ yields the integral:

$$\frac{1}{5}\int u^2\,du = \frac{1}{15}u^3 + C = \frac{1}{15}\sec^3(5x) + C$$

25. This is quite similar to Problem 23:

$$\int \sec^4(\theta)\tan(\theta)\,d\theta = \frac{1}{4}\sec^4(\theta) + C$$

27. Write the integrand as $\sec^2(\theta)\left[1 + \tan^2(\theta)\right]\tan^4(\theta)$ and use $u = \tan(\theta) \Rightarrow du = \sec^2(\theta)\,d\theta$ so that:

$$\int \left[1 + u^2\right]u^4\,du = \frac{1}{5}u^5 + \frac{1}{7}u^7 + C = \frac{1}{5}\tan^5(\theta) + \frac{1}{7}\tan^7(\theta) + C$$

29. The integrand is just $\tan^2(\theta) = \sec^2(\theta) - 1$ so the integral evaluates to $\tan(\theta) - \theta + C$.

31. The integrand simplifies to $\sin^4(\theta)$, so use the result of Example 1.

33. Use a product-to-sum identity to write:

$$\sin(x)\cos(3x) = \frac{1}{2}\left[\sin(4x) + \sin(2x)\right]$$

and integrate to get: $-\frac{1}{8}\cos(8x) - \frac{1}{4}\cos(4x) + C$

35. Use a product-to-sum identity to write:

$$\sin(x)\sin(3x) = \frac{1}{2}\left[\cos(2x) - \cos(4x)\right]$$

and integrate to get: $\frac{1}{4}\sin(2x) - \frac{1}{8}\sin(4x) + C$

37. If n is odd, we can write $n = 2k+1$ where k is some other integer. Then write:

$$\begin{aligned}\sin^{2k+1}(x) &= \sin^{2k}(x)\cdot\sin(x)\\ &= \left[\sin^2(x)\right]^k \sin(x)\\ &= \left[1-\cos^2(x)\right]^k \sin(x)\end{aligned}$$

and use $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$. This substitution changes the limits of integration from $x = 0$ to $u = \cos(0) = 1$ and from $x = 2\pi$ to $u = \cos(2\pi) = 1$. The integral thus becomes:

$$\int_1^1 \left[1-u^2\right]\,du = 0$$

39. Use a product-to-sum formula to rewrite the integrand $\sin(mx)\cdot\sin(nx)$ as:

$$\frac{1}{2}\left[\sin\left((m+n)x\right) + \sin\left((m-n)x\right)\right]$$

If $k = m+n$ or $k = m-n$ (neither of which is 0:

$$\int_0^{2\pi} \sin(kx)\,dx = \left[-\frac{1}{k}\cos(kx)\right]_0^{2\pi} = 0$$

so the original integral must equal 0 as well.

41. The integrand is just $\sin^2(mx)$, so:

$$\int_0^{2\pi} \sin^2(mx)\,dx = \left[\frac{x}{2} - \frac{\sin(2mx)}{4m}\right]_0^{2\pi} = \pi$$

43. Integrating the product of $\sin(2x)$ and any term of $P(x)$ other than $-4\sin(2x)$ yields 0, so a_2 is:

$$\frac{1}{\pi}\int_0^{2\pi} \sin(2x)\cdot\left[-4\sin(2x)\right]\,dx = \frac{-4}{\pi}\cdot\pi = -4$$

45. $a_4 = 0$ because $P(x)$ has no $\sin(4x)$ term.

47. On your own.

Section 8.6

In place of full solutions, the "answers" for this section suggest an integration method, or a first step.

1. Substitute $u = 1 - x$.
3. Substitute $u = a^2 - x^2$.
5. Substitute $u = a + bx$.
7. Substitute $u = x^2$, or factor the denomiator:

$$1 - x^4 = (1-x^2)(1+x^2) = (1-x)(1+x)(1+x^2)$$

and then use partial fractions.

9. Divide first.
11. Substitute $u = a^{\frac{2}{3}} - x^{\frac{2}{3}}$.
13. Expand by squaring.
15. Substitute $u = 1 + x^3$.
17. Substitute $u = e^x + x^{-x}$. (Or note that the integrand is equivalent to $\tanh(x)$.)
19. Start with long division.
21. Start with long division.
23. Substitute $u = a + b\cos(2\theta)$.
25. Substitute $u = \cos(\theta)$.
27. Complete the square.
29. Start with long division.
31. Substitute $u = 1 - e^{2t}$.
33. Substitute $x = u^2$.
35. Substitute $u = 1 + x^2$.
37. Start with integration by parts.
39. Substitute $u = a^2 - x^2$.
41. Substitute $u = \sin(\theta)$.
43. Use integration by parts with $u = x$.
45. Substitute $x = u^2$.
47. Substitute $u = 1 + 2x$.
49. Substitute $y = e^x$.
51. Use integration by parts with $u = x^2$.

53. Substitute $u = 1 - x$.
55. Start with long division.
57. Substitute $u = 1 + \cos\left(\frac{\theta}{2}\right)$.
59. Substitute $u = 1 + e^{-x}$.
61. Substitute $u = 1 - x$.
63. Substitute $u = \sin(\theta)$.
65. Substitute $u = x^3$.
67. Substitute $u = \frac{x}{2}$.
69. Start with long division.
71. Substitute $u = x^2$.
73. Use integration by parts with $u = \theta$.
75. Substitute $x = t^2$.
77. Start with long division.
79. Expand by squaring.
81. Complete the square.
83. Substitute $u = 1 - 3e^x$.
85. Use integration by parts with $u = [\ln(x)]^2$.
87. Substitute $t = -x^3$.
89. Factor the denominator:

$$x^2(x+1) - 4(x+1) = (x^2-4)(x+1)$$
$$= (x-2)(x+2)(x+1)$$

and use partial fractions.
91. Start with long division.
93. Substitute $x = e^t$.
95. Use partial fractions.
97. Use partial fractions.
99. Use partial fractions (graph the polynomial in the denominator to assist with factoring).
101. Start with long division.
103. Use partial fractions.
105. Complete the square.
107. Substitute $u = 1 + x^2$.
109. Write $\cot(\theta) = \dfrac{1}{\tan(\theta)}$ and simplify.
111. Expand the denominator and use long division.
113. Substitute $x = e^t - 1$.
115. Substitute $u = x^2$.
117. Complete the square.
119. Complete the square.
121. Use a trigonometric identity.
123. Use a trigonometric identity.
125. Integration by parts.
127. Integration by parts (twice).
129. Write $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)}$.
131. Integration by parts.
133. Integration by parts.

Section 8.7

1. $P(x) = 5 + 3x$
3. $P(x) = 4 - x$
5. $P(x) = 4$
7. $P(0) = A$, $P'(0) = B$
9. $P(x) = -2 + 7x + 3x^2$
11. $P(x) = 8 + 5x + 5x^2$
13. $P(x) = -3 - 2x + 2x^2$
15. $P(x) = 5 + 3x + 2x^2 + x^3$
17. $P(x) = 4 - x - x^2 - 2x^3$
19. $P(x) = 4 - 2x^2 + 6x^3$
21. $P(0) = A$, $P'(0) = B$, $P''(0) = 2C$, $P'''(0) = 6D$
23. To five decimal places:

x	$f(x))$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0.0	0.00000	0.00000	0.00000
0.1	0.09983	0.09983	0.00000
0.2	0.19867	0.19867	0.00000
0.3	0.29552	0.29552	0.00000
1.0	0.84147	0.84167	0.00020
2.0	0.90930	0.93333	0.02404

25. To five decimal places:

x	$f(x))$	$P(x)$	$\lvert f(x) - P(x)\rvert$
0.0	1.00000	1.00000	0.00000
0.1	0.99500	0.99500	0.00000
0.2	0.98007	0.98007	0.00000
0.3	0.95534	0.95534	0.00000
1.0	0.54030	0.54167	0.00136
2.0	-0.41615	-0.33333	0.08281

27. To five decimal places:

x	$f(x))$	$P(x)$	$\|f(x) - P(x)\|$
0.0	1.00000	1.00000	0.00000
0.1	1.10517	1.10517	0.00000
0.2	1.22140	1.22133	0.00007
0.3	1.34986	1.34950	0.00036
1.0	2.71828	2.66667	0.05162
2.0	7.38906	6.33333	1.05572

29. Starting with:

$$e^u \approx 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3$$

and substituting $u = 2x$ yields:

$$e^{2x} \approx 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

31. $f(x) = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f'''(x) = 6(1-x)^{-4}$ so $f(0) = 1$, $f'(0) = 1$, $f''(0) = 2$ and $f'''(0) = 6$:

$$P(x) = 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 = 1 + x + x^2 + x^3$$

33. $f(x) = \ln(1+x) \Rightarrow f'(x) = (1+x)^{-1} \Rightarrow f''(x) = -(1+x)^{-2} \Rightarrow f'''(x) = 2(1+x)^{-3} \Rightarrow f^{(4)}(x) = -6(1+x)^{-4}$ so $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$ and $f^{(4)}(0) = -6$:

$$\begin{aligned} P(x) &= 0 + x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \end{aligned}$$

35. Starting with:

$$\cos(u) \approx 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 - \frac{1}{6!}u^6$$

and substituting $u = x^2$ yields:

$$\cos(x^2) \approx 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 - \frac{1}{720}x^{12}$$

37. Starting with:

$$\sin(u) \approx u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7$$

and substituting $u = x^2$ yields:

$$\sin(x^2) \approx x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10} - \frac{1}{5040}x^{14}$$

so multiplying both sides by x^3 results in:

$$x^3 \cdot \sin(x^2) \approx x^5 - \frac{1}{6}x^9 + \frac{1}{120}x^{13} - \frac{1}{5040}x^{17}$$

39. Starting with:

$$\sin(u) \approx u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7$$

and substituting $u = x^3$ yields:

$$\sin(x^3) \approx x^3 - \frac{1}{6}x^9 + \frac{1}{120}x^{15} - \frac{1}{5040}x^{21}$$

so integrating both sides from 0 to 1 results in:

$$\begin{aligned} \int_0^1 \sin(x^3)\,dx &\approx \left[\frac{1}{4}x^4 - \frac{1}{60}x^{10} + \frac{1}{1920}x^{16} - \frac{1}{110880}x^{22}\right]_0^1 \\ &= \frac{1}{4} - \frac{1}{60} + \frac{1}{1920} - \frac{1}{110880} \approx 0.2338 \end{aligned}$$

41. Starting with:

$$e^u \approx 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3$$

and substituting $u = -x^3$ yields:

$$e^{-x^3} \approx 1 - x^3 + \frac{1}{2}x^6 - \frac{1}{6}x^9$$

so integrating both sides from 0 to $\frac{1}{2}$ results in:

$$\begin{aligned} \int_0^1 e^{-x^3}\,dx &\approx \left[x - \frac{1}{4}x^4 + \frac{1}{14}x^7 - \frac{1}{60}x^{10}\right]_0^{\frac{1}{2}} \\ &= \frac{1}{2} - \frac{1}{64} + \frac{1}{1792} - \frac{1}{61440} \approx 0.4849 \end{aligned}$$