# 1
# Limits and Continuity

## 1.0 Tangent Lines, Velocities, Growth

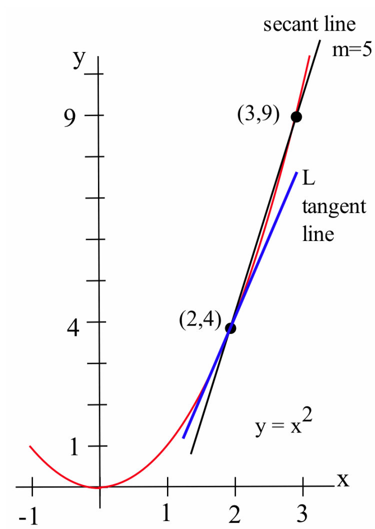
In Section 0.2, we estimated the slope of a line tangent to the graph of a function *at a point*. At the end of Section 0.3, we constructed a new function that gave the slope of the line tangent to the graph of a given function *at each point*. In both cases, before we could calculate a slope, we had to **estimate** the tangent line from the graph of the given function, a method that required an accurate graph and good estimating. In this section we will begin to look at a more precise method of finding the slope of a tangent line that does not require a graph or any estimation by us. We will start with a non-applied problem and then look at two applications of the same idea.

### The Slope of a Line Tangent to a Function at a Point

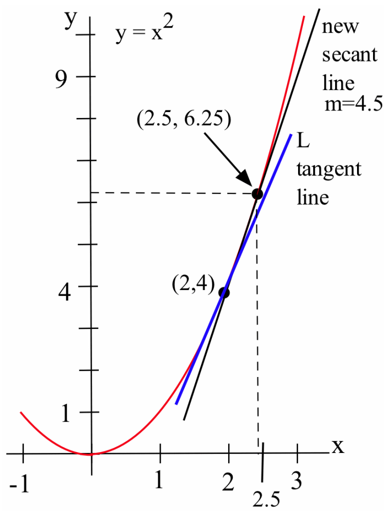
Our goal is to find a way of exactly determining the slope of the line that is tangent to a function (to the graph of the function) at a point in a way that does not require us to actually have the graph of the function.

Let's start with the problem of finding the slope of the line $L$ (see margin figure), which is tangent to $f(x) = x^2$ at the point $(2,4)$. We could estimate the slope of $L$ from the graph, but we won't. Instead, we can see that the line through $(2,4)$ and $(3,9)$ on the graph of $f$ is an approximation of the slope of the tangent line, and we can calculate that slope exactly:

$$m = \frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5$$




But $m = 5$ is only an *estimate* of the slope of the tangent line—and not a very good estimate. It's too big. We can get a better estimate by picking a second point on the graph of $f$ closer to $(2,4)$—the point $(2,4)$ is fixed and it must be one of the two points we use. From the figure in the margin, we can see that the slope of the line through the points $(2,4)$ and $(2.5, 6.25)$ is a better approximation of the slope of the




tangent line at $(2,4)$:

$$m = \frac{\Delta y}{\Delta x} = \frac{6.25-4}{2.5-2} = \frac{2.25}{0.5} = 4.5$$

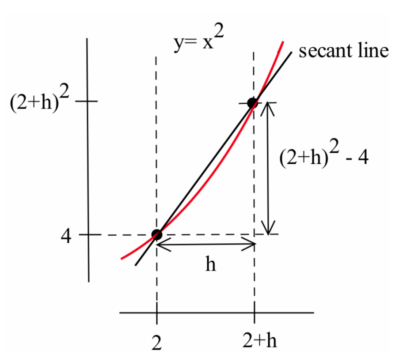
This is a better estimate, but still an approximation.

We can continue picking points closer and closer to $(2,4)$ on the graph of $f$, and then calculating the slopes of the lines through each of these points $(x,y)$ and the point $(2,4)$:

| points to the left of $(2,4)$ | | | points to the right of $(2,4)$ | | |
|---|---|---|---|---|---|
| $x$ | $y=x^2$ | slope | $x$ | $y=x^2$ | slope |
| 1.5 | 2.25 | 3.5 | 3 | 9 | 5 |
| 1.9 | 3.61 | 3.9 | 2.5 | 6.25 | 4.5 |
| 1.99 | 3.9601 | 3.99 | 2.01 | 4.0401 | 4.01 |

The only thing special about the $x$-values we picked is that they are numbers close—and very close—to $x=2$. Someone else might have picked other nearby values for $x$. As the points we pick get closer and closer to the point $(2,4)$ on the graph of $y=x^2$, the slopes of the lines through the points and $(2,4)$ are better approximations of the slope of the tangent line, and these slopes are getting closer and closer to 4.

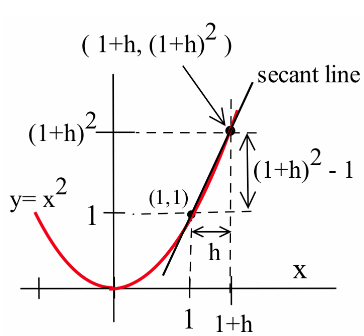
**Practice 1.** What is the slope of the line through $(2,4)$ and $(x,y)$ for $y=x^2$ and $x=1.994$? For $x=2.0003$?




We can bypass much of the calculating by not picking the points one at a time: let's look at a general point near $(2,4)$. Define $x=2+h$ so $h$ is the increment from 2 to $x$ (see margin figure). If $h$ is small, then $x=2+h$ is close to 2 and the point $(2+h, f(2+h)) = \left(2+h,(2+h)^2\right)$ is close to $(2,4)$. The slope $m$ of the line through the points $(2,4)$ and $\left(2+h,(2+h)^2\right)$ is a good approximation of the slope of the tangent line at the point $(2,4)$:

$$m = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2-4}{(2+h)-2} = \frac{(4+4h+h^2)-4}{h}$$
$$= \frac{4h+h^2}{h} = \frac{h(4+h)}{h} = 4+h$$

If $h$ is very small, then $m=4+h$ is a very good approximation to the slope of the tangent line, and $m=4+h$ also happens to be very close to the value 4. The value $m=4+h$ is called the slope of the **secant line** through the two points $(2,4)$ and $\left(2+h,(2+h)^2\right)$. The limiting value 4 of $m=4+h$ as $h$ gets smaller and smaller is called the slope of the **tangent line** to the graph of $f$ at $(2,4)$.




**Example 1.** Find the slope of the line tangent to $f(x)=x^2$ at the point $(1,1)$ by evaluating the slope of the secant line through $(1,1)$ and $(1+h, f(1+h))$ and then determining what happens as $h$ gets very small (see margin figure).

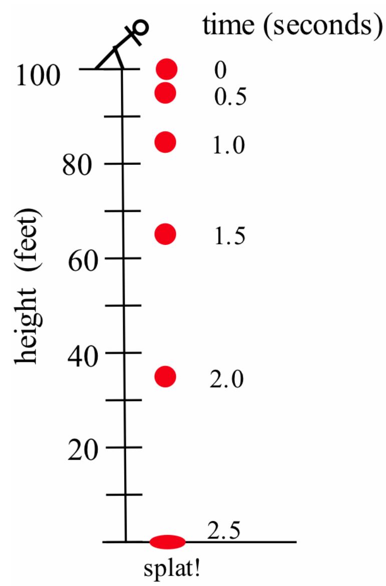
**Solution.** The slope of the secant line through the points $(1,1)$ and $(1+h, f(1+h))$ is:

$$m = \frac{f(1+h)-1}{(1+h)-1} = \frac{(1+h)^2-1}{h} = \frac{(1+2h+h^2)-1}{h}$$
$$= \frac{2h+h^2}{h} = \frac{h(2+h)}{h} = 2+h$$

As $h$ gets very small, the value of $m$ approaches the value 2, the slope of tangent line at the point $(1,1)$. ◀

**Practice 2.** Find the slope of the line tangent to the graph of $y = f(x) = x^2$ at the point $(-1,1)$ by finding the slope of the secant line, $m_{\text{sec}}$, through the points $(-1,1)$ and $(-1+h, f(-1+h))$ and then determining what happens to $m_{\text{sec}}$ as $h$ gets very small.

*Falling Tomato*




Suppose we drop a tomato from the top of a 100-foot building (see margin figure) and record its position at various times during its fall:

| time (sec) | height (ft) |
|---|---|
| 0.0 | 100 |
| 0.5 | 96 |
| 1.0 | 84 |
| 1.5 | 64 |
| 2.0 | 36 |
| 2.5 | 0 |

Some questions are easy to answer directly from the table:

(a) How long did it take for the tomato to drop 100 feet? (2.5 seconds)

(b) How far did the tomato fall during the first second? $(100 - 84 = 16 \text{ feet})$

(c) How far did the tomato fall during the last second? $(64 - 0 = 64 \text{ feet})$

(d) How far did the tomato fall between $t = 0.5$ and $t = 1$? $(96 - 84 = 12 \text{ feet})$

Other questions require a little calculation:

(e) What was the average velocity of the tomato during its fall?

$$\text{average velocity} = \frac{\text{distance fallen}}{\text{total time}} = \frac{\Delta\text{position}}{\Delta\text{time}} = \frac{-100 \text{ ft}}{2.5 \text{ s}} = -40\,\frac{\text{ft}}{\text{sec}}$$

(f) What was the average velocity between $t = 1$ and $t = 2$ seconds?

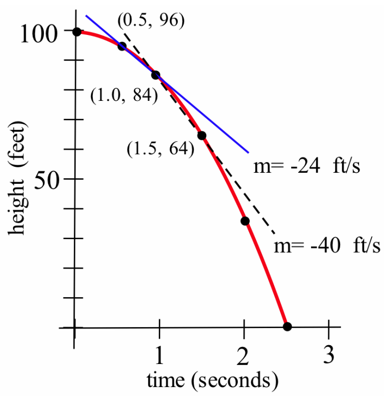
$$\text{average velocity} = \frac{\Delta\text{position}}{\Delta\text{time}} = \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} = \frac{-48 \text{ ft}}{1 \text{ s}} = -48\,\frac{\text{ft}}{\text{sec}}$$

Some questions are more difficult.

(g) How fast was the tomato falling 1 second after it was dropped?

This question is significantly different from the previous two questions about average velocity. Here we want the **instantaneous velocity**, the velocity at an instant in time. Unfortunately, the tomato is not equipped with a speedometer, so we will have to give an approximate answer.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during the entire fall, $-40\frac{\text{ft}}{\text{sec}}$. But the tomato fell slowly at the beginning and rapidly near the end, so this estimate may or may not be a good answer.




We can get a better approximation of the instantaneous velocity at $t = 1$ by calculating the average velocities over a short time interval near $t = 1$. The average velocity between $t = 0.5$ and $t = 1$ is:

$$\frac{-12 \text{ feet}}{0.5 \text{ sec}} = -24 \frac{\text{ft}}{\text{sec}}$$

and the average velocity between $t = 1$ and $t = 1.5$ is

$$\frac{-20 \text{ feet}}{0.5 \text{ sec}} = -40 \frac{\text{ft}}{\text{sec}}$$

so we can be reasonably sure that the instantaneous velocity is between $-24 \frac{\text{ft}}{\text{sec}}$ and $-40 \frac{\text{ft}}{\text{sec}}$.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity. The average velocity over a time interval is:

$$\frac{\Delta\text{position}}{\Delta\text{time}}$$

which is the slope of the secant line through two points on the graph of height versus time (see margin figure).

$$\boxed{\begin{aligned}\text{average velocity} &= \frac{\Delta\text{position}}{\Delta\text{time}} \\ &= \text{slope of the secant line through two points}\end{aligned}}$$

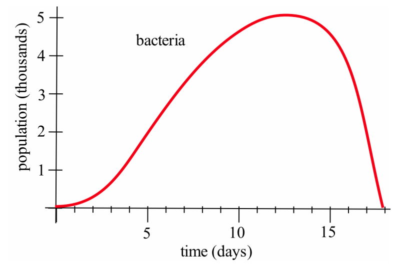
The instantaneous velocity at a particular time and height is the slope of the tangent line to the graph at the point given by that time and height.

$$\boxed{\text{instantaneous velocity} = \text{slope of the line tangent to the graph}}$$

**Practice 3.** Estimate the instantaneous velocity of the tomato 2 seconds after it was dropped.

*Growing Bacteria*

Suppose we set up a machine to count the number of bacteria growing on a Petri plate (see margin figure). At first there are few bacteria, so the population grows slowly. Then there are more bacteria to divide, so the population grows more quickly. Later, there are more bacteria and less room and nutrients available for the expanding population, so the population grows slowly again. Finally, the bacteria have used up most of the nutrients and the population declines as bacteria die.




The population graph can be used to answer a number of questions:

(a) What is the bacteria population at time $t = 3$ days? (about 500 bacteria)

(b) What is the population increment from $t = 3$ to $t = 10$ days? (about $4,000$ bacteria)

(c) What is the **rate** of population growth from $t = 3$ to $t = 10$ days?

To answer this last question, we compute the average change in population during that time:

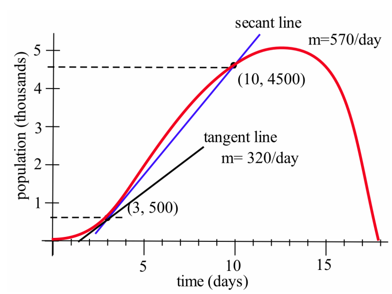
$$\begin{aligned}\text{average change in population} &= \frac{\text{change in population}}{\text{change in time}}\\ &= \frac{\Delta\text{population}}{\Delta\text{time}} = \frac{4000 \text{ bacteria}}{7 \text{ days}} \approx 570\,\frac{\text{bacteria}}{\text{day}}\end{aligned}$$

This is the slope of the secant line through $(3, 500)$ and $(10, 4500)$.

$$\begin{aligned}\text{average population growth rate} &= \frac{\Delta\text{population}}{\Delta\text{time}}\\ &= \text{slope of the secant line through two points}\end{aligned}$$

Now for a more difficult question:

(d) What is the rate of population growth on the third day, at $t = 3$?




This question asks for the instantaneous rate of population change, the slope of the line tangent to the population curve at $(3, 500)$. If we sketch a line approximately tangent to the curve at $(3, 500)$ and pick two points near the ends of the tangent line segment (see margin figure), we can estimate that the instantaneous rate of population growth is approximately $320\,\frac{\text{bacteria}}{\text{day}}$.

$$\begin{aligned}&\text{instantaneous population growth rate} =\\ &\text{slope of the line tangent to the graph}\end{aligned}$$

**Practice 4.** Find approximate values for:

(a) the average change in population between $t = 9$ and $t = 13$.

(b) the rate of population growth at $t = 9$ days.

The tangent line problem, the instantaneous velocity problem and the instantaneous growth rate problem are all similar. In each problem we wanted to know how rapidly something was **changing at an instant in time**, and each problem turned out to involve finding the **slope of a tangent line**. The approach in each problem was also the same: find an approximate solution and then examine what happens to the approximate solution over shorter and shorter intervals. We will often use this approach of finding a limiting value, but before we can use it effectively we need to describe the concept of a limit with more precision.

## *1.0 Problems*

1. (a) What is the slope of the line through $(3, 9)$ and $(x, y)$ for $y = x^2$ when:
   i. $x = 2.97$?
   ii. $x = 3.001$?
   iii. $x = 3 + h$?

   (b) What happens to this last slope when $h$ is very small (close to 0)?

   (c) Sketch the graph of $y = x^2$ for $x$ near 3.

2. (a) What is the slope of the line through $(-2, 4)$ and $(x, y)$ for $y = x^2$ when:
   i. $x = -1.98$?
   ii. $x = -2.03$?
   iii. $x = -2 + h$?

   (b) What happens to this last slope when $h$ is very small (close to 0)?

   (c) Sketch the graph of $y = x^2$ for $x$ near $-2$.

3. (a) What is the slope of the line through $(2, 4)$ and $(x, y)$ for $y = x^2 + x - 2$ when:
   i. $x = 1.99$?
   ii. $x = 2.004$?
   iii. $x = 2 + h$?

   (b) What happens to this last slope when $h$ is very small (close to 0)?

   (c) Sketch the graph of $y = x^2 + x - 2$ for $x$ near 2.

4. (a) What is the slope of the line through $(-1, -2)$ and $(x, y)$ for $y = x^2 + x - 2$ when:
   i. $x = -0.98$?
   ii. $x = -1.03$?
   iii. $x = -1 + h$?

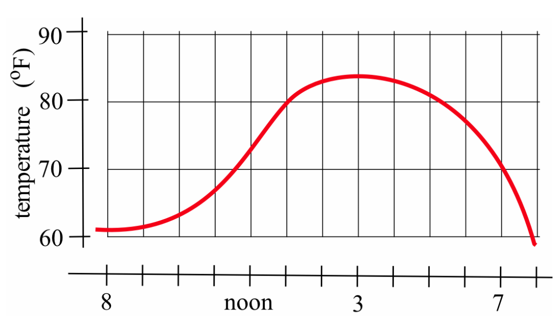
   (b) What happens to this last slope when $h$ is very small (close to 0)?

   (c) Sketch the graph of $y = x^2 + x - 2$ for $x$ near $-1$.

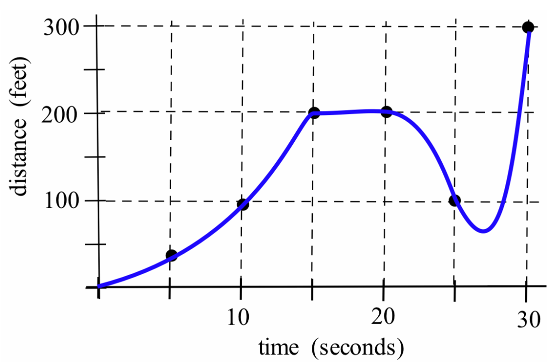
5. The figure below shows the temperature during a day in Ames.

   (a) What was the average change in temperature from 9 a.m. to 1 p.m.?

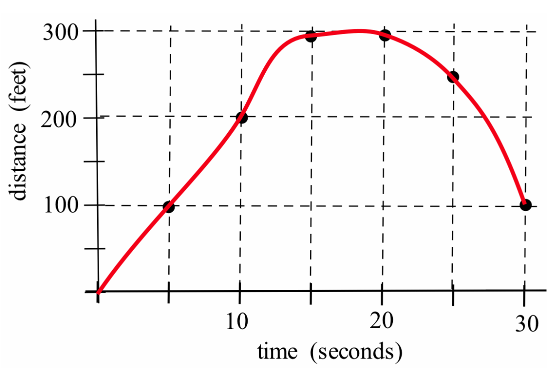
   (b) Estimate how fast the temperature was rising **at** 10 a.m. and **at** 7 p.m.



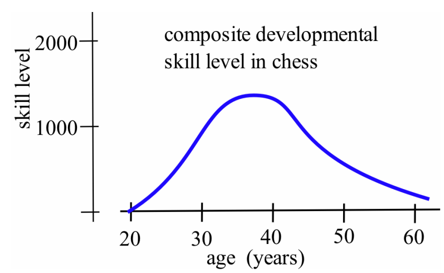
6. The figure below shows the distance of a car from a measuring position located on the edge of a straight road.
   (a) What was the average velocity of the car from $t = 0$ to $t = 30$ seconds?
   (b) What was the average velocity from $t = 10$ to $t = 30$ seconds?
   (c) About how fast was the car traveling **at** $t = 10$ seconds? **At** $t = 20$? **At** $t = 30$?
   (d) What does the horizontal part of the graph between $t = 15$ and $t = 20$ seconds tell you?
   (e) What does the negative velocity at $t = 25$ represent?



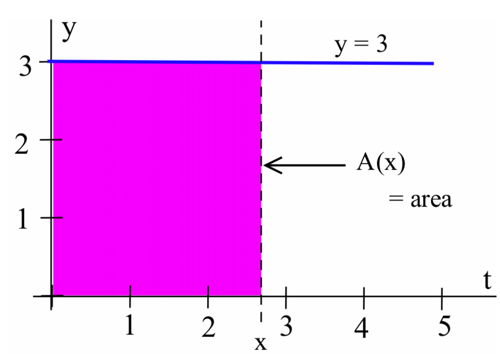

7. The figure below shows the distance of a car from a measuring position located on the edge of a straight road.
   (a) What was the average velocity of the car from $t = 0$ to $t = 20$ seconds?
   (b) What was the average velocity from $t = 10$ to $t = 30$ seconds?
   (c) About how fast was the car traveling **at** $t = 10$ seconds? **At** $t = 20$? **At** $t = 30$?



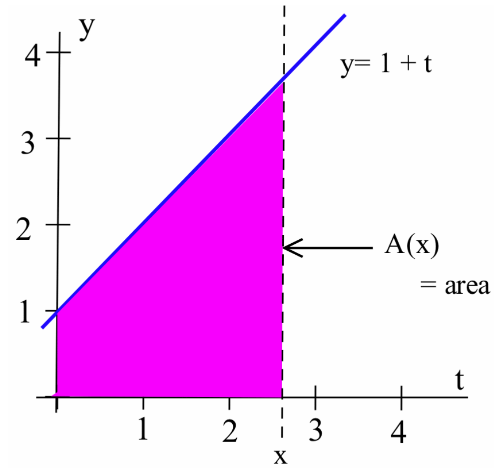

8. The figure below shows the composite developmental skill level of chessmasters at different ages as determined by their performance against other chessmasters. (From "Rating Systems for Human Abilities," by W.H. Batchelder and R.S. Simpson, 1988. UMAP Module 698.)
   (a) At what age is the "typical" chessmaster playing the best chess?
   (b) At approximately what age is the chessmaster's skill level increasing most rapidly?
   (c) Describe the development of the "typical" chessmaster's skill in words.
   (d) Sketch graphs that you **think** would reasonably describe the performance levels versus age for an athlete, a classical pianist, a rock singer, a mathematician and a professional in your major field.




9. Define $A(x)$ to be the **area** bounded by the $t$- (horizontal) and $y$-axes, the horizontal line $y = 3$, and the vertical line at $x$ (see figure below). For example, $A(4) = 12$ is the area of the $4 \times 3$ rectangle.
   (a) Evaluate $A(0)$, $A(1)$, $A(2)$, $A(2.5)$ and $A(3)$.
   (b) What area would $A(4) - A(1)$ represent?
   (c) Graph $y = A(x)$ for $0 \le x \le 4$.



10. Define $A(x)$ to be the **area** bounded by the $t$-(horizontal) and $y$-axes, the line $y = t + 1$, and the vertical line at $x$ (see figure). For example, $A(4) = 12$.




(a) Evaluate $A(0)$, $A(1)$, $A(2)$, $A(2.5)$ and $A(3)$.

(b) What area would $A(3) - A(1)$ represent in the figure?

(c) Graph $y = A(x)$ for $0 \le x \le 4$.

*1.0 Practice Answers*

1. If $x = 1.994$, then $y = 3.976036$, so the slope between $(2,4)$ and $(x,y)$ is:
$$\frac{4-y}{2-x} = \frac{4-3.976036}{2-1.994} = \frac{0.023964}{0.006} \approx 3.994$$
If $x = 2.0003$, then $y \approx 4.0012$, so the slope between $(2,4)$ and $(x,y)$ is:
$$\frac{4-y}{2-x} = \frac{4-4.0012}{2-2.0003} = \frac{-0.0012}{-0.0003} \approx 4.0003$$

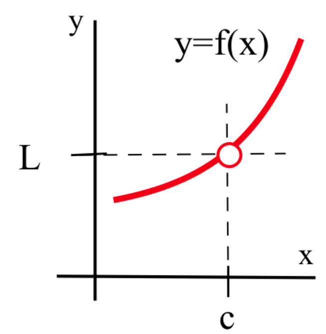
2. Computing $m_{\text{sec}}$:
$$\frac{f(-1+h)-(1)}{(-1+h)-(-1)} = \frac{(-1+h)^2-1}{h} = \frac{1-2h+h^2-1}{h} = \frac{h(-2+h)}{h} = -2+h$$
As $h \to 0$, $m_{\text{sec}} = -2+h \to -2$.

3. The average velocity between $t = 1.5$ and $t = 2.0$ is:
$$\frac{36-64 \text{ feet}}{2.0-1.5 \text{ sec}} = -56\,\frac{\text{feet}}{\text{sec}}$$
The average velocity between $t = 2.0$ and $t = 2.5$ is:
$$\frac{0-36 \text{ feet}}{2.5-2.0 \text{ sec}} = -72\,\frac{\text{feet}}{\text{sec}}$$
The velocity at $t = 2.0$ is somewhere between $-56\,\frac{\text{feet}}{\text{sec}}$ and $-72\,\frac{\text{feet}}{\text{sec}}$, probably around the middle of this interval:
$$\frac{(-56)+(-72)}{2} = -64\,\frac{\text{feet}}{\text{sec}}$$

4. (a) When $t = 9$ days, the population is approximately $P = 4,200$ bacteria. When $t = 13$, $P \approx 5,000$. The average change in population is approximately:
$$\frac{5000-4200 \text{ bacteria}}{13-9 \text{ days}} = \frac{800 \text{ bacteria}}{4 \text{ days}} = 200\,\frac{\text{bacteria}}{\text{day}}$$
(b) To find the rate of population growth at $t = 9$ days, sketch the line tangent to the population curve at the point $(9,4200)$ and then use $(9,4200)$ and another point on the tangent line to calculate the slope of the line. Using the approximate values $(5,2800)$ and $(9,4200)$, the slope of the tangent line at the point $(9,4200)$ is approximately:
$$\frac{4200-2800 \text{ bacteria}}{9-5 \text{ days}} = \frac{1400 \text{ bacteria}}{4 \text{ days}} \approx 350\,\frac{\text{bacteria}}{\text{day}}$$

## 1.1 *The Limit of a Function*

Calculus has been called the study of continuous change, and the **limit** is the basic concept that allows us to describe and analyze such change. An understanding of limits is necessary to understand derivatives, integrals and other fundamental topics of calculus.




The symbol $\to$ means "approaches" or "gets very close to."

### *The Idea (Informally)*

The limit of a function at a point helps describe the behavior of the function when the input variable is near—**but does not equal**—a specified number (see margin figure). If the values of $f(x)$ are all "very close"—as close as we want—to one number $L$ as we restrict values of $x$ to be "very close" to (but not equal to) a number $c$, then

> we say: "the limit of $f(x)$, as $x$ approaches $c$, is $L$"
> and we write: $\lim_{x \to c} f(x) = L$

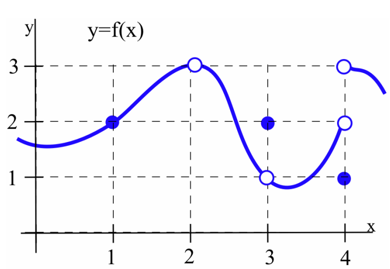
It is very important to note that:

> $f(c)$ is a single number that describes the behavior (value) of $f$ **at** the point $x = c$

while:

> $\lim_{x \to c} f(x)$ is a single number that describes the behavior of $f$ **near**, **but not at** the point $x = c$

If we have a graph of the function $f(x)$ near $x = c$, then it is often easy to estimate $\lim_{x \to c} f(x)$.




**Example 1.** Use the graph of $y = f(x)$ given in the margin to estimate the following limits:

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 2} f(x)$ (c) $\lim_{x \to 3} f(x)$ (d) $\lim_{x \to 4} f(x)$

**Solution.** Each of these limits involves a different issue, as you may be able to tell from the graph.

(a) $\lim_{x \to 1} f(x) = 2$: When $x$ is very close to 1, the values of $f(x)$ appear to be very close to $y = 2$. In this example, it happens that $f(1) = 2$, but that is irrelevant for the limit. The only thing that matters is what happens for $x$ *close to* 1 but with $x \neq 1$.

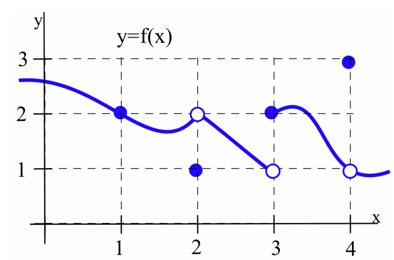
(b) $f(2)$ is undefined, but we only care about the behavior of $f(x)$ for $x$ *close to* 2, where the values of $f(x)$ appear to be close to 3. If we restrict $x$ close enough to 2, the values of $f(x)$ will be as close to 3 as we want, so $\lim_{x\to 2} f(x) = 3$.

(c) When $x$ is close to 3, the values of $f(x)$ appear to be close to 1, so $\lim_{x\to 3} f(x) = 1$. For this limit it is completely irrelevant that $f(3) = 2$: we only care about what happens to $f(x)$ for $x$ close to (but not equal to) 3.

(d) This one is harder and we need to be careful. When $x$ is close to 4 and slightly *less than* 4 ($x$ is just to the left of 4 on the $x$-axis) then the values of $f(x)$ are close to 2. But if $x$ is close to 4 and slightly *larger than* 4 then the values of $f(x)$ are close to 3.

If we know only that $x$ is very close to 4, then we cannot say whether $y = f(x)$ will be close to 2 or close to 3—it depends on whether $x$ is on the right or the left side of 4. In this situation, the $f(x)$ values are not all close to a single number when $x$ is close to 4, so we say that $\lim_{x\to 4} f(x)$ **does not exist**.

In (d), it is irrelevant that $f(4) = 1$. The limit, as $x$ approaches 4, would still cease to exist if $f(4)$ was 3 or 2 or anything else. ◀




**Practice 1.** Use the graph of $y = f(x)$ in the margin to estimate the following limits:

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{t\to 2} f(t)$ (c) $\lim_{x\to 3} f(x)$ (d) $\lim_{w\to 4} f(w)$

**Example 2.** Determine the value of $\lim_{x\to 3} \dfrac{2x^2 - x - 1}{x - 1}$.

**Solution.** We need to investigate the values of $f(x) = \frac{2x^2-x-1}{x-1}$ when $x$ is close to 3. If the $f(x)$ values are all arbitrarily close to—or even equal to—some number $L$, then $L$ will be the limit.

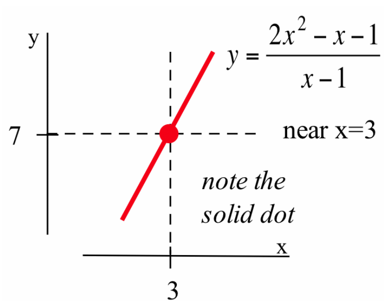
One way to keep track of both the $x$ and the $f(x)$ values is to set up a table and to pick several $x$ values that get closer and closer (but not equal) to 3.

We can pick some values of $x$ that approach 3 from the left, say $x = 2.91$, 2.9997, 2.999993 and 2.9999999, and some values of $x$ that approach 3 from the right, say $x = 3.1$, 3.004, 3.0001 and 3.000002. The only thing important about these particular values for $x$ is that they get closer and closer to 3 without actually equaling 3. You should try some other values "close to 3" to see what happens. Our table of values is:

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 2.9 | 6.82 | 3.1 | 7.2 |
| 2.9997 | 6.9994 | 3.004 | 7.008 |
| 2.999993 | 6.999986 | 3.0001 | 7.0002 |
| 2.9999999 | 6.9999998 | 3.000002 | 7.000004 |
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| 3 | 7 | 3 | 7 |

As the $x$ values get closer and closer to 3, the $f(x)$ values are all close to 7. In fact, we can get $f(x)$ as close to 7 as we want ("arbitrarily close") by taking the values of $x$ very close ("sufficiently close") to 3. We write:

$$\lim_{x \to 3} \frac{2x^2 - x - 1}{x - 1} = 7$$




Instead of using a table of values, we could have graphed $y = f(x)$ for $x$ close to 3 (see margin) and used the graph to answer the limit question. This graphical approach is easier, particularly if you have a calculator or computer do the graphing work for you, but it is really very similar to the "table of values" method: in each method you need to evaluate $y = f(x)$ at many values of $x$ near 3. ◀

In the previous example, you might have noticed that if we just evaluate $f(3)$, then we get the correct answer, 7. That works for this particular problem, but it often fails. The next example (identical to the previous one, except $x \to 1$) illustrates one such difficulty.

**Example 3.** Find $\displaystyle\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1}$.

**Solution.** You might try to evaluate $f(x) = \frac{2x^2-x-1}{x-1}$ at $x = 1$, but $f(1) = \frac{0}{0}$, so $f$ is not defined at $x = 1$.

It is tempting—*but wrong*—to conclude that this function does not have a limit as $x$ approaches 1.

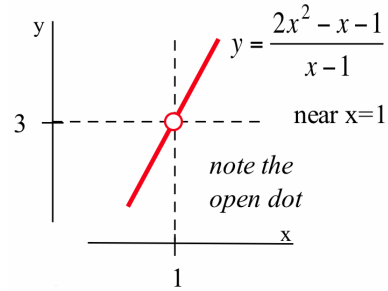
**Table Method**: Trying some "test" values for $x$ that get closer and closer to 1 from both the left and the right, we get:

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| 0.9 | 2.82 | 1.1 | 3.2 |
| 0.9998 | 2.9996 | 1.003 | 3.006 |
| 0.999994 | 2.999988 | 1.0001 | 3.0002 |
| 0.9999999 | 2.9999998 | 1.000007 | 3.000014 |
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| 1 | 3 | 1 | 3 |

The function $f$ is not defined at $x = 1$, but when $x$ gets close to 1, the values of $f(x)$ are very close to 3. We can ensure that $f(x)$ is as close to 3 as we want by restricting $x$ to be very close to 1, so:

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = 3$$

**Graph Method**: We can graph $y = f(x) = \frac{2x^2-x-1}{x-1}$ for $x$ close to 1 (see margin) and notice that whenever $x$ is close to 1, the values of $y = f(x)$ are close to 3; $f$ is not defined at $x = 1$, so the graph has a hole above $x = 1$, but we only care about what $f(x)$ is doing for $x$ *close to* but *not equal to* 1.




**Algebra Method**: We could have found the same result by noting:

$$f(x) = \frac{2x^2 - x - 1}{x - 1} = \frac{(2x+1)(x-1)}{x-1} = 2x + 1$$

as long as $x \neq 1$. The "$x \to 1$" part of the limit means that $x$ is *close to* 1 but *not equal to* 1, so our division step is valid and:

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = \lim_{x \to 1} [2x + 1] = 3$$

which is the same answer we obtained using the first two methods. ◀

## *Three Methods for Evaluating Limits*

The previous example utilized three different methods, each of which led us to the same answer for the limit.

### The Algebra Method

The algebra method involves algebraically simplifying the function before trying to evaluate its limit. Often, this simplification just means factoring and dividing, but sometimes more complicated algebraic or even trigonometric steps are needed.

### The Table Method

To evaluate a limit of a function $f(x)$ as $x$ approaches $c$, the table method involves calculating the values of $f(x)$ for "enough" values of $x$ very close to $c$ so that we can "confidently" determine a limiting value of $f(x)$. If $f(x)$ is well behaved, we may not need to use very many values for $x$. However, this method is usually used with complicated functions, and then we need to evaluate $f(x)$ for lots of values of $x$.

A computer or calculator can often make the function evaluations easier, but their calculations are subject to "round off" errors. The result of any computer calculation that involves both large and small numbers should be viewed with some suspicion. For example, the function

$$f(x) = \frac{((0.1)^x + 1) - 1}{(0.1)^x} = \frac{(0.1)^x}{(0.1)^x} = 1$$

for every value of $x$, and my calculator gives the correct answer for some values of $x$: $f(3) = 1$, and $f(8)$ and $f(9)$ both equal 1.

But my calculator says $\left((0.1)^{10} + 1\right) - 1 = 0$, so it evaluates $f(10)$ to be 0, definitely an incorrect value.

Your calculator may evaluate $f(10)$ correctly, but try $f(35)$ or $f(107)$.

**Calculators are too handy to be ignored, but they are too prone to these types of errors to be believed uncritically. Be careful.**

**The Graph Method**

The graph method is closely related to the table method, but we create a graph of the function instead of a table of values, and then we use the graph to determine the limiting value of $f(x)$ (if there is one).

**Which Method Should You Use?**

In general, the algebraic method is preferred because it is precise and does not depend on which values of $x$ we chose or the accuracy of our graph or precision of our calculator. **If you can evaluate a limit algebraically, you should do so.** Sometimes, however, it will be very difficult to evaluate a limit algebraically, and the table or graph methods offer worthwhile alternatives. Even when you can algebraically evaluate the limit of a function, it is still a good idea to graph the function or evaluate it at a few points just to check that your algebraic answer is reasonable.

The table and graph methods have the same advantages and disadvantages. Both can be used on complicated functions that are difficult to handle algebraically or whose algebraic properties you don't know.

Often both methods can be easily programmed on a calculator or computer. However, these two methods are very time-consuming by hand and are prone to round-off errors on computers. You need to know how to use these methods when you can't figure out how to use the algebraic method, but you need to use these two methods warily.

**Example 4.** Evaluate each limit.

(a) $\displaystyle\lim_{x \to 0} \frac{x^2 + 5x + 6}{x^2 + 3x + 2}$ (b) $\displaystyle\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 + 3x + 2}$

**Solution.** The function in each limit is the same but $x$ is approaching a different number in each of them.

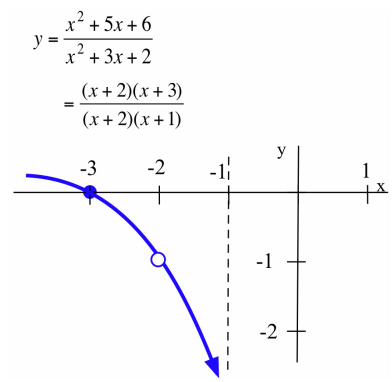
(a) Because $x \to 0$, we know that $x$ is getting closer and closer to 0, so the values of the $x^2$, $5x$ and $3x$ terms get as close to 0 as we want. The numerator approaches 6 and the denominator approaches 2, so the values of the whole function get arbitrarily close to $\frac{6}{2} = 3$, the limit.

(b) As $x$ approaches $-2$, the numerator and denominator approach 0, and a small number divided by a small number can be almost anything — the ratio depends on the size of the top compared to the size of the bottom. More investigation is needed.

**Table Method**: If we pick some values of $x$ close to (but not equal to) $-2$, we get the table:

| $x$ | $x^2+5x+6$ | $x^2+3x+2$ | $\frac{x^2+5x+6}{x^2+3x+2}$ |
|---|---|---|---|
| $-1.97$ | $0.0309$ | $-0.0291$ | $-1.061856$ |
| $-2.005$ | $-0.004975$ | $0.005025$ | $-0.990050$ |
| $-1.9998$ | $0.00020004$ | $-0.00019996$ | $-1.00040008$ |
| $-2.00003$ | $-0.00002999$ | $0.0000300009$ | $-0.9996666$ |
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| $-2$ | $0$ | $0$ | $-1$ |

Even though the numerator and denominator are each getting closer and closer to 0, their ratio is getting arbitrarily close to $-1$, which appears to be the limit.




**Graph Method**: The graph of $y = f(x) = \frac{x^2+5x+6}{x^2+3x+2}$ in the margin appears to show that the values of $f(x)$ are very close to $-1$ when the $x$-values are close to $-2$.

**Algebra Method**: Factoring the numerator and denominator:

$$f(x) = \frac{x^2+5x+6}{x^2+3x+2} = \frac{(x+2)(x+3)}{(x+2)(x+1)}$$

We know $x \to -2$ so $x \neq -2$ and we can divide the top and bottom by $(x+2)$. Then

$$f(x) = \frac{(x+3)}{(x+1)} \to \frac{1}{-1} = -1$$

as $x \to -2$. ◀

You should remember the technique used in the previous example:

$$\text{If } \lim_{x\to c} \frac{\text{polynomial}}{\text{another polynomial}} = \frac{0}{0},$$
try dividing the top and bottom by $x - c$.

**Practice 2.** Evaluate each limit.

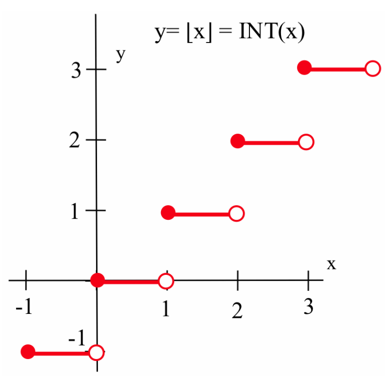
(a) $\displaystyle\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2}$

(b) $\displaystyle\lim_{t\to 0} \frac{t \cdot \sin(t)}{t^2 + 3t}$

(c) $\displaystyle\lim_{w\to 2} \frac{w - 2}{\ln(\frac{w}{2})}$

*One-Sided Limits*

Sometimes, what happens to us at a place depends on the direction we use to approach that place. If we approach Niagara Falls from the upstream side, then we will be 182 feet higher and have different worries than if we approach from the downstream side. Similarly, the values of a function near a point may depend on the direction we use to approach that point.




If we let $x$ approach 3 from the left ($x$ is close to 3 and $x < 3$) then the values of $\lfloor x \rfloor = \text{INT}(x)$ equal 2 (see margin).

If we let $x$ approach 3 from the right ($x$ is close to 3 and $x > 3$) then the values of $\lfloor x \rfloor = \text{INT}(x)$ equal 3.

On the number line we can approach a point from the left or the right, and that leads to **one-sided limits**.

**Definition of Left and Right Limits**:
The **left limit** as $x$ approaches $c$ of $f(x)$ is $L$ if the values of $f(x)$ are as close to $L$ as we want when $x$ is very close to but left of $c$ ($x < c$):

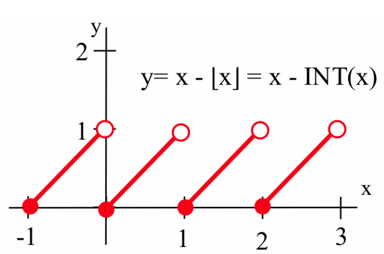
$$\lim_{x\to c^-} f(x) = L$$

The **right limit**, $\displaystyle\lim_{x\to c^+} f(x)$, requires that $x$ lie to the right of $c$ ($x > c$).

**Example 5.** Evaluate $\displaystyle\lim_{x\to 2^-} x - \lfloor x \rfloor$ and $\displaystyle\lim_{x\to 2^+} x - \lfloor x \rfloor$.

**Solution.** The left-limit notation $x \to 2^-$ requires that $x$ be close to 2 and that $x$ be to the left of 2, so $x < 2$. If $1 < x < 2$, then $\lfloor x \rfloor = 1$ and:

$$\lim_{x\to 2^-} x - \lfloor x \rfloor = \lim_{x\to 2^-} x - 1 = 2 - 1 = 1$$




If $x$ is close to 2 and is to the right of 2, then $2 < x < 3$, so $\lfloor x \rfloor = 2$ and:

$$\lim_{x\to 2^+} x - \lfloor x \rfloor = \lim_{x\to 2^+} x - 2 = 2 - 2 = 0$$

A graph of $f(x) = x - \lfloor x \rfloor$ appears in the margin. ◀

If the left and right limits of $f(x)$ have the same value at $x = c$:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

then the value of $f(x)$ is close to $L$ whenever $x$ is close to $c$, and it does not matter whether $x$ is left or right of $c$, so

$$\lim_{x \to c} f(x) = L$$

Similarly, if:

$$\lim_{x \to c} f(x) = L$$

then $f(x)$ is close to $L$ whenever $x$ is close to $c$ and less than $c$, and whenever $x$ is close to $c$ and greater than $c$, so:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

We can combine these two statements into a single theorem.

**One-Sided Limit Theorem**:

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$

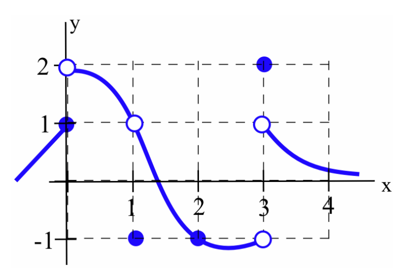
This theorem has an important corollary.

**Corollary**:

If $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$, then $\lim_{x \to c} f(x)$ does not exist.

One-sided limits are particularly useful for describing the behavior of functions that have steps or jumps.

To determine the limit of a function involving the greatest integer or absolute value or a multiline definition, definitely consider both the left and right limits.




**Practice 3.** Use the graph in the margin to evaluate the one- and two-sided limits of $f$ at $x = 0$, 1, 2 and 3.

**Practice 4.** Defining $f(x)$ as:

$$f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x < 3 \\ 2 & \text{if } 3 < x \end{cases}$$

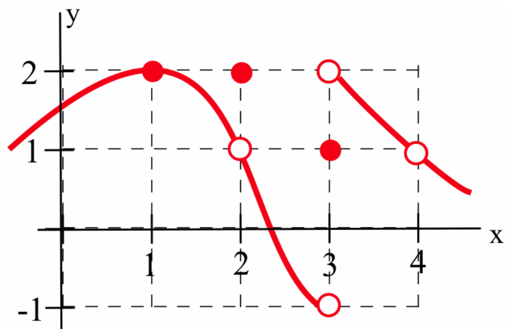
find the one- and two-sided limits of $f$ at 1 and 3.

## *1.1 Problems*

1. Use the graph below to estimate the limits.

   (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$

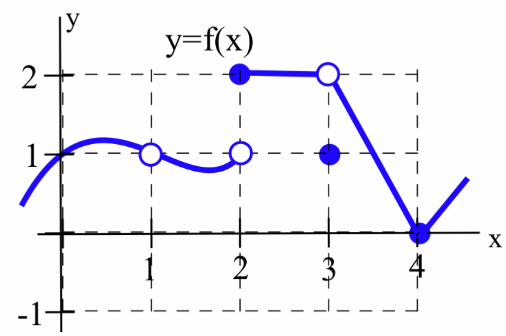
   (c) $\lim_{x\to 3} f(x)$ (d) $\lim_{x\to 4} f(x)$




2. Use the graph below to estimate the limits.

   (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$

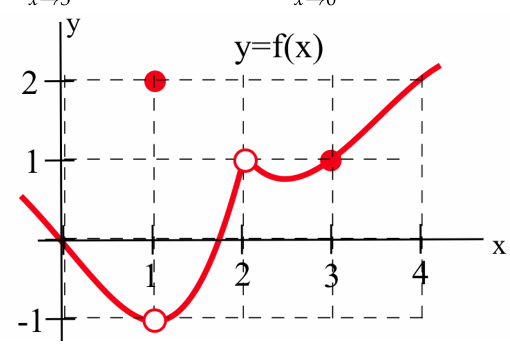
   (c) $\lim_{x\to 3} f(x)$ (d) $\lim_{x\to 4} f(x)$




3. Use the graph below to estimate the limits.

   (a) $\lim_{x\to 1} f(2x)$ (b) $\lim_{x\to 2} f(x-1)$

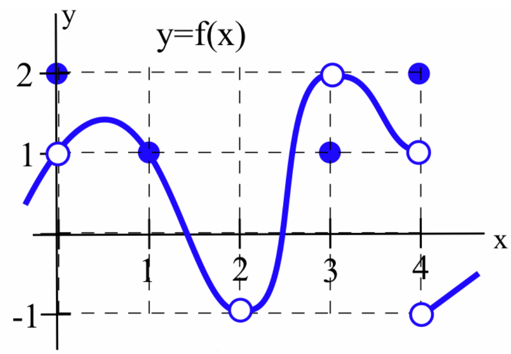
   (c) $\lim_{x\to 3} f(2x-5)$ (d) $\lim_{x\to 0} f(4+x)$




4. Use the graph below to estimate the limits.

   (a) $\lim_{x\to 1} f(3x)$ (b) $\lim_{x\to 2} f(x+1)$

   (c) $\lim_{x\to 3} f(2x-4)$ (d) $\lim_{x\to 0} |f(4+x)|$




In Problems 5–11, evaluate (or estimate) each limit.

5. (a) $\lim_{x\to 1} \dfrac{x^2+3x+3}{x-2}$ (b) $\lim_{x\to 2} \dfrac{x^2+3x+3}{x-2}$

6. (a) $\lim_{x\to 0} \dfrac{x+7}{x^2+9x+14}$ (b) $\lim_{x\to 3} \dfrac{x+7}{x^2+9x+14}$

   (c) $\lim_{x\to -4} \dfrac{x+7}{x^2+9x+14}$ (d) $\lim_{x\to -7} \dfrac{x+7}{x^2+9x+14}$

7. (a) $\lim_{x\to 1} \dfrac{\cos(x)}{x}$ (b) $\lim_{x\to \pi} \dfrac{\cos(x)}{x}$

   (c) $\lim_{x\to -1} \dfrac{\cos(x)}{x}$

8. (a) $\lim_{x\to 7} \sqrt{x-3}$ (b) $\lim_{x\to 9} \sqrt{x}-3$

   (c) $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

9. (a) $\lim_{x\to 0^-} |x|$ (b) $\lim_{x\to 0^+} |x|$

   (c) $\lim_{x\to 0} |x|$

10. (a) $\lim_{x\to 0^-} \dfrac{|x|}{x}$ (b) $\lim_{x\to 0^+} \dfrac{|x|}{x}$

    (c) $\lim_{x\to 0} \dfrac{|x|}{x}$

11. (a) $\lim_{x\to 5} |x-5|$ (b) $\lim_{x\to 3} \frac{|x-5|}{x-5}$

(c) $\lim_{x\to 5} \frac{|x-5|}{x-5}$

12. Find the one- and two-sided limits of:

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ \sin(x) & \text{if } 0 < x \le 2 \\ 1 & \text{if } 2 < x \end{cases}$$

as $x \to 0$, 1 and 2.

13. Find the one- and two-sided limits of:

$$g(x) = \begin{cases} 1 & \text{if } x \le 2 \\ \frac{8}{x} & \text{if } 2 < x < 4 \\ 6-x & \text{if } 4 < x \end{cases}$$

as $x \to 1$, 2, 4 and 5.

In 14–17, use a calculator or computer to get approximate answers accurate to 2 decimal places.

14. (a) $\lim_{x\to 0} \frac{2^x-1}{x}$ (b) $\lim_{x\to 1} \frac{\log_{10}(x)}{x-1}$

15. (a) $\lim_{x\to 0} \frac{3^x-1}{x}$ (b) $\lim_{x\to 1} \frac{\ln(x)}{x-1}$

16. (a) $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5}$ (b) $\lim_{x\to 0} \frac{\sin(3x)}{5x}$

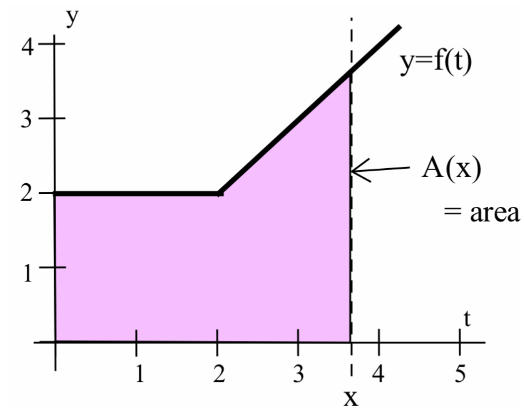
17. (a) $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$ (b) $\lim_{x\to 0} \frac{\sin(7x)}{2x}$

18. Define $A(x)$ to be the **area** bounded by the $t$- and $y$-axes, the "bent line" in the figure below, and the vertical line $t = x$. For example, $A(4) = 10$.

(a) Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.

(b) Graph $y = A(x)$ for $0 \le x \le 4$.

(c) What area does $A(3) - A(1)$ represent?

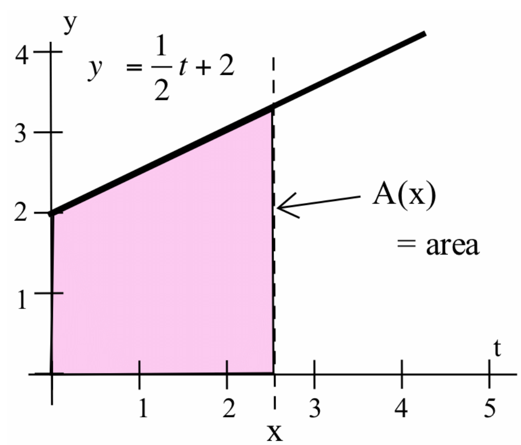



19. Define $A(x)$ to be the **area** bounded by the $t$- and $y$-axes, the line $y = \frac{1}{2}t + 2$ and the vertical line $t = x$ (See figure below). For example, $A(4) = 12$.

(a) Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.

(b) Graph $y = A(x)$ for $0 \le x \le 4$.

(c) What area does $A(3) - A(1)$ represent?




20. Sketch the graph of $f(t) = \sqrt{4t - t^2}$ for $0 \le t \le 4$ (you should get a semicircle). Define $A(x)$ to be the area bounded below by the $t$-axis, above by the graph $y = f(t)$ and on the right by the vertical line at $t = x$.

(a) Evaluate $A(0)$, $A(2)$ and $A(4)$.

(b) Sketch a graph $y = A(x)$ for $0 \le x \le 4$.

(c) What area does $A(3) - A(1)$ represent?

## 1.1 Practice Answers

1. (a) 2

   (b) 2

   (c) does not exist (no limit)

   (d) 1

2. (a) $\lim_{x \to 2} \dfrac{(x+1)(x-2)}{x-2} = \lim_{x \to 2} (x+1) = 3$

   (b) $\lim_{t \to 0} \dfrac{t \sin(t)}{t(t+3)} = \lim_{t \to 0} \dfrac{\sin(t)}{t+3} = \dfrac{0}{3} = 0$

   (c) $\lim_{w \to 2} \dfrac{w-2}{\ln(\frac{w}{2})} = 2$ To see this, make a graph or a table:

| $w$ | $\frac{w-2}{\ln(\frac{w}{2})}$ | $w$ | $\frac{w-2}{\ln(\frac{w}{2})}$ |
|---|---|---|---|
| 2.2 | 2.098411737 | 1.9 | 1.949572575 |
| 2.01 | 2.004995844 | 1.99 | 1.994995823 |
| 2.003 | 2.001499625 | 1.9992 | 1.999599973 |
| 2.0001 | 2.00005 | 1.9999 | 1.99995 |
| $\downarrow$ | $\downarrow$ | $\downarrow$ | $\downarrow$ |
| 2 | 2 | 2 | 2 |

3. $\lim_{x \to 0^-} f(x) = 1$ $\quad \lim_{x \to 0^+} f(x) = 2$ $\quad \lim_{x \to 0} f(x)$ DNE

   $\lim_{x \to 1^-} f(x) = 1$ $\quad \lim_{x \to 1^+} f(x) = 1$ $\quad \lim_{x \to 1} f(x) = 1$

   $\lim_{x \to 2^-} f(x) = -1$ $\quad \lim_{x \to 2^+} f(x) = -1$ $\quad \lim_{x \to 2} f(x) = -1$

   $\lim_{x \to 3^-} f(x) = -1$ $\quad \lim_{x \to 3^+} f(x) = 1$ $\quad \lim_{x \to 3} f(x)$ DNE

4. $\lim_{x \to 1^-} f(x) = 1$ $\quad \lim_{x \to 1^+} f(x) = 1$ $\quad \lim_{x \to 1} f(x) = 1$

   $\lim_{x \to 3^-} f(x) = 3$ $\quad \lim_{x \to 3^+} f(x) = 2$ $\quad \lim_{x \to 3} f(x)$ DNE

## 1.2 *Properties of Limits*

This section presents results that make it easier to calculate limits of combinations of functions or to show that a limit does not exist. The main result says we can determine the limit of "elementary combinations" of functions by calculating the limit of each function separately and recombining these results to get our final answer.

**Main Limit Theorem**:

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

then (a) $\lim_{x \to a} [f(x) + g(x)] = L + M$

(b) $\lim_{x \to a} [f(x) - g(x)] = L - M$

(c) $\lim_{x \to a} k \cdot f(x) = k \cdot L$

(d) $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$

(e) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ (if $M \neq 0$)

(f) $\lim_{x \to a} [f(x)]^n = L^n$

(g) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$

When $n$ is an even integer in part (g) of the Main Limit Theorem, we need $L \geq 0$ and $f(x) \geq 0$ for $x$ near $a$.

The Main Limit Theorem says we get the same result if we first perform the algebra and then take the limit or if we take the limits first and then perform the algebra: for example, (a) says that the limit of the sum equals the sum of the limits.

A proof of the Main Limit Theorem is not inherently difficult, but it requires a more precise definition of the limit concept than we have at the moment, and it then involves a number of technical difficulties.

**Practice 1.** For $f(x) = x^2 - x - 6$ and $g(x) = x^2 - 2x - 3$, evaluate:

(a) $\lim_{x \to 1} [f(x) + g(x)]$

(b) $\lim_{x \to 1} f(x) \cdot g(x)$

(c) $\lim_{x \to 1} \dfrac{f(x)}{g(x)}$

(d) $\lim_{x \to 3} [f(x) + g(x)]$

(e) $\lim_{x \to 3} f(x) \cdot g(x)$

(f) $\lim_{x \to 3} \dfrac{f(x)}{g(x)}$

(g) $\lim_{x \to 2} [f(x)]^3$

(h) $\lim_{x \to 2} \sqrt{1 - g(x)}$

*Limits of Some Very Nice Functions: Substitution*

As you may have noticed in the previous example, for some functions $f(x)$ it is possible to calculate the limit as $x$ approaches $a$ simply by substituting $x = a$ into the function and then evaluating $f(a)$, but sometimes this method does not work. The following results help to (partially) answer the question about when such a substitution is valid.

**Two Easy Limits**:

$$\lim_{x \to a} k = k \quad \text{and} \quad \lim_{x \to a} x = a$$

We can use the preceding Two Easy Limits and the Main Limit Theorem to prove the following Substitution Theorem.

**Substitution Theorem For Polynomial and Rational Functions**:

If $P(x)$ and $Q(x)$ are polynomials and $a$ is any number

then $\lim_{x \to a} P(x) = P(a)$ and $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$

as long as $Q(a) \neq 0$.

The Substitution Theorem says that we can calculate the limits of polynomials and rational functions by substituting (as long as the substitution does not result in a division by 0).

**Practice 2.** Evaluate each limit.

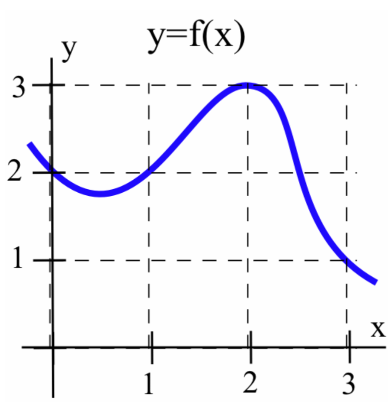
(a) $\lim_{x \to 2} \left[5x^3 - x^2 + 3\right]$

(b) $\lim_{x \to 2} \frac{x^3 - 7x}{x^2 + 3x}$

(c) $\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - x - 2}$

*Limits of Other Combinations of Functions*

So far we have concentrated on limits of single functions and elementary combinations of functions. If we are working with limits of other combinations or compositions of functions, the situation becomes slightly more difficult, but sometimes these more complicated limits have useful geometric interpretations.




**Example 1.** Use the graph in the margin to estimate each limit.

(a) $\lim_{x \to 1} [3 + f(x)]$

(b) $\lim_{x \to 1} f(2 + x)$

(c) $\lim_{x \to 0} f(3 - x)$

(d) $\lim_{x \to 2} [f(x+1) - f(x)]$

**Solution.** (a) $\lim_{x \to 1} [3 + f(x)]$ requires a straightforward application of part (a) of the Main Limit Theorem:

$$\lim_{x \to 1} [3 + f(x)] = \lim_{x \to 1} 3 + \lim_{x \to 1} f(x) = 3 + 2 = 5$$

(b) We first need to examine what happens to the quantity $2 + x$ as $x \to 1$ before we can consider the limit of $f(2 + x)$. When $x$ is very close to 1, the value of $2 + x$ is very close to 3, so the limit of $f(2 + x)$ as $x \to 1$ is equivalent to the limit of $f(w)$ as $w \to 3$ (where $w = 2 + x$) and it is clear from the graph that $\lim_{w \to 3} f(w) = 1$, so:

$$\lim_{x \to 1} f(2 + x) = \lim_{w \to 3} f(w) = 1$$

In most situations it is not necessary to formally substitute a new variable $w$ for the quantity $2 + x$, but it is still necessary to think about what happens to the quantity $2 + x$ as $x \to 1$.

(c) As $x \to 0$ the quantity $3 - x$ will approach 3, so we want to know what happens to the values of $f$ when the input variable is approaching 3:

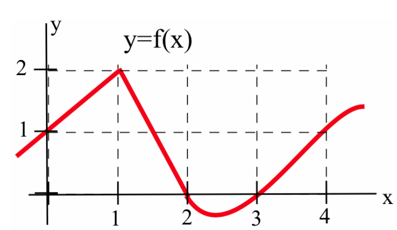
$$\lim_{x \to 0} f(3 - x) = 1$$

(d) Using part (b) of the Main Limit Theorem:

$$\begin{aligned} \lim_{x \to 2} [f(x + 1) - f(x)] &= \lim_{x \to 2} f(x + 1) - \lim_{x \to 2} f(x) \\ &= \lim_{w \to 3} f(w) - \lim_{x \to 2} f(x) = 1 - 3 = -2 \end{aligned}$$

Notice the use of the substitution $w = x + 1$ above. ◀

**Practice 3.** Use the graph in the margin to estimate each limit.



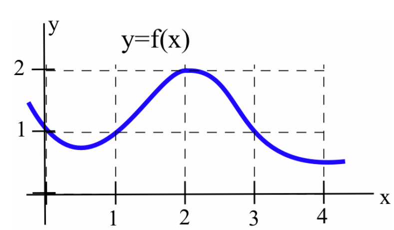

(a) $\lim_{x \to 1} f(2x)$

(b) $\lim_{x \to 2} f(x - 1)$

(c) $\lim_{x \to 0} 3 \cdot f(4 + x)$

(d) $\lim_{x \to 2} f(3x - 2)$

**Example 2.** Use the graph in the margin to estimate each limit.




(a) $\lim_{h \to 0} f(3 + h)$

(b) $\lim_{h \to 0} f(3)$

(c) $\lim_{h \to 0} [f(3 + h) - f(3)]$

(d) $\lim_{h \to 0} \dfrac{f(3 + h) - f(3)}{h}$

**Solution.** The last limit is a special type of limit we will encounter often in this book, while the first three parts are the steps we need to evaluate it.

(a) As $h \to 0$, the quantity $w = 3 + h$ will approach 3, so

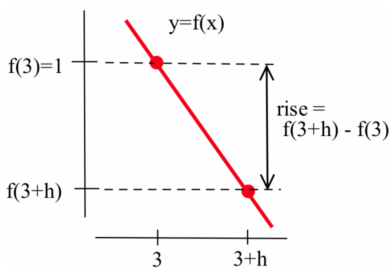
$$\lim_{h\to 0} f(3+h) = \lim_{w\to 3} f(w) = 1$$

(b) $f(3)$ is a constant (equal to 1) and does not depend on $h$ in any way, so:

$$\lim_{h\to 0} f(3) = f(3) = 1$$

(c) This limit is just an algebraic combination of the first two limits:

$$\lim_{h\to 0} [f(3+h) - f(3)] = \lim_{h\to 0} f(3+h) - \lim_{h\to 0} f(3) = 1 - 1 = 0$$




The quantity $f(3+h) - f(3)$ also has a geometric interpretation: it is the change in the $y$-coordinates, the $\Delta y$, between the points $(3, f(3))$ and $(3+h, f(3+h))$ (see margin figure).

(d) As $h \to 0$, the numerator and denominator of $\dfrac{f(3+h) - f(3)}{h}$ both approach 0, so we cannot immediately determine the value of the limit. But if we recognize that $f(3+h) - f(3) = \Delta y$ for the two points $(3, f(3))$ and $(3+h, f(3+h))$ and that $h = \Delta x$ for the same two points, then we can interpret $\dfrac{f(3+h) - f(3)}{h}$ as $\frac{\Delta y}{\Delta x}$, which is the slope of the secant line through the two points:

$$\begin{aligned}\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{\Delta x\to 0} [\text{slope of the secant line}] \\ &= \text{slope of the tangent line at } (3, f(3)) \\ &\approx -1\end{aligned}$$

This last limit represents the slope of line tangent to the graph of $f$ at the point $(3, f(3))$.

It is a pattern we will encounter often. ◀

### *Tangent Lines as Limits*

If we have two points on the graph of the function $y = f(x)$:

$$(x, f(x)) \text{ and } (x+h, f(x+h))$$

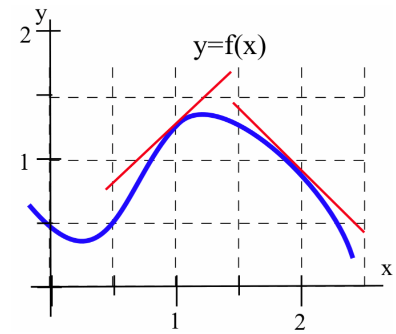
then $\Delta y = f(x+h) - f(x)$ and $\Delta x = (x+h) - (x) = h$, so the slope of the secant line through those points is:

$$m_{\text{sec}} = \frac{\Delta y}{\Delta x}$$

and the slope of the line tangent to the graph of $f$ at the point $(x, f(x))$ is, by definition,

$$m_{\text{tan}} = \lim_{\Delta x\to 0} [\text{slope of the secant line}] = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

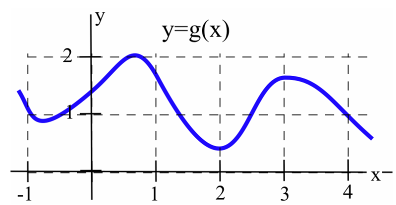
**Example 3.** Give a geometric interpretation for the following limits and **estimate** their values for the function whose graph appears in the margin.




(a) $\lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$ (b) $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$

**Solution.** (a) The limit represents the slope of the line tangent to the graph of $f(x)$ at the point $(1, f(1))$, so $\lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h} \approx 1$. (b) The limit represents the slope of the line tangent to the graph of $f(x)$ at the point $(2, f(2))$, so $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} \approx -1$. ◀

**Practice 4.** Give a geometric interpretation for the following limits and estimate their values for the function whose graph appears in the margin.




(a) $\lim_{h\to 0} \dfrac{g(1+h)-g(1)}{h}$

(b) $\lim_{h\to 0} \dfrac{g(3+h)-g(3)}{h}$

(c) $\lim_{h\to 0} \dfrac{g(h)-g(0)}{h}$

*Comparing the Limits of Functions*

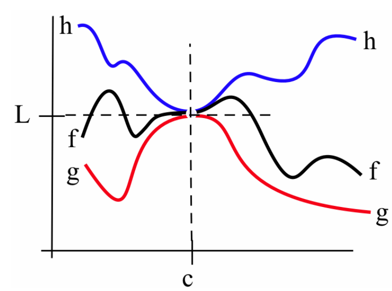
Sometimes it is difficult to work directly with a function. However, if we can compare our complicated function with simpler ones, then we can use information about the simpler functions to draw conclusions about the complicated one. If the complicated function is always between two functions whose limits are equal, then we know the limit of the complicated function.

**Squeezing Theorem**:

If $g(x) \le f(x) \le h(x)$ for all $x$ near (but not equal to) $c$

and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$

then $\lim_{x\to c} f(x) = L$.



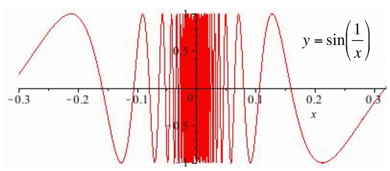

The margin figure shows the idea behind the proof of this theorem: the function $f(x)$ gets "squeezed" between the smaller function $g(x)$ and the bigger function $h(x)$. Because $g(x)$ and $h(x)$ converge to the same limit, $L$, so must $f(x)$.

We can use the Squeezing Theorem to evaluate some "hard" limits by squeezing a "complicated" function in between two "simpler" functions with "easier" limits.

**Example 4.** Use the inequality $-|x| \leq \sin(x) \leq |x|$ to determine:

(a) $\lim_{x \to 0} \sin(x)$ (b) $\lim_{x \to 0} \cos(x)$

**Solution.** (a) $\lim_{x \to 0} |x| = 0$ and $\lim_{x \to 0} -|x| = 0$ so, by the Squeezing Theorem, $\lim_{x \to 0} \sin(x) = 0$. (b) If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $\cos(x) = \sqrt{1 - \sin^2(x)}$, so $\lim_{x \to 0} \cos(x) = \lim_{x \to 0} \sqrt{1 - \sin^2(x)} = \sqrt{1 - 0^2} = 1$. ◀




**Example 5.** Evaluate $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right)$.

**Solution.** In the graph of $\sin\left(\frac{1}{x}\right)$ (see margin), the $y$-values change very rapidly for values of $x$ near 0, but they all lie between $-1$ and 1:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

so, if $x > 0$, multiplying this inequality by $x$ we get:

$$-x \leq x \cdot \sin\left(\frac{1}{x}\right) \leq x$$

which we can rewrite as:

$$-|x| \leq x \cdot \sin\left(\frac{1}{x}\right) \leq |x|$$

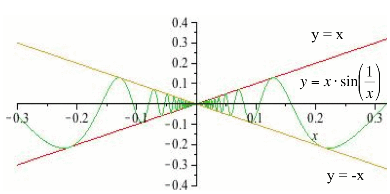
because $|x| = x$ when $x > 0$.

If $x < 0$, when we multiply the original inequality by $x$ we get:

$$-x \geq x \cdot \sin\left(\frac{1}{x}\right) \geq x \quad \Rightarrow \quad |x| \geq x \cdot \sin\left(\frac{1}{x}\right) \geq -|x|$$

because $|x| = -x$ when $x < 0$. Either way we have:

$$-|x| \leq x \cdot \sin\left(\frac{1}{x}\right) \leq |x|$$




for all $x \neq 0$, and in particular for $x$ near 0. Both "simple" functions ($-|x|$ and $|x|$) approach 0 as $x \to 0$, so

$$\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$$

by the Squeezing Theorem. ◀

**Practice 5.** If $f(x)$ is always between $x^2 + 2$ and $2x + 1$, what can you say about $\lim_{x \to 1} f(x)$?

Problem 27 guides you through the steps to prove this relation.

**Practice 6.** Use the relation $\cos(x) \leq \frac{\sin(x)}{x} \leq 1$ to show that:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

*List Method for Showing that a Limit Does Not Exist*

If the limit of $f(x)$, as $x$ approaches $c$, exists and equals $L$, then we can guarantee that the values of $f(x)$ are as close to $L$ as we want by restricting the values of $x$ to be very, very close to $c$. To show that a limit, as $x$ approaches $c$, does **not** exist, we need to show that no matter how closely we restrict the values of $x$ to $c$, the values of $f(x)$ are not **all** close to a single, finite value $L$.

One way to demonstrate that $\lim_{x \to c} f(x)$ does not exist is to show that the left and right limits exist but are not equal.

Another method of showing that $\lim_{x \to c} f(x)$ does not exist uses two (infinite) lists of numbers, $\{a_1, a_2, a_3, a_4, \ldots\}$ and $\{b_1, b_2, b_3, b_4, \ldots\}$, with entries that become arbitrarily close to the value $c$ as the subscripts get larger, but for which the corresponding lists of function values, $\{f(a_1), f(a_2), f(a_3), f(a_4), \ldots\}$ and $\{f(b_1), f(b_2), f(b_3), f(b_4), \ldots\}$ approach two different numbers as the subscripts get larger.

**Example 6.** For $f(x)$ defined as:

$$f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x < 3 \\ 2 & \text{if } 3 < x \end{cases}$$

show that $\lim_{x \to 3} f(x)$ does not exist.

**Solution.** We could use one-sided limits to show that this limit does not exist, but instead we will use the list method.

One way to define values of $\{a_1, a_2, a_3, a_4, \ldots\}$ that approach 3 from the right is to define $a_1 = 3 + 1$, $a_2 = 3 + \frac{1}{2}$, $a_3 = 3 + \frac{1}{3}$, $a_4 = 3 + \frac{1}{4}$ and, in general, $a_n = 3 + \frac{1}{n}$. Then $a_n > 3$ so $f(a_n) = 2$ for all subscripts $n$, and the values in the list $\{f(a_1), f(a_2), f(a_3), f(a_4), \ldots\}$ are all close to 2 — in fact, all of the $f(a_n)$ values *equal* 2.

We can define values of $\{b_1, b_2, b_3, b_4, \ldots\}$ that approach 3 from the left by $b_1 = 3 - 1$, $b_2 = 3 - \frac{1}{2}$, $b_3 = 3 - \frac{1}{3}$, $b_4 = 3 - \frac{1}{4}$, and, in general, $b_n = 3 - \frac{1}{n}$. Then $b_n < 3$ so $f(b_n) = b_n = 3 - \frac{1}{n}$ for each subscript $n$, and the values in the list $\{f(b_1), f(b_2), f(b_3), f(b_4), \ldots\} = \left\{2, 2.5, 2\frac{2}{3}, 2\frac{3}{4}, 2\frac{4}{5}, \ldots, 3 - \frac{1}{n}, \ldots\right\}$ are all close to 3 for large values of $n$.

Because the values in the lists $\{f(a_1), f(a_2), f(a_3), f(a_4), \ldots\}$ and $\{f(b_1), f(b_2), f(b_3), f(b_4), \ldots\}$ have two different limiting values, we can conclude that $\lim_{x \to 3} f(x)$ does not exist. ◀

**Example 7.** Define $h(x)$ as:

$$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

(the "holey" function introduced in Section 0.4). Use the list method to show that $\lim_{x \to 3} h(x)$ does not exist.

**Solution.** Let $\{a_1, a_2, a_3, a_4, \dots\}$ be a list of rational numbers that approach 3: for example, $a_1 = 3+1$, $a_2 = 3+\frac{1}{2}$, $a_3 = 3+\frac{1}{3}, \dots, a_n = 3+\frac{1}{n}$. Then $f(a_n) = 2$ for all $n$, so:

$$\{f(a_1), f(a_2), f(a_3), f(a_4), \dots\} = \{2, 2, 2, 2, \dots\}$$

and the $f(a_n)$ values are all "close to" (in fact, equal) 2.

If $\{b_1, b_2, b_3, b_4, \dots\}$ is a list of irrational numbers that approach 3 (for example, $b_1 = 3+\pi$, $b_2 = 3+\frac{\pi}{2}, \dots, b_n = 3+\frac{\pi}{n}$) then:

$$\{f(b_1), f(b_2), f(b_3), f(b_4), \dots\} = \{1, 1, 1, 1, \dots\}$$

and the $f(b_n)$ values are all close to 1 for large values of $n$.

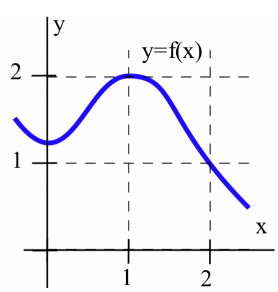
Because the $f(a_n)$ and $f(b_n)$ values become close to two different numbers, the limit of $f(x)$ as $x \to 3$ does not exist. A similar argument will work as $x$ approaches any number $c$, so for every $c$ we can show that $\lim_{x\to c} (x)$ does not exist. The "holey" function does not have a limit as $x$ approaches *any* value $c$. ◀

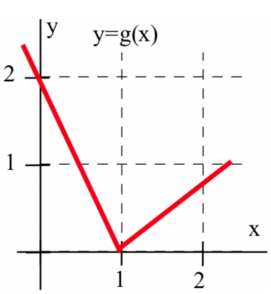
## *1.2 Problems*

1. Use the functions $f$ and $g$ defined by the graphs below to determine the following limits.

   (a) $\lim_{x\to 1} [f(x)+g(x)]$ (b) $\lim_{x\to 1} f(x)\cdot g(x)$

   (c) $\lim_{x\to 1} \dfrac{f(x)}{g(x)}$ (d) $\lim_{x\to 1} f(g(x))$




2. Use the functions $f$ and $g$ defined by the graphs above to determine the following limits.

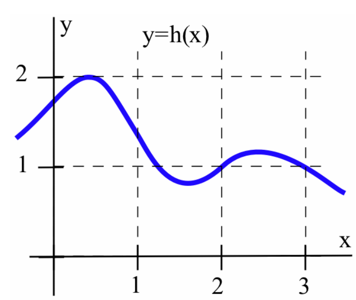
   (a) $\lim_{x\to 2} [f(x)+g(x)]$ (b) $\lim_{x\to 2} f(x)\cdot g(x)$

   (c) $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$ (d) $\lim_{x\to 2} f(g(x))$

3. Use the function $h$ defined by the graph below to determine the following limits.

   (a) $\lim_{x\to 2} h(2x-2)$ (b) $\lim_{x\to 2} [x+h(x)]$

   (c) $\lim_{x\to 2} h(1+x)$ (d) $\lim_{x\to 3} h\left(\dfrac{x}{2}\right)$




4. Use the function $h$ defined by the graph above to determine the following limits.

   (a) $\lim_{x\to 2} h(5-x)$ (b) $\lim_{x\to 0} [h(3+x)-h(3)]$

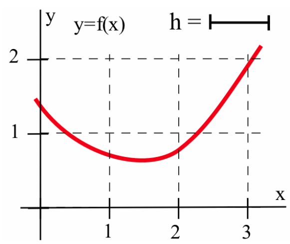
   (c) $\lim_{x\to 2} x\cdot h(x-1)$ (d) $\lim_{x\to 0} \dfrac{h(3+x)-h(3)}{x}$

5. Label the parts of the graph of $f$ (below) that are described by

(a) $2+h$ (b) $f(2)$

(c) $f(2+h)$ (d) $f(2+h)-f(2)$

(e) $\dfrac{f(2+h)-f(2)}{(2+h)-2}$ (f) $\dfrac{f(2-h)-f(2)}{(2-h)-2}$

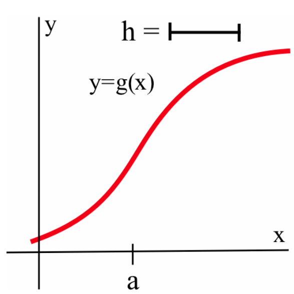



6. Label the parts of the graph of $g$ (below) that are described by

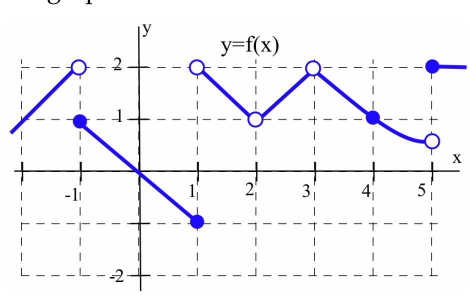
(a) $a+h$ (b) $g(a)$

(c) $g(a+h)$ (d) $g(a+h)-g(a)$

(e) $\dfrac{g(a+h)-g(a)}{(a+h)-a}$ (f) $\dfrac{g(a-h)-g(a)}{(a-h)-a}$




7. Use the graph below estimate:




(a) $\lim\limits_{x\to 1^+} f(x)$ (b) $\lim\limits_{x\to 1^-} f(x)$ (c) $\lim\limits_{x\to 1} f(x)$

(d) $\lim\limits_{x\to 3^+} f(x)$ (e) $\lim\limits_{x\to 3^-} f(x)$ (f) $\lim\limits_{x\to 3} f(x)$

(g) $\lim\limits_{x\to -1^+} f(x)$ (h) $\lim\limits_{x\to -1^-} f(x)$ (i) $\lim\limits_{x\to -1} f(x)$

8. Use the graph from Problem 7 to estimate:

(a) $\lim\limits_{x\to 2^+} f(x)$ (b) $\lim\limits_{x\to 2^-} f(x)$ (c) $\lim\limits_{x\to 2} f(x)$

(d) $\lim\limits_{x\to 4^+} f(x)$ (e) $\lim\limits_{x\to 4^-} f(x)$ (f) $\lim\limits_{x\to 4} f(x)$

(g) $\lim\limits_{x\to -2^+} f(x)$ (h) $\lim\limits_{x\to -2^-} f(x)$ (i) $\lim\limits_{x\to -2} f(x)$

9. The Lorentz contraction formula in relativity theory says the length $L$ of an object moving at $v$ miles per second with respect to an observer is:

$$L = A\cdot\sqrt{1-\frac{v^2}{c^2}}$$

where $c$ is the speed of light (a constant).

(a) Determine the object's "rest length" ($v=0$).

(b) Determine: $\lim\limits_{v\to c^-} L$

10. Evaluate each limit.

(a) $\lim\limits_{x\to 2^+} \lfloor x\rfloor$ (b) $\lim\limits_{x\to 2^-} \lfloor x\rfloor$

(c) $\lim\limits_{x\to -2^+} \lfloor x\rfloor$ (d) $\lim\limits_{x\to -2^-} \lfloor x\rfloor$

(e) $\lim\limits_{x\to -2.3} \lfloor x\rfloor$ (f) $\lim\limits_{x\to 3} \left\lfloor \dfrac{x}{2}\right\rfloor$

(g) $\lim\limits_{x\to 3} \dfrac{\lfloor x\rfloor}{2}$ (h) $\lim\limits_{x\to 0^+} \dfrac{\lfloor 2+x\rfloor - \lfloor 2\rfloor}{x}$

11. For $f(x)$ and $g(x)$ defined as:

$$f(x)=\begin{cases} 1 & \text{if } x<1 \\ x & \text{if } 1<x \end{cases} \qquad g(x)=\begin{cases} x & \text{if } x\neq 2 \\ 3 & \text{if } x=2 \end{cases}$$

determine the following limits:

(a) $\lim\limits_{x\to 2} [f(x)+g(x)]$ (b) $\lim\limits_{x\to 2} \dfrac{f(x)}{g(x)}$

(c) $\lim\limits_{x\to 2} f(g(x))$ (d) $\lim\limits_{x\to 0} \dfrac{g(x)}{f(x)}$

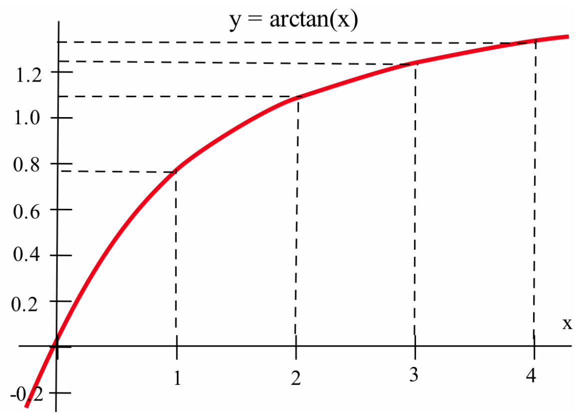
(e) $\lim\limits_{x\to 1} \dfrac{f(x)}{g(x)}$ (f) $\lim\limits_{x\to 1} g(f(x))$

12. Give geometric interpretations for each limit and use a calculator to estimate its value.

(a) $\displaystyle\lim_{h\to 0} \frac{\arctan(0+h) - \arctan(0)}{h}$

(b) $\displaystyle\lim_{h\to 0} \frac{\arctan(1+h) - \arctan(1)}{h}$

(c) $\displaystyle\lim_{h\to 0} \frac{\arctan(2+h) - \arctan(2)}{h}$




13. (a) What does $\displaystyle\lim_{h\to 0} \frac{\cos(h) - 1}{h}$ represent in relation to the graph of $y = \cos(x)$? It may help to recognize that:

$$\frac{\cos(h) - 1}{h} = \frac{\cos(0+h) - \cos(0)}{h}$$

(b) Graphically and using your calculator, estimate $\displaystyle\lim_{h\to 0} \frac{\cos(h) - 1}{h}$.

14. (a) What does the ratio $\dfrac{\ln(1+h)}{h}$ represent in relation to the graph of $y = \ln(x)$? It may help to recognize that:

$$\frac{\ln(1+h)}{h} = \frac{\ln(1+h) - \ln(1)}{h}$$

(b) Graphically and using your calculator, determine $\displaystyle\lim_{h\to 0} \frac{\ln(1+h)}{h}$.

15. Use your calculator (to generate a table of values) to help you estimate the value of each limit.

(a) $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h}$

(b) $\displaystyle\lim_{c\to 0} \frac{\tan(1+c) - \tan(1)}{c}$

(c) $\displaystyle\lim_{t\to 0} \frac{g(2+t) - g(2)}{t}$ when $g(t) = t^2 - 5$.

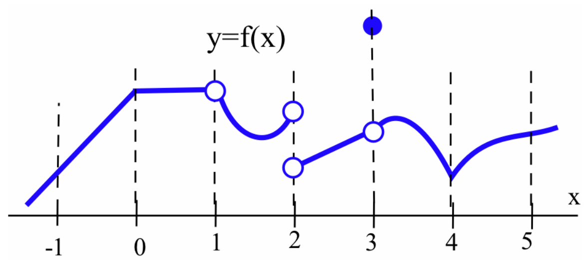
16. (a) For $h > 0$, find the slope of the line through the points $(h, |h|)$ and $(0, 0)$.

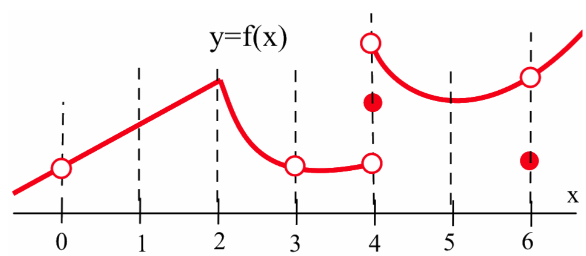
(b) For $h < 0$, find the slope of the line through the points $(h, |h|)$ and $(0, 0)$.

(c) Evaluate $\displaystyle\lim_{h\to 0^-} \frac{|h|}{h}$, $\displaystyle\lim_{h\to 0^+} \frac{|h|}{h}$ and $\displaystyle\lim_{h\to 0} \frac{|h|}{h}$.

In 17–18, describe the behavior at each integer of the function $y = f(x)$ in the figure provided, using one of these phrases:

- "connected and smooth"
- "connected with a corner"
- "not connected because of a simple hole that could be plugged by adding or moving one point"
- "not connected because of a vertical jump that could not be plugged by moving one point"

17. 


18. 


19. Use the list method to show that $\displaystyle\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist .

20. Show that $\displaystyle\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist. (Suggestion: Let $f(x) = \sin\left(\frac{1}{x}\right)$ and let $a_n = \frac{1}{n\pi}$ so that $f(a_n) = \sin\left(\frac{1}{a_n}\right) = \sin(n\pi) = 0$ for every $n$. Then pick $b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ so that $f(b_n) = \sin\left(\frac{1}{b_n}\right) = \sin(2n\pi + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$ for all $n$.)

In Problems 21–26, use the Squeezing Theorem to help evaluate each limit.

21. $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right)$

22. $\lim_{x\to 0} \sqrt[3]{x} \sin\left(\frac{1}{x^3}\right)$

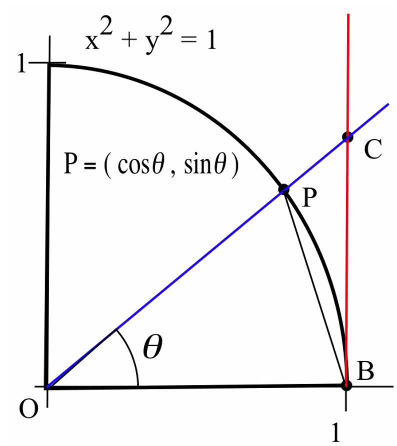
23. $\lim_{x\to 0} 3 + x^2 \sin\left(\frac{1}{x}\right)$

24. $\lim_{x\to 1^-} \sqrt{1-x^2} \cos\left(\frac{1}{x-1}\right)$

25. $\lim_{x\to 0} x^2 \cdot \left\lfloor \frac{1}{x^2} \right\rfloor$

26. $\lim_{x\to 0} (-1)^{\lfloor \frac{1}{x} \rfloor} (1-\cos(x))$

27. This problem outlines the steps of a proof that $\lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1$. Refer to the margin figure, assume that $0 < \theta < \frac{\pi}{2}$, and justify why each statement must be true.




(a) Area of $\triangle OPB = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\sin(\theta)$

(b) $\dfrac{\text{area of the sector (the pie shaped region) } OPB}{\text{area of the whole circle}} = \dfrac{\theta}{2\pi}$

(c) area of the sector $OPB = \pi \cdot \dfrac{\theta}{2\pi} = \dfrac{\theta}{2}$

(d) The line $L$ through the points $(0,0)$ and $P = (\cos(\theta), \sin(\theta))$ has slope $m = \dfrac{\sin(\theta)}{\cos(\theta)}$, so $C = (1, \dfrac{\sin(\theta)}{\cos(\theta)})$

(e) area of $\triangle OCB = \dfrac{1}{2}(\text{base})(\text{height}) = \dfrac{1}{2}(1)\dfrac{\sin(\theta)}{\cos(\theta)}$

(f) area of $\triangle OPB$ < area of sector $OPB$ < area of $\triangle OCB$

(g) $\dfrac{1}{2}\sin(\theta) < \dfrac{\theta}{2} < \dfrac{1}{2}(1)\dfrac{\sin(\theta)}{\cos(\theta)} \Rightarrow \sin(\theta) < \theta < \dfrac{\sin(\theta)}{\cos(\theta)}$

(h) $1 < \dfrac{\theta}{\sin(\theta)} < \dfrac{1}{\cos(\theta)} \Rightarrow 1 > \dfrac{\sin(\theta)}{\theta} > \cos(\theta)$

(i) $\lim_{\theta\to 0^+} 1 = 1$ and $\lim_{\theta\to 0^+} \cos(\theta) = 1$.

(j) $\lim_{\theta\to 0^+} \dfrac{\sin(\theta)}{\theta} = 1$

## *1.2 Practice Answers*

1. (a) $-10$ (b) 24 (c) $\frac{3}{2}$ (d) 0 (e) 0 (f) $\frac{5}{4}$ (g) $-64$ (h) 2

2. (a) 39 (b) $-\frac{3}{5}$ (c) $\frac{2}{3}$

3. (a) 0 (b) 2 (c) 3 (d) 1

4. (a) slope of the line tangent to the graph of $g$ at the point $(1, g(1))$; estimated slope $\approx -2$
   (b) slope of the line tangent to the graph of $g$ at the point $(3, g(3))$; estimated slope $\approx 0$
   (c) slope of the line tangent to the graph of $g$ at the point $(0, g(0))$; estimated slope $\approx 1$

5. $\lim_{x\to 1} \left[x^2+2\right] = 3$ and $\lim_{x\to 1} [2x+1] = 3$ so $\lim_{x\to 1} f(x) = 3$

6. $\lim_{x\to 0} \cos(x) = 1$ and $\lim_{x\to 0} 1 = 1$ so $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$

## 1.3 Continuous Functions

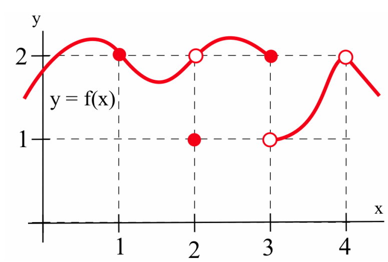
In Section 1.2 we saw a few "nice" functions whose limits as $x \to a$ simply involved substituting $a$ into the function: $\lim_{x \to a} f(x) = f(a)$. Functions whose limits have this substitution property are called **continuous functions** and such functions possess a number of other useful properties.

In this section we will examine what it means graphically for a function to be continuous (or not continuous), state some properties of continuous functions, and look at a few applications of these properties—including a way to solve horrible equations such as $\sin(x) = \frac{2x+1}{x-2}$.

### Definition of a Continuous Function

We begin by formally stating the definition of this new concept.

> **Definition of Continuity at a Point**:
>
> A function $f$ is **continuous** at $x = a$ if and only if
> $$\lim_{x \to a} f(x) = f(a).$$




| $a$ | $f(a)$ | $\lim_{x \to a} f(x)$ |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 1 | 2 |
| 3 | 2 | DNE |
| 4 | undefined | 2 |

The graph in the margin illustrates some of the different ways a function can behave at and near a point, and the accompanying table contains some numerical information about the example function $f$ and its behavior. We can conclude from the information in the table that $f$ is continuous at 1 because $\lim_{x \to 1} f(x) = 2 = f(1)$.

We can also conclude that $f$ is not continuous at 2 or 3 or 4, because $\lim_{x \to 2} f(x) \neq f(2)$, $\lim_{x \to 3} f(x) \neq f(3)$ and $\lim_{x \to 4} f(x) \neq f(4)$.

### Graphical Meaning of Continuity

When $x$ is close to 1, the values of $f(x)$ are close to the value $f(1)$, and the graph of $f$ does not have a hole or break at $x = 1$. The graph of $f$ is "connected" at $x = 1$ and can be drawn without lifting your pencil. At $x = 2$ and $x = 4$ the graph of $f$ has "holes," and at $x = 3$ the graph has a "break." The function $f$ is also continuous at 1.7 (why?) and at every point shown *except* at 2, 3 and 4.

> **Informally**, we can say:
>
> - A function is **continuous** at a point if the graph of the function is **connected** there.
> - A function is **not continuous** at a point if its graph has a **hole** or **break** at that point.

Sometimes the definition of "continuous" (the substitution condition for limits) is easier to use if we chop it into several smaller pieces and then check whether or not our function satisfies each piece.

$f$ is continuous at $a$ if and only if:

(i) $f$ is defined at $a$

(ii) the limit of $f(x)$, as $x \to a$, exists
(so the left limit and right limits exist and are equal)

(iii) the value of $f$ at $a$ equals the value of the limit as $x \to a$:

$$\lim_{x \to a} f(x) = f(a)$$

If $f$ satisfies conditions (i), (ii) and (iii), then $f$ is continuous at $a$. If $f$ does not satisfy one or more of the three conditions at $a$, then $f$ is not continuous at $a$.

For $f(x)$ in the figure on the previous page, all three conditions are satisfied for $a = 1$, so $f$ is continuous at 1. For $a = 2$, conditions (i) and (ii) are satisfied but not (iii), so $f$ is not continuous at 2. For $a = 3$, condition (i) is satisfied but (ii) is violated, so $f$ is not continuous at 3. For $a = 4$, condition (i) is violated, so $f$ is not continuous at 4.

A function is **continuous on an interval** if it is continuous at every point in the interval.

A function $f$ is **continuous from the left** at $a$ if $\lim_{x \to a^-} f(x) = f(a)$ and is **continuous from the right** at $a$ if $\lim_{x \to a^+} f(x) = f(a)$.

**Example 1.** Is the function

$$f(x) = \begin{cases} x+1 & \text{if } x \leq 1 \\ 2 & \text{if } 1 < x \leq 2 \\ \frac{1}{x-3} & \text{if } x > 2 \end{cases}$$

continuous at $x = 1$? At $x = 2$? At $x = 3$?

**Solution.** We could answer these questions by examining a graph of $f(x)$, but let's try them without the benefit of a graph. At $x = 1$, $f(1) = 2$ and the left and right limits are equal:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} [x+1] = 2 = \lim_{x \to 1^+} 2 = \lim_{x \to 1^+} f(x)$$

and their common limit matches the value of the function at $x = 1$:

$$\lim_{x \to 1} f(x) = 2 = f(1)$$

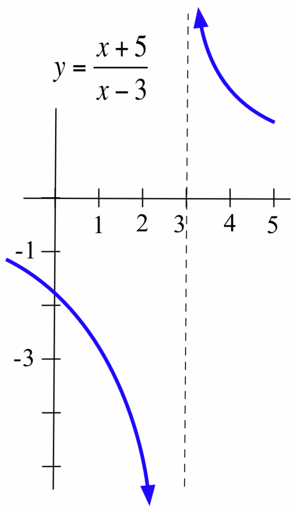
so $f$ is continuous at 1.

At $x = 2$, $f(2) = 2$, but the left and right limits are not equal:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 1^-} 2 = 2 \neq -1 = \lim_{x\to 2^+} \frac{1}{x-3} = \lim_{x\to 2^+} f(x)$$

so $f$ fails condition (ii), hence is not continuous at 2. We can, however, say that $f$ is continuous from the left (but not from the right) at 2.

At $x = 3$, $f(3) = \frac{1}{0}$, which is undefined, so $f$ is not continuous at 3 because it fails condition (i). ◀

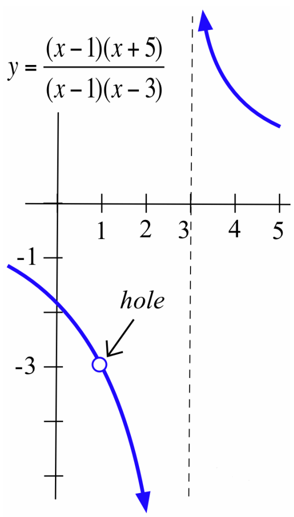



**Example 2.** Where is $f(x) = 3x^2 - 2x$ continuous?

**Solution.** By the Substitution Theorem for Polynomial and Rational Functions, $\lim_{x\to a} P(x) = P(a)$ for any polynomial $P(x)$ at any point $a$, so every polynomial is continuous everywhere. In particular, $f(x) = 3x^2 - 2x$ is continuous everywhere. ◀

**Example 3.** Where is the function $g(x) = \frac{x+5}{x-3}$ continuous? Where is $h(x) = \frac{x^2+4x-5}{x^2-4x+3}$ continuous?

**Solution.** Because $g(x)$ is a rational function, the Substitution Theorem for Polynomial and Rational Functions says it is continuous everywhere except where its denominator is 0: $g$ is continuous everywhere except at $x = 3$. The graph of $g$ (see margin) is "connected" everywhere except at $x = 3$, where it has a vertical asymptote.



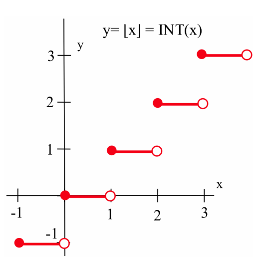

We can rewrite the rational function $h(x)$ as:

$$h(x) = \frac{(x-1)(x+5)}{(x-1)(x-3)}$$

and note that its denominator is 0 at $x = 1$ and $x = 3$, so $h$ is continuous everywhere except 3 and 1. The graph of $h$ (see margin) is "connected" everywhere except at 3, where it has a vertical asymptote, and 1, where it has a hole: $f(1) = \frac{0}{0}$ is undefined. ◀

**Example 4.** Where is $f(x) = \lfloor x \rfloor$ continuous?

**Solution.** The graph of $y = \lfloor x \rfloor$ seems to be "connected" except at each integer, where there is a "jump" (see margin).




If $a$ is an integer, then $\lim_{x\to a^-} \lfloor x \rfloor = a - 1$ and $\lim_{x\to a^+} \lfloor x \rfloor = a$ so $\lim_{x\to a} \lfloor x \rfloor$ is undefined, and $\lfloor x \rfloor$ is not continuous at $x = a$.

If $a$ is *not* an integer, then the left and right limits of $\lfloor x \rfloor$, as $x \to a$, both equal $\lfloor a \rfloor$ so: $\lim_{x\to a} \lfloor x \rfloor = \lfloor a \rfloor$, hence $\lfloor x \rfloor$ is continuous at $x = a$.

Summarizing: $\lfloor x \rfloor$ is continuous everywhere except at the integers. In fact, $f(x) = \lfloor x \rfloor$ is continuous from the right everywhere and is continuous from the left everywhere except at the integers. ◀

**Practice 1.** Where is $f(x) = \frac{|x|}{x}$ continuous?

*Why Do We Care Whether a Function Is Continuous?*

There are several reasons for us to examine continuous functions and their properties:

- Many applications in engineering, the sciences and business are continuous or are modeled by continuous functions or by pieces of continuous functions.
- Continuous functions share a number of useful properties that do not necessarily hold true if the function is not continuous. If a result is true of all continuous functions and we have a continuous function, then the result is true for our function. This can save us from having to show, one by one, that each result is true for each particular function we use. Some of these properties are given in the remainder of this section.
- Differential calculus has been called the study of **continuous** change, and many of the results of calculus are guaranteed to be true only for continuous functions. If you look ahead into Chapters 2 and 3, you will see that many of the theorems have the form "If $f$ is continuous and (some additional hypothesis), then (some conclusion)."

*Combinations of Continuous Functions*

Not only are most of the basic functions we will encounter continuous at most points, so are basic combinations of those functions.

> **Theorem**:
>
> If $f(x)$ and $g(x)$ are continuous at $a$
> and $k$ is any constant
>
> then the elementary combinations of $f$ and $g$
>
> - $k \cdot f(x)$
> - $f(x) + g(x)$
> - $f(x) - g(x)$
> - $f(x) \cdot g(x)$
> - $\dfrac{f(x)}{g(x)}$ (as long as $g(a) \neq 0$)
>
> are continuous at $a$.

The continuity of a function is defined using limits, and all of these results about simple combinations of continuous functions follow from the results about combinations of limits in the Main Limit Theorem.

Our hypothesis is that $f$ and $g$ are both continuous at $a$, so we can assume that

$$\lim_{x\to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x\to a} g(x) = g(a)$$

and then use the appropriate part of the Main Limit Theorem.

For example,

$$\lim_{x\to a} [f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = f(a)+g(a)$$

so $f+g$ is continuous at $a$.

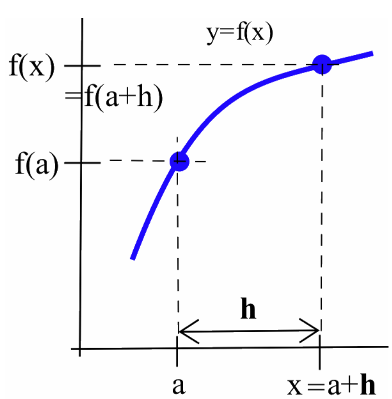
**Practice 2.** Prove: If $f$ and $g$ are continuous at $a$, then $k \cdot f$ and $f-g$ are continuous at $a$ (where $k$ a constant).

> **Composition of Continuous Functions**:
>
> If $\quad g(x)$ is continuous at $a$ and
> $\quad f(x)$ is continuous at $g(a)$
>
> then $\quad \lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(g(a))$
> so $f \circ g(x) = f(g(x))$ is continuous at $a$.

The proof of this result involves some technical details, but just formalizes the following line of reasoning:

The hypothesis that "$g$ is continuous at $a$" means that if $x$ is close to $a$ then $g(x)$ will be close to $g(a)$. Similarly, "$f$ is continuous at $g(a)$" means that if $g(x)$ is close to $g(a)$ then $f(g(x)) = f \circ g(x)$ will be close to $f(g(a)) = f \circ g(a)$. Finally, we can conclude that if $x$ is close to $a$, then $g(x)$ is close to $g(a)$ so $f \circ g(x)$ is close to $f \circ g(a)$ and therefore $f \circ g$ is continuous at $x = a$.

The next theorem presents an alternate version of the limit condition for continuity, which we will use occasionally in the future.




> **Theorem**:
>
> $$\lim_{x\to a} f(x) = f(a) \quad \text{if and only if} \quad \lim_{h\to 0} f(a+h) = f(a)$$

*Proof.* Let's define a new variable $h$ by $h = x-a$ so that $x = a+h$ (see margin figure). Then $x \to a$ if and only if $h = x-a \to 0$, so $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$ and therefore $\lim_{x\to a} f(x) = f(a)$ if and only if $\lim_{h\to 0} f(a+h) = f(a)$. □

We can restate the result of this theorem as:

> A function $f$ is continuous at $a$ if and only if $\lim_{h\to 0} f(a+h) = f(a)$.

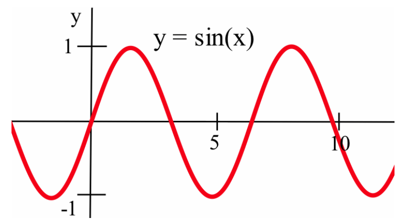
*Which Functions Are Continuous?*

Fortunately, the functions we encounter most often are either continuous everywhere or continuous everywhere except at a few places.

**Theorem**: The following functions are continuous everywhere

(a) polynomials (b) $\sin(x)$ and $\cos(x)$ (c) $|x|$

*Proof.* (a) This follows from the Substitution Theorem for Polynomial and Rational Functions and the definition of continuity.

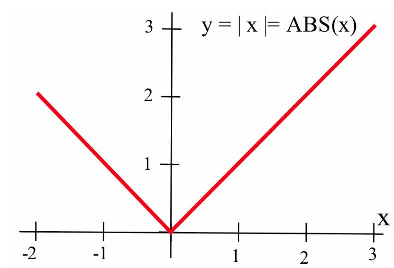



(b) The graph of $y = \sin(x)$ (see margin) clearly has no holes or breaks, so it is reasonable to think that $\sin(x)$ is continuous everywhere. Justifying this algebraically, for every real number $a$:

$$\begin{aligned}\lim_{h\to 0} \sin(a+h) &= \lim_{h\to 0} [\sin(a)\cos(h) + \cos(a)\sin(h)] \\ &= \lim_{h\to 0} \sin(a) \cdot \lim_{h\to 0} \cos(h) + \lim_{h\to 0} \cos(a) \cdot \lim_{h\to 0} \sin(h) \\ &= \sin(a)\cdot 1 + \cos(a)\cdot 0 = \sin(a)\end{aligned}$$

Recall the angle addition formula for $\sin(\theta)$ and the results from Section 1.2 that $\lim_{h\to 0} \cos(h) = 1$ and $\lim_{h\to 0} \sin(h) = 0$.

so $f(x) = \sin(x)$ is continuous at every point. The justification for $f(x) = \cos(x)$ is similar.




(c) For $f(x) = |x|$, when $x > 0$, then $|x| = x$ and its graph (see margin) is a straight line and is continuous because $x$ is a polynomial. When $x < 0$, then $|x| = -x$ and it is also continuous. The only questionable point is the "corner" on the graph when $x = 0$, but the graph there is only bent, not broken:

$$\lim_{h\to 0^+} |0+h| = \lim_{h\to 0^+} h = 0$$

and:

$$\lim_{h\to 0^-} |0+h| = \lim_{h\to 0^-} -h = 0$$

so:

$$\lim_{h\to 0} |0+h| = 0 = |0|$$

and $f(x) = |x|$ is also continuous at 0. □

**A continuous function can have corners but not holes or breaks.**

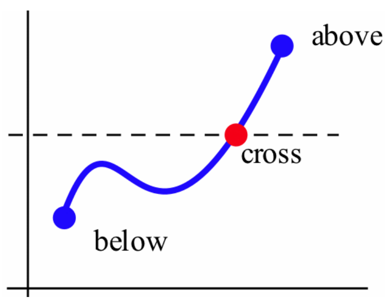
Even functions that fail to be continuous at some points are often continuous most places:

- A rational function is continuous *except* where the denominator is 0.
- The trig functions $\tan(x)$, $\cot(x)$, $\sec(x)$ and $\csc(x)$ are continuous *except* where they are undefined.

- The greatest integer function $\lfloor x \rfloor$ is continuous *except* at each integer.
- But the "holey" function

$$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

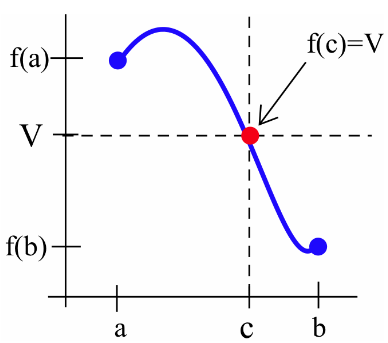
  is **discontinuous everywhere**.




*Intermediate Value Property of Continuous Functions*

Because the graph of a continuous function is connected and does not have any holes or breaks in it, the values of the function can not "skip" or "jump over" a horizontal line (see margin figure). If one value of the continuous function is below the line and another value of the function is above the line, then **somewhere** the graph will cross the line. The next theorem makes this statement more precise. The result seems obvious, but its proof is technically difficult and is not given here.

> **Intermediate Value Theorem for Continuous Functions**:
>
> If $f$ is continuous on the interval $[a, b]$ and $V$ is any value between $f(a)$ and $f(b)$
>
> then there is a number $c$ between $a$ and $b$ so that $f(c) = V$. (That is, $f$ actually takes on each intermediate value between $f(a)$ and $f(b)$.)




If the graph of $f$ connects the points $(a, f(a))$ and $(b, f(b))$ and $V$ is any number between $f(a)$ and $f(b)$, then the graph of $f$ must cross the horizontal line $y = V$ somewhere between $x = a$ and $x = b$ (see margin figure). Since $f$ is continuous, its graph cannot "hop" over the line $y = V$.

We often take this theorem for granted in some common situations:

- If a child's temperature rose from 98.6°F to 101.3°F, then there was an instant when the child's temperature was exactly 100°F. (In fact, every temperature between 98.6°F and 101.3°F occurred at some instant.)
- If you dove to pick up a shell 25 feet below the surface of a lagoon, then at some instant in time you were 17 feet below the surface. (Actually, you want to be at 17 feet twice. Why?)
- If you started driving from a stop (velocity = 0) and accelerated to a velocity of 30 kilometers per hour, then there was an instant when your velocity was exactly 10 kilometers per hour.

But we cannot apply the Intermediate Value Theorem if the function is not continuous:

- In 1987 it cost 22¢ to mail a first-class letter inside the United States, and in 1990 it cost 25¢ to mail the same letter, but we cannot conclude that there was a time when it cost 23¢ or 24¢ or 24.7¢ to send the letter. (Postal rates did not increase in a continuous fashion. They jumped directly from 22¢ to 25¢.)
- Prices, taxes and rates of pay change in jumps — discrete steps — without taking on the intermediate values.

The Intermediate Value Theorem (IVT) is an example of an "existence theorem": it concludes that something exists (a number $c$ so that $f(c) = V$). But like many existence theorems, it does not tell us how to find the the thing that exists (the value of $c$) and is of no use in actually finding those numbers or objects.

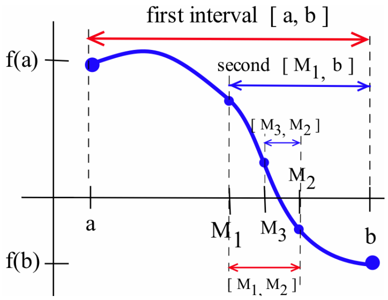
## *Bisection Algorithm for Approximating Roots*

The IVT can help us finds roots of functions and solve equations. If $f$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs (one is positive and one is negative), then 0 is an intermediate value between $f(a)$ and $f(b)$ so $f$ will have a root $c$ between $x = a$ and $x = b$ where $f(c) = 0$.

While the IVT does not tell us how to find $c$, it lays the groundwork for a method commonly used to approximate the roots of continuous functions.

**Bisection Algorithm for Finding a Root of $f(x)$**

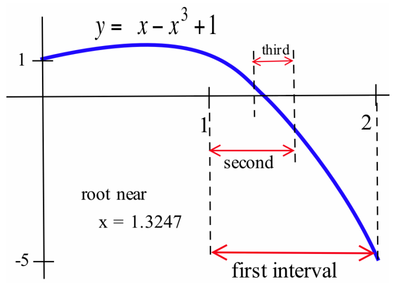
1. Find two values of $x$ (call them $a$ and $b$) so that $f(a)$ and $f(b)$ have opposite signs. (The IVT will then guarantee that $f(x)$ has a root between $a$ and $b$.)
2. Calculate the midpoint (or **bisection point**) of the interval $[a, b]$, using the formula $m = \dfrac{a+b}{2}$, and evaluate $f(m)$.
3. (a) If $f(m) = 0$, then $m$ is a root of $f$ and we are done.
   (b) If $f(m) \neq 0$, then $f(m)$ has the sign opposite $f(a)$ or $f(b)$:
      i. if $f(a)$ and $f(m)$ have opposite signs, then $f$ has a root in $[a, m]$ so put $b = m$
      ii. if $f(b)$ and $f(m)$ have opposite signs, then $f$ has a root in $[m, b]$ so put $a = m$
4. Repeat steps 2 and 3 until a root is found exactly or is approximated closely enough.




The length of the interval known to contain a root is cut in half each time through steps 2 and 3, so the Bisection Algorithm quickly "squeezes" in on a root (see margin figure).

The steps of the Bisection Algorithm can be done "by hand," but it is tedious to do very many of them that way. Computers are very good with this type of tedious repetition, and the algorithm is simple to program.

**Example 5.** Find a root of $f(x) = -x^3 + x + 1$.

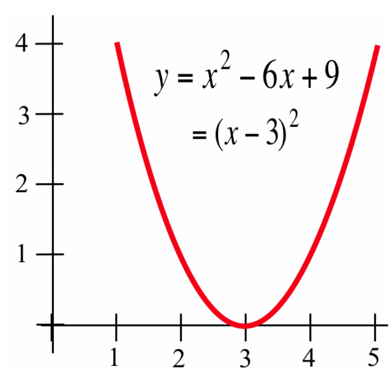



**Solution.** $f(0) = 1$ and $f(1) = 1$ so we cannot conclude that $f$ has a root between 0 and 1. $f(1) = 1$ and $f(2) = -5$ have opposite signs, so by the IVT (this function is a polynomial, so it is continuous everywhere and the IVT applies) we know that there is a number $c$ between 1 and 2 such that $f(c) = 0$ (see figure). The midpoint of the interval $[1,2]$ is $m = \frac{1+2}{2} = \frac{3}{2} = 1.5$ and $f(\frac{3}{2}) = -\frac{7}{8}$ so $f$ changes sign between 1 and 1.5 and we can be sure that there is a root between 1 and 1.5. If we repeat the operation for the interval $[1, 1.5]$, the midpoint is $m = \frac{1+1.5}{2} = 1.25$, and $f(1.25) = \frac{19}{64} > 0$ so $f$ changes sign between 1.25 and 1.5 and we know $f$ has a root between 1.25 and 1.5.

Repeating this procedure a few more times, we get:

| $a$ | $b$ | $m = \frac{b+a}{2}$ | $f(a)$ | $f(b)$ | $f(m)$ | root between | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | | 1 | $-5$ | | 1 | 2 |
| 1 | 2 | 1.5 | 1 | $-5$ | $-0.875$ | 1 | 1.5 |
| 1 | 1.5 | 1.25 | 1 | $-0.875$ | 0.2969 | 1.25 | 1.5 |
| 1.25 | 1.5 | 1.375 | 0.2969 | $-0.875$ | $-0.2246$ | 1.25 | 1.375 |
| 1.25 | 1.375 | 1.3125 | 0.2969 | $-0.2246$ | 0.0515 | 1.3125 | 1.375 |
| 1.3125 | 1.375 | 1.34375 | | | | | |

If we continue the table, the interval containing the root will squeeze around the value 1.324718. ◀




The Bisection Algorithm has one major drawback: there are some roots it does not find. The algorithm requires that the function take on both positive and negative values near the root so that the graph actually crosses the $x$-axis. The function $f(x) = x^2 - 6x + 9 = (x-3)^2$ has the root $x = 3$ but is never negative (see margin figure). We cannot find two starting points $a$ and $b$ so that $f(a)$ and $f(b)$ have opposite signs, so we cannot use the Bisection Algorithm to find the root $x = 3$. In Chapter 2 we will see another method—Newton's Method—that does find roots of this type.

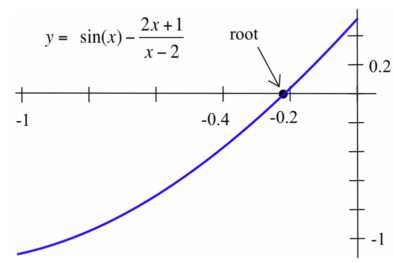
The Bisection Algorithm requires that we supply two starting $x$-values, $a$ and $b$, at which the function has opposite signs. These values can often be found with a little "trial and error," or we can examine the graph of the function to help pick the two values.

Finally, the Bisection Algorithm can also be used to solve equations, because the solution of any equation can always be transformed into an equivalent problem of finding roots by moving everything to one side of the equal sign. For example, the problem of solving the equation $x^3 = x + 1$ can be transformed into the equivalent problem of solving $x^3 - x - 1 = 0$ or of finding the roots of $f(x) = x^3 - x - 1$, which is equivalent to the problem we solved in the previous example.

**Example 6.** Find all solutions of $\sin(x) = \dfrac{2x+1}{x-2}$ (with $x$ in radians.)

**Solution.** We can convert this problem of solving an equation to the problem of finding the roots of

$$f(x) = \sin(x) - \frac{2x+1}{x-2} = 0$$

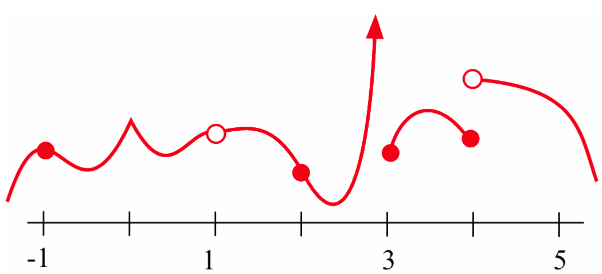



The function $f(x)$ is continuous everywhere except at $x = 2$, and the graph of $f(x)$ (in the margin) can help us find two starting values for the Bisection Algorithm. The graph shows that $f(-1)$ is negative and $f(0)$ is positive, and we know $f(x)$ is continuous on the interval $[-1, 0]$. Using the algorithm with the starting interval $[-1, 0]$, we know that a root is contained in the shrinking intervals $[-0.5, 0]$, $[-0.25, 0]$, $[-0.25, -0.125]$,…, $[-0.238281, -0.236328]$,…, $[-0.237176, -0.237177]$ so the root is approximately $-0.237177$.

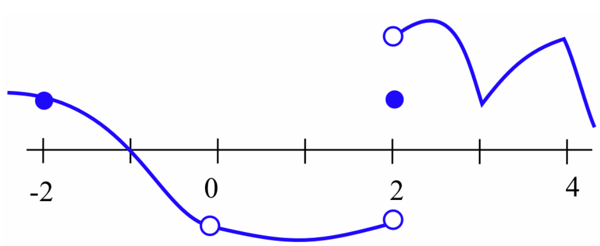
We might notice that $f(0) = 0.5 > 0$ while $f(\pi) = 0 - \frac{2\pi+1}{\pi-2} \approx -6.38 < 0$. Why is it wrong to conclude that $f(x)$ has another root between $x = 0$ and $x = \pi$? ◀

## *1.3 Problems*

1. At which points is the function in the graph below discontinuous?




2. At which points is the function in the graph below discontinuous?



3. Find at least one point at which each function is not continuous and state which of the three conditions in the definition of continuity is violated at that point.

(a) $\dfrac{x+5}{x-3}$ (b) $\dfrac{x^2+x-6}{x-2}$

(c) $\sqrt{\cos(x)}$ (d) $\lfloor x^2 \rfloor$

(e) $\dfrac{x}{\sin(x)}$ (f) $\dfrac{x}{x}$

(g) $\ln(x^2)$ (h) $\dfrac{\pi}{x^2-6x+9}$

(i) $\tan(x)$

4. Which *two* of the following functions are not continuous? Use appropriate theorems to justify that each of the other functions is continuous.

(a) $\dfrac{7}{\sqrt{2+\sin(x)}}$ (b) $\cos^2(x^5-7x+\pi)$

(c) $\dfrac{x^2-5}{1+\cos^2(x)}$ (d) $\dfrac{x^2-5}{1+\cos(x)}$

(e) $\lfloor 3+0.5\sin(x) \rfloor$ (f) $\lfloor 0.3\sin(x)+1.5 \rfloor$

(g) $\sqrt{\cos(\sin(x))}$ (h) $\sqrt{x^2-6x+10}$

(i) $\sqrt[3]{\cos(x)}$ (j) $2^{\sin(x)}$

(k) $1-3^{-x}$ (l) $\arctan(1-x^2)$

5. A continuous function $f$ has the values:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f(x)$ | 5 | 3 | $-2$ | $-1$ | 3 | $-2$ |

(a) $f$ has at least ______ roots between 0 and 5.

(b) $f(x) = 4$ in at least ______ places between $x = 0$ and $x = 5$.

(c) $f(x) = 2$ in at least ______ places between $x = 0$ and $x = 5$.

(d) $f(x) = 3$ in at least ______ places between $x = 0$ and $x = 5$.

(e) Is it possible for $f(x)$ to equal 7 for some $x$-value(s) between 0 and 5?

6. A continuous function $g$ has the values:

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $g(x)$ | $-3$ | 1 | 4 | $-1$ | 3 | $-2$ | $-1$ |

(a) $g$ has at least ______ roots between 1 and 5.

(b) $g(x) = 3.2$ in at least ______ places between $x = 1$ and $x = 7$.

(c) $g(x) = -0.7$ in at least ______ places between $x = 3$ and $x = 7$.

(d) $g(x) = 1.3$ in at least ______ places between $x = 2$ and $x = 6$.

(e) Is it possible for $g(x)$ to equal $\pi$ for some $x$-value(s) between 5 and 6?

7. This problem asks you to verify that the Intermediate Value Theorem is true for some particular functions, intervals and intermediate values. In each problem you are given a function $f$, an interval $[a,b]$ and a value $V$. Verify that $V$ is between $f(a)$ and $f(b)$ and find a value of $c$ in the given interval so that $f(c) = V$.

(a) $f(x) = x^2$ on $[0,3]$, $V = 2$

(b) $f(x) = x^2$ on $[-1,2]$, $V = 3$

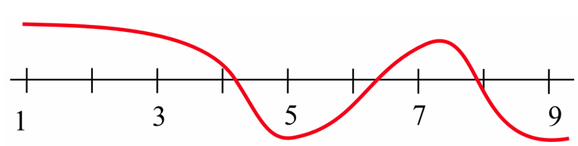
(c) $f(x) = \sin(x)$ on $\left[0, \frac{\pi}{2}\right]$, $V = \frac{1}{2}$

(d) $f(x) = x$ on $[0,1]$, $V = \frac{1}{3}$

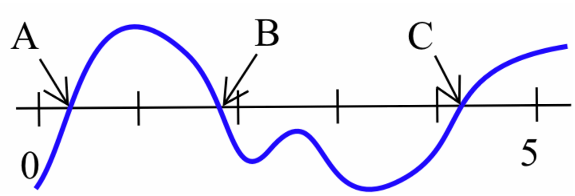
(e) $f(x) = x^2 - x$ on $[2,5]$, $V = 4$

(f) $f(x) = \ln(x)$ on $[1,10]$, $V = 2$

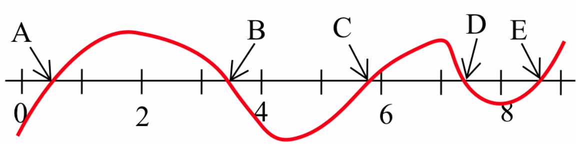
8. Two students claim that they both started with the points $x = 1$ and $x = 9$ and applied the Bisection Algorithm to the function graphed below. The first student says that the algorithm converged to the root near $x = 8$, but the second claims that the algorithm will converge to the root near $x = 4$. Who is correct?



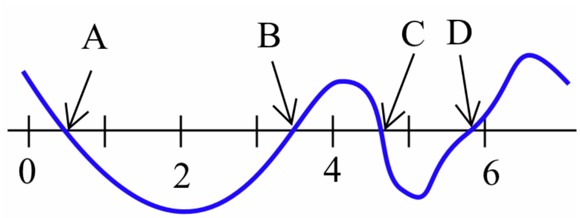
9. Two students claim that they both started with the points $x = 0$ and $x = 5$ and applied the Bisection Algorithm to the function graphed below. The first student says that the algorithm converged to the root labeled $A$, but the second claims that the algorithm will converge to the root labeled $B$. Who is correct?




10. If you apply the Bisection Algorithm to the function graphed below, which root does the algorithm find if you use:
    (a) starting points 0 and 9?
    (b) starting points 1 and 5?
    (c) starting points 3 and 5?

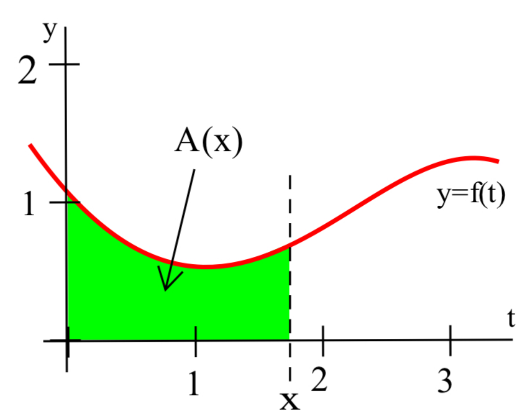



11. If you apply the Bisection Algorithm to the function graphed below, which root does the algorithm find if you use:
    (a) starting points 3 and 7?
    (b) starting points 5 and 6?
    (c) starting points 1 and 6?




In 12–17, use the IVT to verify each function has a root in the given interval(s). Then use the Bisection Algorithm to narrow the location of that root to an interval of length less than or equal to 0.1.

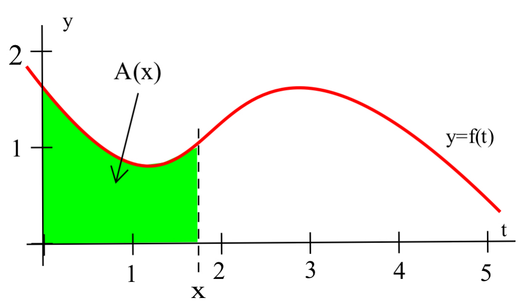
12. $f(x) = x^2 - 2$ on $[0, 3]$
13. $g(x) = x^3 - 3x^2 + 3$ on $[-1, 0]$, $[1, 2]$, $[2, 4]$
14. $h(t) = t^5 - 3t + 1$ on $[1, 3]$
15. $r(x) = 5 - 2^x$ on $[1, 3]$
16. $s(x) = \sin(2x) - \cos(x)$ on $[0, \pi]$
17. $p(t) = t^3 + 3t + 1$ on $[-1, 1]$
18. Explain what is wrong with this reasoning: If $f(x) = \frac{1}{x}$ then
$$f(-1) = -1 < 0 \quad \text{and} \quad f(1) = 1 > 0$$
so $f$ must have a root between $x = -1$ and $x = 1$.
19. Each of the following statements is false for some functions. For each statement, sketch the graph of a counterexample.
    (a) If $f(3) = 5$ and $f(7) = -3$, then $f$ has a root between $x = 3$ and $x = 7$.
    (b) If $f$ has a root between $x = 2$ and $x = 5$, then $f(2)$ and $f(5)$ have opposite signs.
    (c) If the graph of a function has a sharp corner, then the function is not continuous there.
20. Define $A(x)$ to be the **area** bounded by the $t$- and $y$-axes, the curve $y = f(t)$, and the vertical line $t = x$ (see figure below). It is clear that $A(1) < 2$ and $A(3) > 2$. Do you think there is a value of $x$ between 1 and 3 so that $A(x) = 2$? If so, justify your conclusion and estimate the location of the value of $x$ that makes $A(x) = 2$. If not, justify your conclusion.



21. Define $A(x)$ to be the **area** bounded by the $t$- and $y$-axes, the curve $y = f(t)$, and the vertical line $t = x$ (see figure below).

    (a) Shade the part of the graph represented by $A(2.1) - A(2)$ and estimate the value of $\dfrac{A(2.1) - A(2)}{0.1}$.

    (b) Shade the part of the graph represented by $A(4.1) - A(4)$ and estimate the value of $\dfrac{A(4.1) - A(4)}{0.1}$.




22. (a) A square sheet of paper has a straight line drawn on it from the lower-left corner to the upper-right corner. Is it possible for you to start on the left edge of the sheet and draw a "connected" line to the right edge that does not cross the diagonal line?

    (b) Prove: If $f$ is continuous on the interval $[0,1]$ and $0 \leq f(x) \leq 1$ for all $x$, then there is a number $c$ with $0 \leq c \leq 1$ such that $f(c) = c$. (The number $c$ is called a "fixed point" of $f$ because the image of $c$ is the same as $c$: $f$ does not "move" $c$.) Hint: Define a new function $g(x) = f(x) - x$ and start by considering the values $g(0)$ and $g(1)$.

    (c) What does part (b) have to do with part (a)?

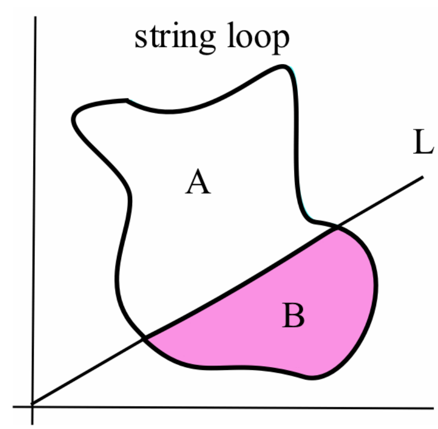
    (d) Is the theorem in part (b) true if we replace the closed interval $[0,1]$ with the open interval $(0,1)$?

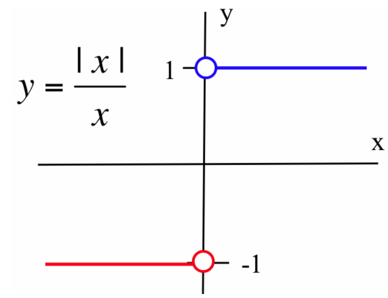
23. A piece of string is tied in a loop and tossed onto quadrant I enclosing a single region (see figure below).

    (a) Is it always possible to find a line $L$ passing through the origin so that $L$ divides the region into two equal areas? (Justify your answer.)

    (b) Is it always possible to find a line $L$ parallel to the $x$-axis so that $L$ divides the region into two equal areas? (Justify your answer.)

    (c) Is it always possible to find two lines, $L$ parallel to the $x$-axis and $M$ parallel to the $y$-axis, so that $L$ and $M$ divide the region into four equal areas? (Justify your answer.)






*1.3 Practice Answers*

1. $f(x) = \frac{|x|}{x}$ (see margin figure) is continuous everywhere **except** at $x = 0$, where this function is not defined.

   If $a > 0$, then $\lim_{x \to a} \frac{|x|}{x} = 1 = f(a)$ so $f$ is continuous at $a$.

   If $a < 0$, then $\lim_{x \to a} \frac{|x|}{x} = -1 = f(a)$ so $f$ is continuous at $a$.

   But $f(0)$ is not defined and

   $$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \neq 1 = \lim_{x \to 0^+} \frac{|x|}{x}$$

   so $\lim_{x \to a} \frac{|x|}{x}$ does not exist.

2. (a) To prove that $k \cdot f$ is continuous at $a$, we need to prove that $k \cdot f$ satisfies the definition of continuity at $a$: $\lim_{x \to a} k \cdot f(x) = k \cdot f(a)$. Using results about limits, we know

   $$\lim_{x \to a} k \cdot f(x) = k \cdot \lim_{x \to a} f(x) = k \cdot f(a)$$

   (because $f$ is continuous at $a$) so $k \cdot f$ is continuous at $a$.

   (b) To prove that $f - g$ is continuous at $a$, we need to prove that $f - g$ satisfies the definition of continuity at $a$: $\lim_{x \to a} [f(x) - g(x)] = f(a) - g(a)$. Again using information about limits:

   $$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = f(a) - g(a)$$

   (because $f$ and $g$ are both continuous at $a$) so $f - g$ is continuous at $a$.

## 1.4 Definition of Limit

It may seem strange that we have been using and calculating the values of limits for quite a while without having a precise definition of "limit," but the history of mathematics shows that many concepts — including limits — were successfully used before they were precisely defined or even fully understood. We have chosen to follow the historical sequence, emphasizing the intuitive and graphical meaning of limit because most students find these ideas and calculations easier than the definition.

This intuitive and graphical understanding of limit was sufficient for the first 100-plus years of the development of calculus (from Newton and Leibniz in the late 1600s to Cauchy in the early 1800s) and it is sufficient for using and understanding the results in beginning calculus.

Mathematics, however, is more than a collection of useful tools, and part of its power and beauty comes from the fact that in mathematics terms are precisely defined and results are rigorously proved. Mathematical tastes (what is mathematically beautiful, interesting, useful) change over time, but because of careful definitions and proofs, the results remain true — everywhere and forever. Textbooks seldom give all of the definitions and proofs, but it is important to mathematics that such definitions and proofs exist.

The goal of this section is to provide a precise definition of the limit of a function. The definition will not help you calculate the values of limits, but it provides a precise statement of what a limit is. The definition of limit is then used to verify the limits of some functions and prove some general results.

### *The Intuitive Approach*

The precise ("formal") definition of limit carefully states the ideas that we have already been using graphically and intuitively. The following side-by-side columns show some of the phrases we have been using to describe limits, and those phrases — particularly the last ones — provide the basis on which to build the definition of limit.

| **A Particular Limit** | **General Limit** |
|---|---|
| $\lim_{x \to 3} 2x - 1 = 5$ | $\lim_{x \to a} f(x) = L$ |
| "as the values of $x$ approach 3, the values of $2x - 1$ approach (are arbitrarily close to) 5" | "as the values of $x$ approach $a$, the values of $f(x)$ approach (are arbitrarily close to) $L$" |
| "when $x$ is close to 3 (but not equal to 3), the value of $2x - 1$ is close to 5" | "when $x$ is close to $a$ (but not equal to $a$), the value of $f(x)$ is close to $L$" |
| "we can guarantee that the values of $2x - 1$ are as close to 5 as we want by restricting the values of $x$ to be sufficiently close to 3 (but not equal to 3)" | "we can guarantee that the values of $f(x)$ are as close to $L$ as we want by restricting the values of $x$ to be sufficiently close to $a$ (but not equal to $a$)" |

Let's examine what the last phrase ("we can. . . ") means for the Particular Limit in the previous discussion.

**Example 1.** We know $\lim\limits_{x \to 3} 2x - 1 = 5$ and need to show that we can guarantee that the values of $f(x) = 2x - 1$ are as close to 5 as we want by restricting the values of $x$ to be sufficiently close to 3.

What values of $x$ guarantee that $f(x) = 2x - 1$ is within:

(a) 1 unit of 5? (b) 0.2 units of 5? (c) $E$ units of 5?

**Solution.** (a) "Within 1 unit of 5" means between $5 - 1 = 4$ and $5 + 1 = 6$, so the question can be rephrased as "for what values of $x$ is $y = 2x - 1$ between 4 and 6: $4 < 2x - 1 < 6$?" We want to know which values of $x$ ensure the values of $y = 2x - 1$ are in the the shaded band in the uppermost margin figure. The algebraic process is straightforward:

$$4 < 2x - 1 < 6 \quad \Rightarrow \quad 5 < 2x < 7 \quad \Rightarrow \quad 2.5 < x < 3.5$$

We can restate this result as follows: "If $x$ is within 0.5 units of 3, then $y = 2x - 1$ is within 1 unit of 5." (See second margin figure) Any smaller distance also satisfies the guarantee: for example, "If $x$ is within 0.4 units of 3, then $y = 2x - 1$ is within 1 unit of 5." (See third margin figure)

(b) "Within 0.2 units of 5" means between $5 - 0.2 = 4.8$ and $5 + 0.2 = 5.2$, so the question can be rephrased as "for which values of $x$ is $y = 2x - 1$ between 4.8 and 5.2: $4.8 < 2x - 1 < 5.2$?" Solving for $x$, we get $5.8 < 2x < 6.2$ and $2.9 < x < 3.1$. "If $x$ is within 0.1 units of 3, then $y = 2x - 1$ is within 0.2 units of 5." (See fourth margin figure.) Any smaller distance also satisfies the guarantee.

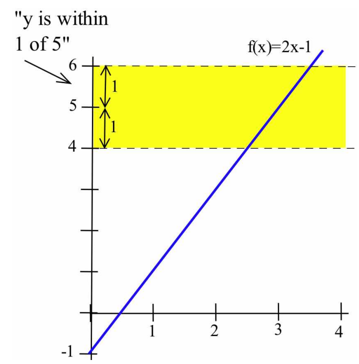
Rather than redoing these calculations for every possible distance from 5, we can do the work once, generally:

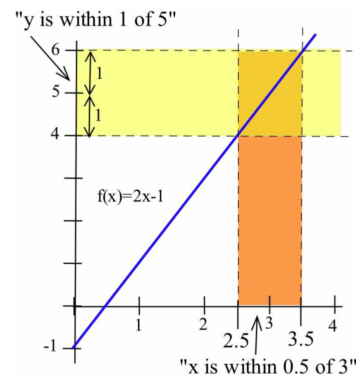
(c) "Within $E$ unit of 5" means between $5 - E$ and $5 + E$, so the question becomes, "For what values of $x$ is $y = 2x - 1$ between $5 - E$ and $5 + E$: $5 - E < 2x - 1 < 5 + E$?" Solving $5 - E < 2x - 1 < 5 + E$ for $x$, we get:

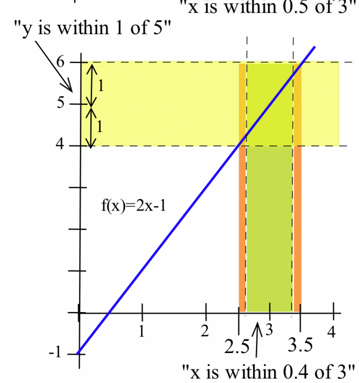
$$6 - E < 2x < 6 + E \quad \Rightarrow \quad 3 - \frac{E}{2} < x < 3 + \frac{E}{2}$$

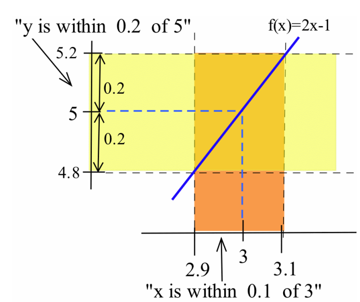
"If $x$ is within $\frac{E}{2}$ units of 3, then $y = 2x - 1$ is within $E$ units of 5." (See last figure.) Any smaller distance also works. ◀

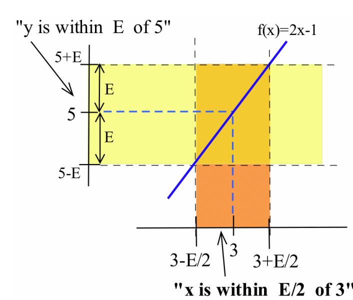
Part (c) of Example 1 illustrates the power of general solutions in mathematics. Rather than redoing similar calculations every time someone demands that $f(x) = 2x - 1$ be within some given distance of 5, we did the calculations once. And then we can quickly respond for any given distance. For the question "What values of $x$ guarantee that $f(x) = 2x - 1$ is within 0.4, 0.1 or 0.006 units of 5?" we can answer, "If $x$ is within 0.2 $(= \frac{0.4}{2})$, 0.05 $(= \frac{0.1}{2})$ or 0.003 $(= \frac{0.006}{2})$ units of 3."

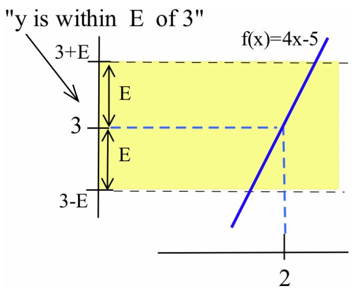


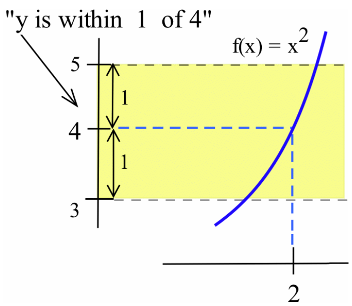






**Practice 1.** Knowing that $\lim_{x\to 2} 4x - 5 = 3$, determine which values of $x$ guarantee that $f(x) = 4x - 5$ is within

(a) 1 unit of 3. (b) 0.08 units of 3. (c) $E$ units of 3.

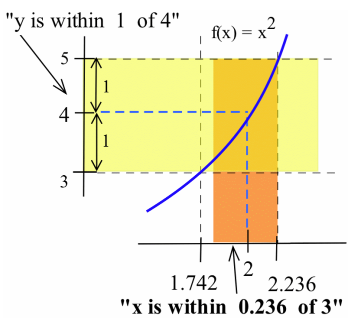
The same ideas work even if the graphs of the functions are not straight lines, but the calculations become more complicated.




**Example 2.** Knowing that $\lim_{x\to 2} x^2 = 4$, determine which values of $x$ guarantee that $f(x) = x^2$ is within:

(a) 1 unit of 4. (b) 0.2 units of 4.

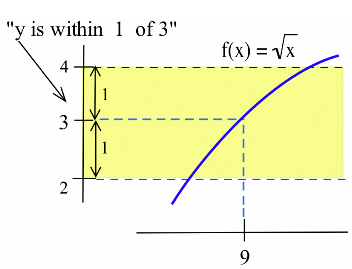
State each answer in the form: "If $x$ is within ______ units of 2, then $f(x)$ is within ______ units of 4."




**Solution.** (a) If $x^2$ is within 1 unit of 4 (and $x$ is near 2, hence positive) then $3 < x^2 < 5$ so $\sqrt{3} < x < \sqrt{5}$ or $1.732 < x < 2.236$. The interval containing these $x$ values extends from $2 - \sqrt{3} \approx 0.268$ units to the left of 2 to $\sqrt{5} - 2 \approx 0.236$ units to the right of 2. Because we want to specify a single distance on each side of 2, we can pick the *smaller* of the two distances, 0.236, and say: "If $x$ is within 0.236 units of 2, then $f(x)$ is within 1 unit of 4."

(b) Similarly, if $x^2$ is within 0.2 units of 4 (and $x$ is near 2, so $x > 0$) then $3.8 < x^2 < 4.2$ so $\sqrt{3.8} < x < \sqrt{4.2}$ or $1.949 < x < 2.049$. The interval containing these $x$ values extends from $2 - \sqrt{3.8} \approx 0.051$ units to the left of 2 to $\sqrt{4.2} - 2 \approx 0.049$ units to the right of 2. Again picking the smaller of the two distances, we can say: "If $x$ is within 0.049 units of 2, then $f(x)$ is within 1 unit of 4." ◀

The situation in Example 2—with different distances on the left and right sides—is very common, and we *always* pick our single distance to be the *smaller* of the distances to the left and right. By using the smaller distance, we can be certain that if $x$ is within that smaller distance on either side, then the value of $f(x)$ is within the specified distance of the value of the limit.



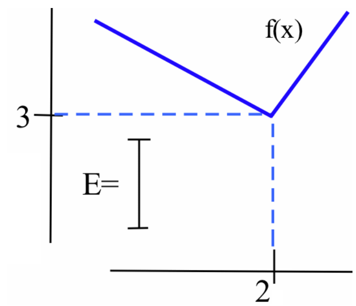

**Practice 2.** Knowing that $\lim_{x\to 9} \sqrt{x} = 3$, determine which values of $x$ guarantee that $f(x) = \sqrt{x}$ is within:

(a) 1 unit of 3. (b) 0.2 units of 3.

State each answer in the form: "If $x$ is within ______ units of 9, then $f(x)$ is within ______ units of 3."

The same ideas can also be used when the function and the specified distance are given graphically, and in that case we can give the answer graphically.

**Example 3.** In the margin figure, $\lim_{x \to 2} f(x) = 3$. Which values of $x$ guarantee that $y = f(x)$ is within $E$ units (given graphically) of 3? State your answer in the form: "If $x$ is within (*show a distance D graphically*) of 2, then $f(x)$ is within $E$ units of 3."




**Solution.** The solution process requires several steps:

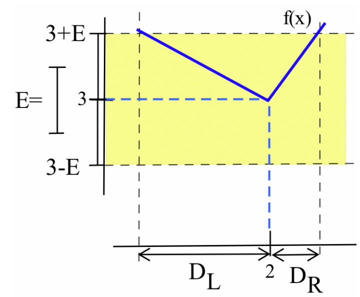
(i) Use the given distance $E$ to find the values $3 - E$ and $3 + E$ on the $y$-axis. (See margin.)

(ii) Sketch the horizontal band with lower edge at $y = 3 - E$ and upper edge at $y = 3 + E$.

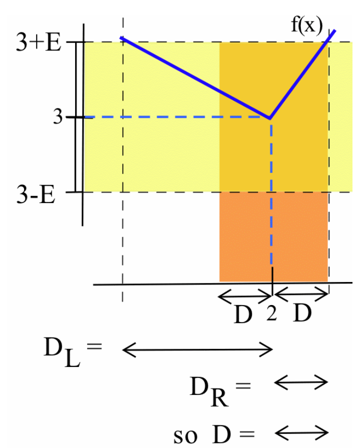
(iii) Find the first locations to the right and left of $x = 2$ where the graph of $y = f(x)$ crosses the lines $y = 3 - E$ and $y = 3 + E$, and at these locations draw vertical line segments extending to the $x$-axis.

(iv) On the $x$-axis, graphically determine the distance from 2 to the vertical line on the left (labeled $D_L$) and from 2 to the vertical line on the right (labeled $D_R$).

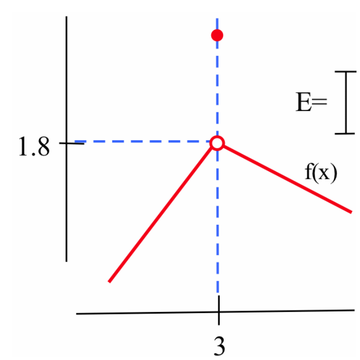
(v) Let the length $D$ be the smaller of the lengths $D_L$ and $D_R$.




If $x$ is within $D$ units of 2, then $f(x)$ is within $E$ units of 3. ◀




**Practice 3.** In the last margin figure, $\lim_{x \to 3} f(x) = 1.8$. Which values of $x$ guarantee that $y = f(x)$ is within $E$ units (given graphically) of 1.8?

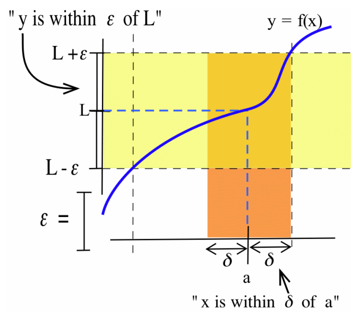


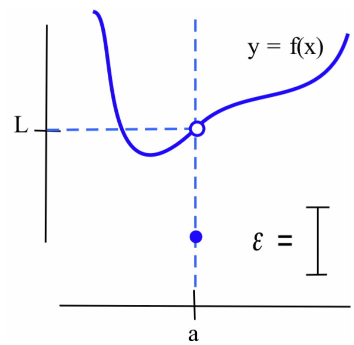

## *The Formal Definition of Limit*

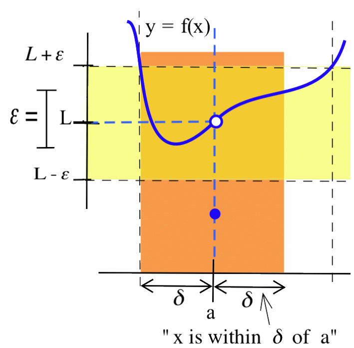
The ideas from the previous Examples and Practice problems, restated for general functions and limits, provide the basis for the definition of limit given below. The use of the lowercase Greek letters $\epsilon$ (epsilon) and $\delta$ (delta) in the definition is standard, and this definition is sometimes called the "epsilon-delta" definition of a limit.

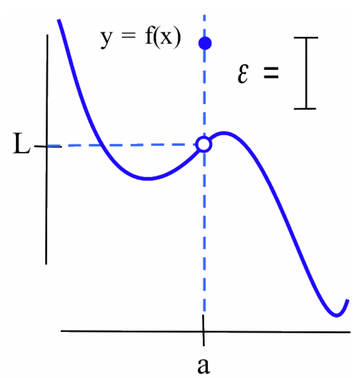
> **Definition of $\lim_{x \to a} f(x) = L$:**
> For every given number $\epsilon > 0$ there is a number $\delta > 0$ so that
>
> if $\quad x \quad$ is within $\delta$ units of $a$ (and $x \neq a$)
> then $\quad f(x) \quad$ is within $\epsilon$ units of $L$
>
> Equivalently: $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

In this definition, $\epsilon$ represents the given distance on either side of the limiting value $y = L$, and $\delta$ is the distance on each side of the point $x = a$ on the $x$-axis that we have been finding in the previous




examples. This definition has the form of a "challenge and response": for any positive challenge $\epsilon$ (make $f(x)$ within $\epsilon$ of $L$), there is a positive response $\delta$ (start with $x$ within $\delta$ of $a$ and $x \neq a$).

**Example 4.** As seen in the second margin figure, $\lim_{x \to a} f(x) = L$, with a value for $\epsilon$ given graphically as a length. Find a length for $\delta$ that satisfies the definition of limit (so "if $x$ is within $\delta$ of $a$, and $x \neq a$, then $f(x)$ is within $\epsilon$ of $L$").

**Solution.** Follow the steps outlined in Example 3. The length for $\delta$ is shown in the third margin figure, and any shorter length for $\delta$ also satisfies the definition. ◀

**Practice 4.** In the bottom margin figure, $\lim_{x \to a} f(x) = L$, with a value for $\epsilon$ given graphically. Find a length for $\delta$ that satisfies the definition of limit.

**Example 5.** Prove that $\lim_{x \to 3} 4x - 5 = 7$.

**Solution.** We need to show that "for every given $\epsilon > 0$ there is a $\delta > 0$ so that if $x$ is within $\delta$ units of 3 (and $x \neq 3$) then $4x - 5$ is within $\epsilon$ units of 7."

Actually, there are two things we need to do. First, we need to find a value for $\delta$ (typically depending on $\epsilon$) and, second, we need to show that our $\delta$ really does satisfy the "if... then..." part of the definition.

Finding $\delta$ is similar to part (c) in Example 1 and Practice 1: Assume $4x - 5$ is within $\epsilon$ units of 7 and solve for $x$. If $7 - \epsilon < 4x - 5 < 7 + \epsilon$ then $12 - \epsilon < 4x < 12 + \epsilon \Rightarrow 3 - \frac{\epsilon}{4} < x < 3 + \frac{\epsilon}{4}$ so $x$ is within $\frac{\epsilon}{4}$ units of 3. Put $\delta = \frac{\epsilon}{4}$.

To show that $\delta = \frac{\epsilon}{4}$ satisfies the definition, we merely reverse the order of the steps in the previous paragraph. Assume that $x$ is within $\delta$ units of 3. Then $3 - \delta < x < 3 + \delta$, so:

$$\begin{aligned} 3 - \frac{\epsilon}{4} < x < 3 + \frac{\epsilon}{4} \quad &\Rightarrow \quad 12 - \epsilon < 4x < 12 + \epsilon \\ &\Rightarrow \quad 7 - \epsilon < 4x - 5 < 7 + \epsilon \end{aligned}$$

so we can conclude that $f(x) = 4x - 5$ is within $\epsilon$ units of 7. This formally verifies that $\lim_{x \to 3} 4x - 5 = 7$. ◀

**Practice 5.** Prove that $\lim_{x \to 4} 5x + 3 = 23$.

The method used to prove the values of the limits for these particular linear functions can also be used to prove the following general result about the limits of linear functions.

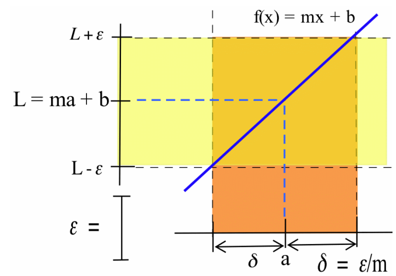
| | |
|---|---|
| **Theorem**: | $\lim_{x \to a} mx + b = ma + b$ |

*Proof.* Let $f(x) = mx + b$.

**Case 1**: $m = 0$. Then $f(x) = 0x + b = b$ is simply a constant function, and any value for $\delta > 0$ satisfies the definition. Given any value of $\epsilon > 0$, let $\delta = 1$ (any positive value for $\delta$ works). If $x$ is is within 1 unit of $a$, then $f(x) - f(a) = b - b = 0 < \epsilon$, so we have shown that for any $\epsilon > 0$ there is a $\delta > 0$ that satisfies the limit definition.

**Case 2**: $m \neq 0$. For any $\epsilon > 0$, put $\delta = \frac{\epsilon}{|m|} > 0$. If $x$ is within $\delta = \frac{\epsilon}{|m|}$ of $a$ then

$$a - \frac{\epsilon}{|m|} < x < a + \frac{\epsilon}{|m|} \Rightarrow \frac{\epsilon}{|m|} < x - a < \frac{\epsilon}{|m|} \Rightarrow |x - a| < \frac{\epsilon}{|m|}$$




Then the distance between $f(x)$ and $L = ma + b$ is:

$$\begin{aligned} |f(x) - L| &= |(mx + b) - (ma + b)| = |mx - ma| \\ &= |m| \cdot |x - a| < |m| \frac{\epsilon}{|m|} = \epsilon \end{aligned}$$

so $f(x)$ is within $\epsilon$ of $L = ma + b$.

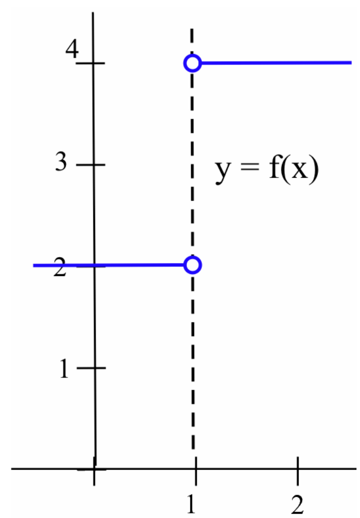
In each case, we have shown that "given any $\epsilon > 0$, there is a $\delta > 0$" that satisfies the rest of the limit definition. $\square$

If there is even a single value of $\epsilon$ for which there is no $\delta$, then we say that the limit "**does not exist**."

**Example 6.** With $f(x)$ defined as:

$$f(x) = \begin{cases} 2 & \text{if } x < 1 \\ 4 & \text{if } x > 1 \end{cases}$$

use the limit definition to prove that $\lim_{x \to 1} f(x)$ does not exist.




The definition says "for every $\epsilon$" so we can certainly pick $\frac{1}{2}$ as our $\epsilon$ value; why we chose this particular value for $\epsilon$ shows up later in the proof.

**Solution.** One common proof technique in mathematics is called "proof by contradiction" and that is the method we use here:

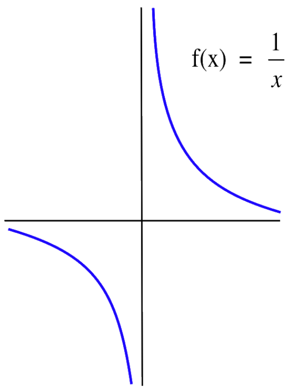
- We assume that the limit does exist and equals some number $L$.
- We show that this assumption leads to a contradiction
- We conclude that the assumption must have been false.

We therefore conclude that the limit does not exist.

First, assume that the limit exists: $\lim_{x \to 1} f(x) = L$ for some value for $L$. Let $\epsilon = \frac{1}{2}$. Then, because we are assuming that the limit exists, there is a $\delta > 0$ so that if $x$ is within $\delta$ of 1 then $f(x)$ is within $\epsilon$ of $L$.

Next, let $x_1$ be between 1 and $1 + \delta$. Then $x_1 > 1$ so $f(x_1) = 4$. Also, $x_1$ is within $\delta$ of 1 so $f(x_1) = 4$ is within $\frac{1}{2}$ of $L$, which means that $L$ is between 3.5 and 4.5: $3.5 < L < 4.5$.

Let $x_2$ be between 1 and $1 - \delta$. Then $x_2 < 1$, so $f(x_2) = 2$. Also, $x_2$ is within $\delta$ of 1 so $f(x_2) = 2$ is within $\frac{1}{2}$ of $L$, which means that $L$ is between 1.5 and 2.5: $1.5 < L < 2.5$.




These inequalities provide the contradiction we hoped to find. There is no value $L$ that satisfies both $3.5 < L < 4.5$ **and** $1.5 < L < 2.5$, so our assumption must be false: $f(x)$ does not have a limit as $x \to 1$. ◀

**Practice 6.** Use the limit definition to prove that $\displaystyle\lim_{x\to 0} \frac{1}{x}$ does not exist.

*Proofs of Two Limit Theorems*

We conclude with proofs of two parts of the Main Limit Theorem so you can see how such proofs proceed — you have already used these theorems to evaluate limits.

There are rigorous proofs of all of the other limit properties in the Main Limit Theorem, but they are somewhat more complicated than the proofs given here.

**Theorem**:

If $\displaystyle\lim_{x\to a} f(x) = L$

then $\displaystyle\lim_{x\to a} k \cdot f(x) = kL$

*Proof.* Case $k = 0$: The theorem is true but not very interesting:

$$\lim_{x\to a} k \cdot f(x) = \lim_{x\to a} 0 \cdot f(x) = \lim_{x\to a} 0 = 0 = 0 \cdot L = kL$$

Case $k \neq 0$: Because $\displaystyle\lim_{x\to a} f(x) = L$, then, by the definition, for every $\epsilon > 0$ there is a $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$. For any $\epsilon > 0$, we know $\frac{\epsilon}{|k|} > 0$, so pick a value of $\delta$ that satisfies $|f(x) - L| < \frac{\epsilon}{|k|}$ whenever $|x - a| < \delta$.

When $|x - a| < \delta$ ("$x$ is within $\delta$ of $a$") then $|f(x) - L| < \frac{\epsilon}{|k|}$ ("$f(x)$ is within $\frac{\epsilon}{|k|}$ of $L$") so $|k| \cdot |f(x) - L| < \epsilon \quad \Rightarrow \quad |k \cdot f(x) - k \cdot L| < \epsilon$ (that is, $k \cdot f(x)$ is within $\epsilon$ of $k \cdot L$). □

**Theorem**:

If $\displaystyle\lim_{x\to a} f(x) = L$ and $\displaystyle\lim_{x\to a} g(x) = M$

then $\displaystyle\lim_{x\to a} [f(x) + g(x)] = L + M$.

*Proof.* Given any $\epsilon > 0$, we know $\frac{\epsilon}{2} > 0$, so there is a number $\delta_f > 0$ such that when $|x - a| < \delta_f$ then $|f(x) - L| < \frac{\epsilon}{2}$ ("if $x$ is within $\delta_f$ of $a$, then $f(x)$ is within $\frac{\epsilon}{2}$ of $L$").

Likewise, there is a number $\delta_g > 0$ such that when $|x - a| < \delta_g$ then $|g(x) - M| < \frac{\epsilon}{2}$ ("if $x$ is within $\delta_g$ of $a$, then $g(x)$ is within $\frac{\epsilon}{2}$ of $M$").

Let $\delta$ be the smaller of $\delta_f$ and $\delta_g$. If $|x - a| < \delta$ then $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$ so:

Here we use the "triangle inequality":

$$|a + b| \leq |a| + |b|$$

$$\begin{aligned} |(f(x) + g(x)) - (L + M))| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

so $f(x) + g(x)$ is within $\epsilon$ of $L + M$ whenever $x$ is within $\delta$ of $a$. □

## 1.4 Problems

In Problems 1–4, state each answer in the form "If $x$ is within ______ units of…"

1. Knowing that $\lim_{x \to 3} 2x + 1 = 7$, what values of $x$ guarantee that $f(x) = 2x + 1$ is within:

   (a) 1 unit of 7? (b) 0.6 units of 7?

   (c) 0.04 units of 7? (d) $\epsilon$ units of 7?

2. Knowing that $\lim_{x \to 1} 3x + 2 = 5$, what values of $x$ guarantee that $f(x) = 3x + 2$ is within:

   (a) 1 unit of 5? (b) 0.6 units of 5?

   (c) 0.09 units of 5? (d) $\epsilon$ units of 5?

3. Knowing that $\lim_{x \to 2} 4x - 3 = 5$, what values of $x$ guarantee that $f(x) = 4x - 3$ is within:

   (a) 1 unit of 5? (b) 0.4 units of 5?

   (c) 0.08 units of 5? (d) $\epsilon$ units of 5?

4. Knowing that $\lim_{x \to 1} 5x - 3 = 2$, what values of $x$ guarantee that $f(x) = 5x - 3$ is within:

   (a) 1 unit of 2? (b) 0.5 units of 2?

   (c) 0.01 units of 2? (d) $\epsilon$ units of 2?

5. For Problems 1–4, list the slope of each function $f$ and the $\delta$ (as a function of $\epsilon$). For these linear functions $f$, how is $\delta$ related to the slope?

6. You have been asked to cut two boards (exactly the same length after the cut) and place them end to end. If the combined length must be within 0.06 inches of 30 inches, then each board must be within how many inches of 15?

7. You have been asked to cut three boards (exactly the same length after the cut) and place them end to end. If the combined length must be within 0.06 inches of 30 inches, then each board must be within how many inches of 10?

8. Knowing that $\lim_{x \to 3} x^2 = 9$, what values of $x$ guarantee that $f(x) = x^2$ is within:

   (a) 1 unit of 9? (b) 0.2 units of 9?

9. Knowing that $\lim_{x \to 2} x^3 = 8$, what values of $x$ guarantee that $f(x) = x^3$ is within:

   (a) 0.5 units of 8? (b) 0.05 units of 8?

10. Knowing that $\lim_{x \to 16} \sqrt{x} = 4$, what values of $x$ guarantee that $f(x) = \sqrt{x}$ is within:

    (a) 1 unit of 4? (b) 0.1 units of 4?

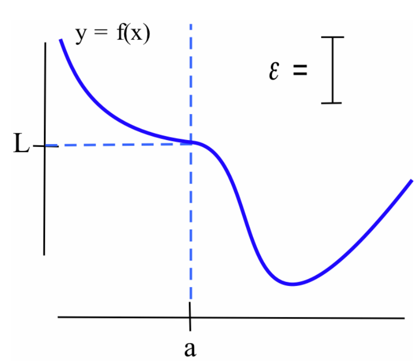
11. Knowing that $\lim_{x \to 3} \sqrt{1 + x} = 2$, what values of $x$ guarantee that $f(x) = \sqrt{1 + x}$ is within:

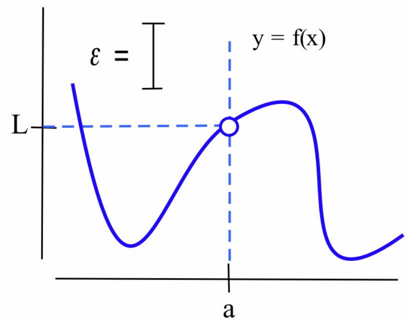
    (a) 1 unit of 2? (b) 0.0002 units of 2?

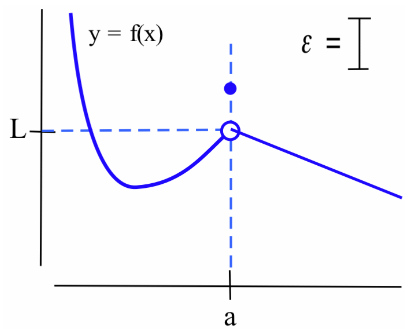
12. You must cut four pieces of wire (all the same length) and form them into a square. If the area of the square must be within 0.06 in$^2$ of 100 in$^2$, then each piece of wire must be within how many inches of 10 in?

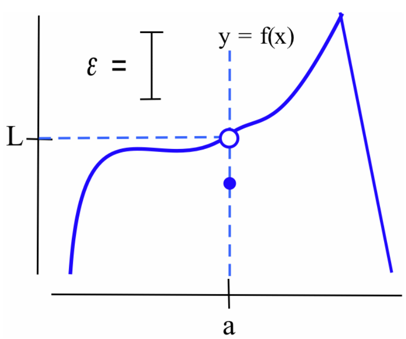
13. You need to cut four pieces of wire (all the same length) and form them into a square. If the area of the square must be within 0.06 in$^2$ of 25 in$^2$, then each piece of wire must be within how many inches of 5 in?

Problems 14–17 give $\lim_{x \to a} f(x) = L$, the function $f$ and a value for $\epsilon$ graphically. Find a length for $\delta$ that satisfies the limit definition for the given function and value of $\epsilon$.

14. 

15. 


16. 


17. 


18. Redo each of Problems 14–17 taking a new value of $\epsilon$ to be half the value of $\epsilon$ given in the problem.

In Problems 19–22, use the limit definition to prove that the given limit does not exist. (Find a value for $\epsilon > 0$ for which there is no $\delta$ that satisfies the definition.)

19. With $f(x)$ defined as:
$$f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 3 & \text{if } x > 2 \end{cases}$$
show that $\lim\limits_{x\to 2} f(x)$ does not exist.

20. Show that $\lim\limits_{x\to 3} \lfloor x \rfloor$ does not exist.

21. With $f(x)$ defined as:
$$f(x) = \begin{cases} x & \text{if } x < 2 \\ 6 - x & \text{if } x > 2 \end{cases}$$
show that $\lim\limits_{x\to 2} f(x)$ does not exist.

22. With $f(x)$ defined as:
$$f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$$
show that $\lim\limits_{x\to 2} f(x)$ does not exist.

23. Prove: If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$ then $\lim\limits_{x\to a} [f(x) - g(x)] = L - M$.

## *1.4 Practice Answers*

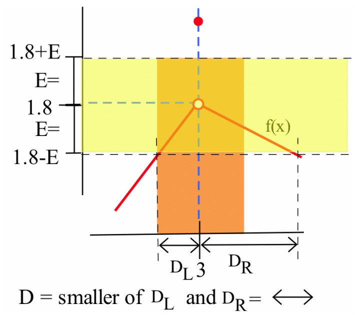
1. (a) $3 - 1 < 4x - 5 < 3 + 1 \Rightarrow 7 < 4x < 9 \Rightarrow 1.75 < x < 2.25$: "$x$ within $\frac{1}{4}$ unit of 2."

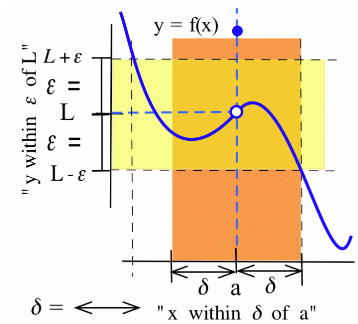
   (b) $3 - 0.08 < 4x - 5 < 3 + 0.08 \Rightarrow 7.92 < 4x < 8.08 \Rightarrow 1.98 < x < 2.02$: "$x$ within 0.02 units of 2."

   (c) $3 - E < 4x - 5 < 3 + E \Rightarrow 8 - E < 4x < 8 + E \Rightarrow 2 - \frac{E}{4} < x < 2 + \frac{E}{4}$: "$x$ within $\frac{E}{4}$ units of 2."

2. "Within 1 unit of 3": If $2 < \sqrt{x} < 4$, then $4 < x < 16$, which extends from 5 units to the left of 9 to 7 units to right of 9. Using the smaller of these two distances from 9: "If $x$ is within 5 units of 9, then $\sqrt{x}$ is within 1 unit of 3."

   "Within 0.2 units of 3": If $2.8 < \sqrt{x} < 3.2$, then $7.84 < x < 10.24$, which extends from 1.16 units to the left of 9 to 1.24 units to the right of 9. "If $x$ is within 1.16 units of 9, then $x$ is within 0.2 units of 3.

3. 


4. 


5. Given any $\epsilon > 0$, take $\delta = \frac{\epsilon}{5}$. If $x$ is within $\delta = \frac{\epsilon}{5}$ of 4, then $4 - \frac{\epsilon}{5} < x < 4 + \frac{\epsilon}{5}$ so:

$$-\frac{\epsilon}{5} < x - 4 < \frac{\epsilon}{5} \Rightarrow -\epsilon < 5x - 20 < \epsilon \Rightarrow -\epsilon < (5x+3) - 23 < \epsilon$$

   so, finally, $f(x) = 5x + 3$ is within $\epsilon$ of $L = 23$.

   We have shown that "for any $\epsilon > 0$, there is a $\delta > 0$ (namely $\delta = \frac{\epsilon}{5}$)" so that the rest of the definition is satisfied.

6. Using "proof by contradiction" as in the solution to Example 6:

This is a much more sophisticated (= harder) problem.

   - Assume that the limit exists: $\lim_{x \to 0} \frac{1}{x} = L$ for some value of $L$. Let $\epsilon = 1$. Since we're assuming that the limit exists, there is a $\delta > 0$ so that if $x$ is within $\delta$ of 0 then $f(x) = \frac{1}{x}$ is within $\epsilon = 1$ of $L$.

The definition says "for every $\epsilon$" so we can pick $\epsilon = 1$. For this particular limit, the definition fails for every $\epsilon > 0$.

   - Let $x_1$ be between 0 and $0 + \delta$ and also require that $x_1 < \frac{1}{2}$. Then $0 < x_1 < \frac{1}{2}$ so $f(x_1) = \frac{1}{x_1} > 2$. Because $x_1$ is within $\delta$ of 0, $f(x_1) > 2$ is within $\epsilon = 1$ of $L$, so $L > 2 - \epsilon = 1$: that is, $1 < L$.

     Let $x_2$ be between 0 and $0 - \delta$ and also require $x_2 > -\frac{1}{2}$. Then $0 > x_2 > \frac{1}{2}$ so $f(x_2) = \frac{1}{x_2} < -2$. Since $x_2$ is within $\delta$ of 0, $f(x_2) < -2$ is within $\epsilon = 1$ of $L$, so $L < -2 + \epsilon = -1 \Rightarrow -1 > L$.

   - The two inequalities derived above provide the contradiction we were hoping to find. There is no value $L$ that satisfies **both** $1 < L$ and $L < -1$, so we can conclude that our assumption was false and that $f(x) = \frac{1}{x}$ does not have a limit as $x \to 0$.