3
Derivatives and Graphs

In this chapter, we explore what the first and second derivatives of a function tell us about the graph of that function and apply this graphical knowledge to locate the extreme values of a function.

3.1 Finding Maximums and Minimums

In theory and applications, we often want to maximize or minimize some quantity. An engineer may want to maximize the speed of a new computer or minimize the heat produced by an appliance. A manufacturer may want to maximize profits and market share or minimize waste. A student may want to maximize a grade in calculus or minimize the hours of study needed to earn a particular grade.

Many natural objects follow minimum or maximum principles, so if we want to model natural phenomena we may need to maximize or minimize. A light ray travels along a "minimum time" path. The shape and surface texture of some animals tend to minimize or maximize heat loss. Systems reach equilibrium when their potential energy is minimized. A basic tenet of evolution is that a genetic characteristic that maximizes the reproductive success of an individual will become more common in a species.

Calculus provides tools for analyzing functions and their behavior and for finding maximums and minimums.

Methods for Finding Maximums and Minimums

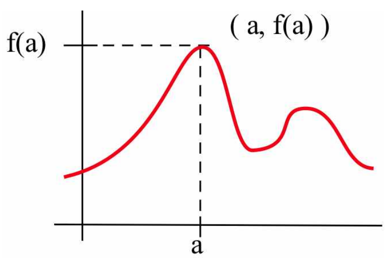
We can try to find where a function f is largest or smallest by evaluating f at lots of values of x, a method that is not very efficient and may not find the exact place where f achieves its extreme value. If we try hundreds or thousands of values for x, however, then we can often find a value of f that is close to the maximum or minimum. In general, this type of exhaustive search is only practical if you have a computer do the work.

The graph of a function provides a visual way of examining lots of values of f, and it is a good method, particularly if you have a computer to do the work for you. It is still inefficient, however, as you (or a computer) still need to evaluate the function at hundreds or thousands of inputs in order to create the graph—and we still may not find the exact location of the maximum or minimum.

Calculus provides ways to drastically narrow the number of points we need to examine to find the exact locations of maximums and minimums. Instead of examining f at thousands of values of x, calculus can often guarantee that the maximum or minimum must occur at one of three or four values of x, a substantial improvement in efficiency.

A Little Terminology

Before we examine how calculus can help us find maximums and minimums, we need to carefully define these concepts.



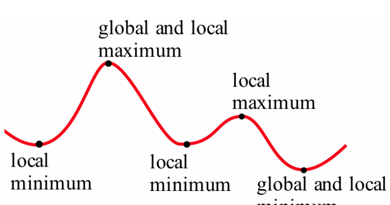
Definitions:

- f has a **maximum** or **global maximum** at $x = a$ if $f(a) \geq f(x)$ for all x in the domain of f.
- The **maximum value** of f is then $f(a)$ and this maximum value of f **occurs at** a.
- The **maximum point** on the graph of f is $(a, f(a))$.

The previous definition involves the overall biggest value a function attains on its entire domain. We are sometimes interested in how a function behaves locally rather than globally.

Definition: f has a **local** or **relative maximum** at $x = a$ if $f(a) \geq f(x)$ for all x "near" a, (that is, in some open interval that contains a).

Global and local **minimums** are defined similarly by replacing the $\geq$ symbol with $\leq$ in the previous definitions.



Definition:

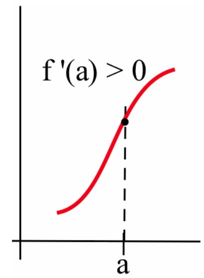
f has a **global extreme** at $x = a$ if $f(a)$ is a global maximum or minimum.

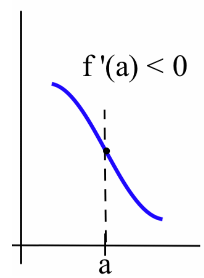
See the margin figure for graphical examples of local and global extremes of a function.

You should notice that every global extreme is also a local extreme, but there are local extremes that are not global extremes. If $h(x)$ is the height of the earth above sea level at location x, then the global maximum of h is $h(\text{summit of Mt. Everest}) = 29{,}028$ feet. The local maximum of h for the United States is $h(\text{summit of Mt. McKinley}) = 20{,}320$ feet. The local minimum of h for the United States is $h(\text{Death Valley}) =$ -282 feet.

Finding Maximums and Minimums of a Function

One way to narrow our search for a maximum value of a function f is to eliminate those values of x that, for some reason, cannot possibly make f maximum.





Theorem:

If $f'(a) > 0$ or $f'(a) < 0$
then $f(a)$ is not a local maximum or minimum.

Proof. Assume that $f'(a) > 0$. By definition:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

so $f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} > 0$. This means that the right and left limits are both positive: $f'(a) = \lim_{\Delta x \to 0^+} \frac{f(a + \Delta x) - f(a)}{\Delta x} > 0$ and $f'(a) = \lim_{\Delta x \to 0^-} \frac{f(a + \Delta x) - f(a)}{\Delta x} > 0$.

Considering the right limit, we know that if we restrict $\Delta x > 0$ to be sufficiently small, we can guarantee that $\frac{f(a + \Delta x) - f(a)}{\Delta x} > 0$ so, multiplying each side of this last inequality by the positive number Δx, we have $f(a + \Delta x) - f(a) > 0 \Rightarrow f(a + \Delta x) > f(a)$ for all sufficiently small values of $\Delta x > 0$, so any open interval containing $x = a$ will also contain values of x with $f(x) > f(a)$. This tell us that $f(a)$ is not a maximum.

Considering the left limit, we know that if we restrict $\Delta x < 0$ to be sufficiently small, we can guarantee that $\frac{f(a + \Delta x) - f(a)}{\Delta x} > 0$ so, multiplying each side of this last inequality by the negative number Δx, we have $f(a + \Delta x) - f(a) < 0 \Rightarrow f(a + \Delta x) < f(a)$ for all sufficiently small values of $\Delta x < 0$, so any open interval containing $x = a$ will also contain values of x with $f(x) < f(a)$. This tell us that $f(a)$ is not a minimum.

The argument for the "$f'(a) < 0$" case is similar. □

When we evaluate the derivative of a function f at a point $x = a$, there are only four possible outcomes: $f'(a) > 0$, $f'(a) < 0$, $f'(a) = 0$ or $f'(a)$ is undefined. If we are looking for extreme values of f, then we can eliminate those points at which f' is positive or negative, and only two possibilities remain: $f'(a) = 0$ or $f'(a)$ is undefined.

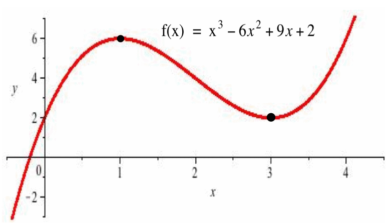
Theorem:

If f is defined on an open interval and $f(a)$ is a local extreme of f
then either $f'(a) = 0$ or f is not differentiable at a.

Example 1. Find the local extremes of $f(x) = x^3 - 6x^2 + 9x + 2$.

Solution. An extreme value of f can occur only where $f'(x) = 0$ or where f is not differentiable; $f(x)$ is a polynomial, so it is differentiable for all values of x, and we can restrict our attention to points where $f'(x) = 0$.

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$$

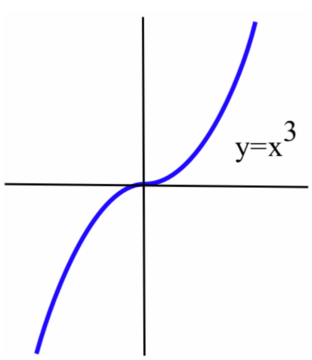


so $f'(x) = 0$ only at $x = 1$ and $x = 3$.

The only possible locations of local extremes of f are at $x = 1$ and $x = 3$. We don't know yet whether $f(1)$ or $f(3)$ is a local extreme of f, but we can be certain that no other point is a local extreme. The graph of f (see margin) shows that $(1, f(1)) = (1, 6)$ appears to be a local maximum and $(3, f(3)) = (3, 2)$ appears to be a local minimum. This function does not have a global maximum or minimum. ◀

Practice 1. Find the local extremes of $f(x) = x^2 + 4x - 5$ and of $g(x) = 2x^3 - 12x^2 + 7$.

It is important to recognize that the two conditions "$f'(a) = 0$" or "f not differentiable at a" do not guarantee that $f(a)$ is a local maximum or minimum. They only say that $f(a)$ **might** be a local extreme or that $f(a)$ is a **candidate** for being a local extreme.

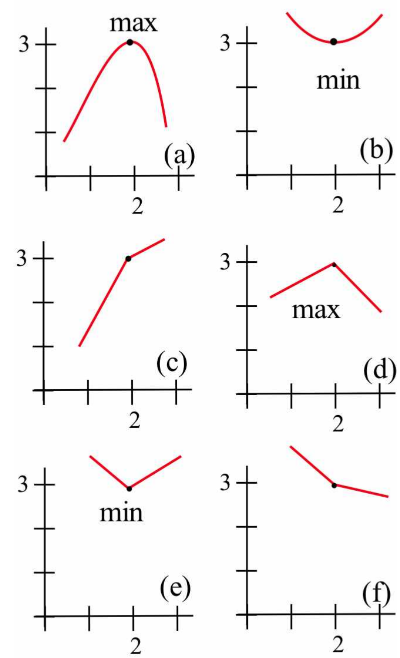


Example 2. Find all local extremes of $f(x) = x^3$.

Solution. $f(x) = x^3$ is differentiable for all x, and $f'(x) = 3x^2$ equals 0 only at $x = 0$, so the only candidate is the point $(0, 0)$. But if $x > 0$ then $f(x) = x^3 > 0 = f(0)$, so $f(0)$ is not a local maximum. Similarly, if $x < 0$ then $f(x) = x^3 < 0 = f(0)$ so $f(0)$ is not a local minimum. The point $(0, 0)$ is the only candidate to be a local extreme of f, but this candidate did not turn out to be a local extreme of f. The function $f(x) = x^3$ does not have any local extremes. ◀

If	$f'(a) = 0$ or f is not differentiable at a
then	the point $(a, f(a))$ is a candidate to be a local extreme but may not actually be a local extreme.

Practice 2. Sketch the graph of a differentiable function f that satisfies the conditions: $f(1) = 5$, $f(3) = 1$, $f(4) = 3$ and $f(6) = 7$; $f'(1) = 0$, $f'(3) = 0$, $f'(4) = 0$ and $f'(6) = 0$; the only local maximums of f are at $(1, 5)$ and $(6, 7)$; and the only local minimum is at $(3, 1)$.



Is $f(a)$ a Maximum or Minimum or Neither?

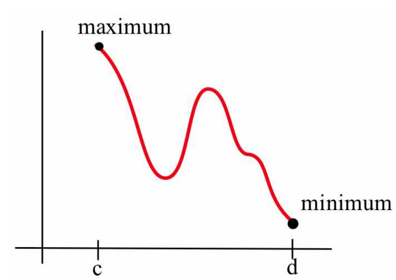
Once we have found the candidates $(a, f(a))$ for extreme points of f, we still have the problem of determining whether the point is a maximum, a minimum or neither.

One method involves graphing (or letting a calculator graph) the function near a, and then drawing a conclusion from the graph. All of the graphs in the margin have $f(2) = 3$, and on each of the graphs $f'(2)$ either equals 0 or is undefined. It is clear from the graphs that the point $(2, 3)$ is: a local maximum in (a) and (d); a local minimum in (b) and (e); and not a local extreme in (c) and (f).

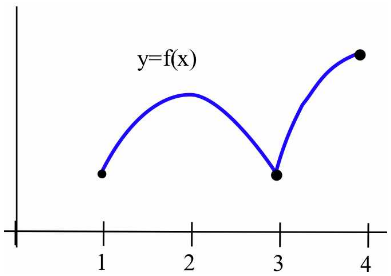
In Sections 3.3 and 3.4, we will investigate how information about the first and **second** derivatives of f can help determine whether the candidate $(a, f(a))$ is a maximum, a minimum or neither.

Endpoint Extremes

So far we have discussed finding extreme values of functions over the entire real number line or on an open interval, but in practice we may need to find the extreme of a function over some closed interval $[c, d]$. If an extreme value of f occurs at $x = a$ between c and d ($c < a < d$) then the previous reasoning and results still apply: either $f'(a) = 0$ or f is not differentiable at a. On a closed interval, however, there is one more possibility: an extreme can occur at an **endpoint** of the closed interval (see margin): at $x = c$ or $x = d$.



We can extend our definition of a local extreme at $x = a$ (which requires $f(a) \geq f(x)$ [or $f(a) \leq f(x)$] for all x in some *open* interval containing a) to include $x = a$ being the endpoint of a closed interval: $f(a) \geq f(x)$ [or $f(a) \leq f(x)$] for all x in an interval of the form $[a, a + h)$ (for left endpoints) or $(a - h, a]$ (for right endpoints), where $h > 0$ is a number small enough to guarantee the "half-open" interval is in the domain of $f(x)$. Using this extended definition, the function in the margin has a local maximum (which is also a global maximum) at $x = c$ and a local minimum (also a global minimum) at $x = d$.



Practice 3. List all of the extremes $(a, f(a))$ of the function in the margin figure on the interval $[1, 4]$ and state whether $f'(a) = 0$, f is not differentiable at a, or a is an endpoint.

Example 3. Find the extreme values of $f(x) = x^3 - 3x^2 - 9x + 5$ for $-2 \le x \le 6$.

Solution. We need to find investigate points where $f'(x) = 0$, points where f is not differentiable, and the endpoints:

- $f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$, so $f'(x) = 0$ only at $x = -1$ and $x = 3$.
- f is a polynomial, so it is differentiable everywhere.
- The endpoints of the interval are $x = -2$ and $x = 6$.

Altogether we have four points in the interval to examine, and any extreme values of f can only occur when x is one of those four points: $f(-2) = 3$, $f(-1) = 10$, $f(3) = -22$ and $f(6) = 59$. The (global) minimum of f on $[-2, 6]$ is -22 when $x = 3$, and the (global) maximum of f on $[-2, 6]$ is 59 when $x = 6$. ◀

Sometimes the function we need to maximize or minimize is more complicated, but the same methods work.

Example 4. Find the extreme values of $f(x) = \frac{1}{3}\sqrt{64 + x^2} + \frac{1}{5}(10 - x)$ for $0 \le x \le 10$.

Solution. This function comes from an application we will examine in section 3.5. The only possible locations of extremes are where $f'(x) = 0$ or $f'(x)$ is undefined or where x is an endpoint of the interval $[0, 10]$.

$$\begin{aligned} f'(x) &= \mathbf{D}\left(\frac{1}{3}\left(64 + x^2\right)^{\frac{1}{2}} + \frac{1}{5}(10 - x)\right) \\ &= \frac{1}{3} \cdot \frac{1}{2}(64 + x^2)^{-\frac{1}{2}} \cdot 2x - \frac{1}{5} \\ &= \frac{x}{3\sqrt{64 + x^2}} - \frac{1}{5} \end{aligned}$$

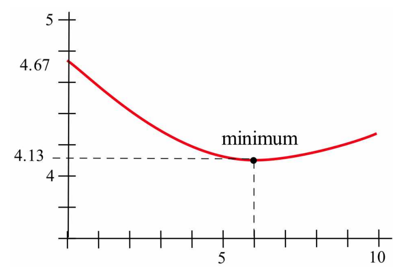
To find where $f'(x) = 0$, set the derivative equal to 0 and solve for x:

$$\frac{x}{3\sqrt{64 + x^2}} - \frac{1}{5} = 0 \Rightarrow \frac{x}{3\sqrt{64 + x^2}} = \frac{1}{5} \Rightarrow \frac{x^2}{576 + 9x^2} = \frac{1}{25}$$
$$\Rightarrow 16x^2 = 576 \Rightarrow x = \pm 6$$

but only $x = 6$ is in the interval $[0, 10]$. Evaluating f at this point gives $f(6) \approx 4.13$.

We can evaluate the formula for $f'(x)$ for any value of x, so the derivative is always defined.

Finally, the interval $[0, 10]$ has two endpoints, $x = 0$ and $x = 10$, and $f(0) \approx 4.67$ while $f(10) \approx 4.27$.



The maximum of f on $[0, 10]$ must occur at one of the points $(0, 4.67)$, $(6, 4.13)$ and $(10, 4.27)$, and the minimum must occur at one of these three points as well.

The maximum value of f is 4.67 at $x = 0$, and the minimum value of f is 4.13 at $x = 6$. ◀

Practice 4. Rework the previous Example to find the extreme values of $f(x) = \frac{1}{3}\sqrt{64 + x^2} + \frac{1}{5}(10 - x)$ for $0 \leq x \leq 5$.

Critical Numbers

The points at which a function **might** have an extreme value are called **critical numbers**.

Definitions: A **critical number** for a function f is a value $x = a$ in the domain of f so that:

- $f'(a) = 0$ or
- f is not differentiable at a or
- a is an **endpoint** of a closed interval to which f is restricted.

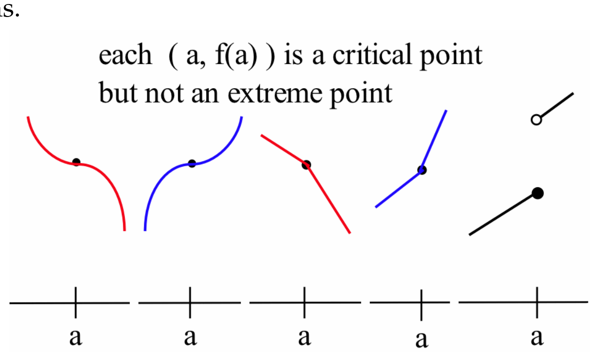
If we are trying to find the extreme values of f on an **open** interval $c < x < d$ or on the entire number line, then the set of inputs to which f is restricted will not include any endpoints, so we will not need to worry about any endpoint critical numbers.

We can now give a very succinct description of where to look for extreme values of a function.

An extreme value of f can only occur at a critical number.

The critical numbers only give **possible** locations of extremes; some critical numbers are not locations of extremes. In other words, critical numbers are the **candidates** for the locations of maximums and minimums.

Section 3.5 is devoted entirely to translating and solving maximum and minimum problems.

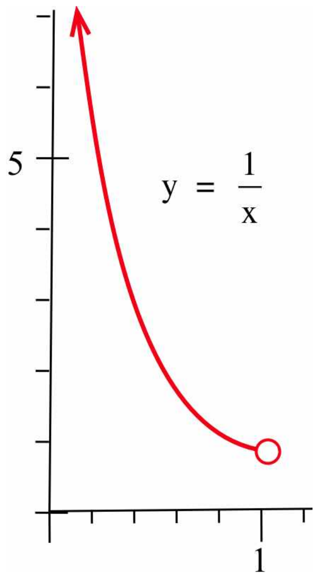


Which Functions Have Extremes?

Some functions don't have extreme values: Example 2 showed that $f(x) = x^3$ (defined on the entire number line) did not have a maximum or minimum.

Example 5. Find the extreme values of $f(x) = x$.

Solution. Because $f'(x) = 1 > 0$ for all x, the first theorem in this section guarantees that f has no extreme values. The function $f(x) = x$ does not have a maximum or minimum on the real number line. ◀



How would the situation change if we changed the interval in this example to $(0, 1]$? To $[1, 2]$?

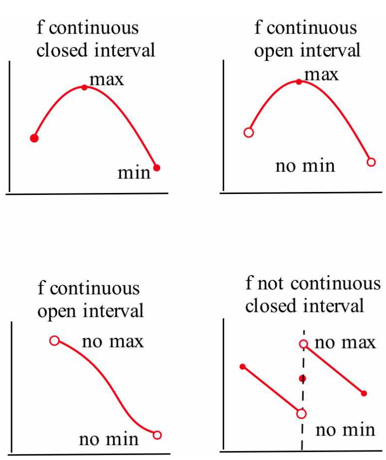
With the previous function, the domain was so large that we could always make the function output larger or smaller than any given value by choosing an appropriate input x. The next example shows that we can encounter the same difficulty even on a "small" interval.

Example 6. Show that $f(x) = \frac{1}{x}$ does not have a maximum or minimum on the interval $(0, 1)$.

Solution. f is continuous for all $x \neq 0$ so f is continuous on the interval $(0, 1)$. For $0 < x < 1$, $f(x) = \frac{1}{x} > 0$ and for any number a strictly between 0 and 1, we can show that $f(a)$ is neither a maximum nor a minimum of f on $(0, 1)$, as follows.

Pick b to be any number between 0 and a: $0 < b < a$. Then $f(b) = \frac{1}{b} > \frac{1}{a} = f(a)$, so $f(a)$ is not a maximum. Similarly, pick c to be any number between a and 1: $a < c < 1$. Then $f(a) = \frac{1}{a} > \frac{1}{c} = f(c)$, so $f(a)$ is not a minimum. The interval $(0, 1)$ is not "large," yet f does not attain an extreme value anywhere in $(0, 1)$. ◀

The Extreme Value Theorem provides conditions that guarantee a function to have a maximum and a minimum.



Extreme Value Theorem:

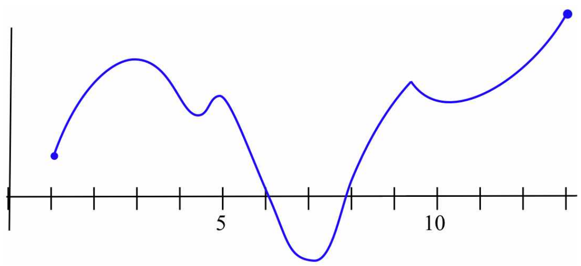
If f is continuous on a **closed** interval $[a, b]$
then f attains both a maximum and minimum on $[a, b]$.

The proof of this theorem is difficult, so we omit it. The margin figure illustrates some of the possibilities for continuous and discontinuous functions on open and closed intervals.

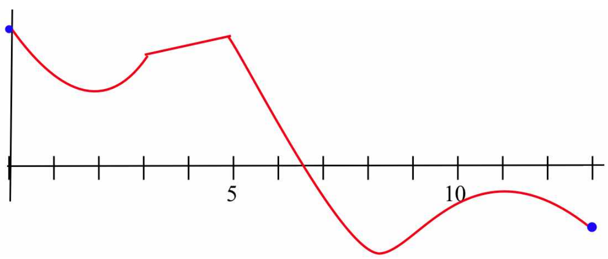
The Extreme Value Theorem guarantees that certain functions (continuous ones) on certain intervals (closed ones) must have maximums and minimums. Other functions on other intervals **may** or **may not** have maximums and minimums.

3.1 Problems

1. Label all of the local maximums and minimums of the function in the figure below. Also label all of the critical points.



2. Label the local extremes and critical points of the function graphed below.



In Problems 3–22, find all of the critical points and local maximums and minimums of each function.

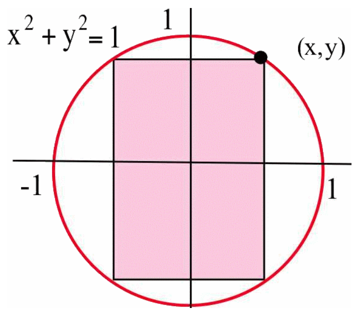
3. $f(x) = x^2 + 8x + 7$
4. $f(x) = 2x^2 - 12x + 7$
5. $f(x) = \sin(x)$
6. $f(x) = x^3 - 6x^2 + 5$
7. $f(x) = \sqrt[3]{x}$
8. $f(x) = 5x - 2$
9. $f(x) = xe^{5x}$
10. $f(x) = \sqrt[3]{1 + x^2}$
11. $f(x) = (x-1)^2(x-3)$
12. $f(x) = \ln(x^2 - 6x + 11)$
13. $f(x) = 2x^3 - 96x + 42$
14. $f(x) = 5x + \cos(2x + 1)$
15. $f(x) = e^{-(x-2)^2}$
16. $f(x) = |x + 5|$
17. $f(x) = \dfrac{x}{1 + x^2}$
18. $f(x) = \dfrac{x^3}{1 + x^4}$
19. $f(x) = (x-2)^{\frac{2}{3}}$
20. $f(x) = \left(x^2 - 1\right)^{\frac{2}{3}}$
21. $f(x) = \sqrt[3]{x^2 - 4}$
22. $f(x) = \sqrt[3]{x - 2}$

23. Sketch the graph of a continuous function f with:
    (a) $f(1) = 3$, $f'(1) = 0$ and the point $(1, 3)$ a relative maximum of f.
    (b) $f(2) = 1$, $f'(2) = 0$ and the point $(2, 1)$ a relative minimum of f.
    (c) $f(3) = 5$, f is not differentiable at $x = 3$, and the point $(3, 5)$ a relative maximum of f.
    (d) $f(4) = 7$, f is not differentiable at $x = 4$, and the point $(4, 7)$ a relative minimum of f.
    (e) $f(5) = 4$, $f'(5) = 0$ and the point $(5, 4)$ not a relative minimum or maximum of f.
    (f) $f(6) = 3$, f not differentiable at 6, and $(6, 3)$ not a relative minimum or maximum of f.

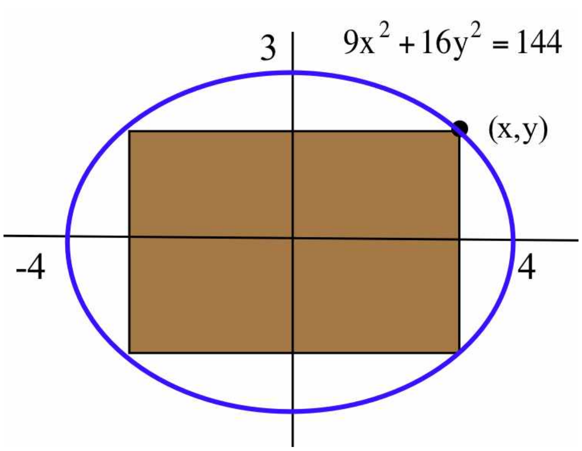
In Problems 24–37, find all critical points and local extremes of each function on the given intervals.

24. $f(x) = x^2 - 6x + 5$ on the entire real number line
25. $f(x) = x^2 - 6x + 5$ on $[-2, 5]$
26. $f(x) = 2 - x^3$ on the entire real number line
27. $f(x) = 2 - x^3$ on $[-2, 1]$
28. $f(x) = x^3 - 3x + 5$ on the entire real number line
29. $f(x) = x^3 - 3x + 5$ on $[-2, 1]$
30. $f(x) = x^5 - 5x^4 + 5x^3 + 7$ on $(-\infty, \infty)$
31. $f(x) = x^5 - 5x^4 + 5x^3 + 7$ on $[0, 2]$
32. $f(x) = \dfrac{1}{x^2 + 1}$ on $(-\infty, \infty)$
33. $f(x) = \dfrac{1}{x^2 + 1}$ on $[1, 3]$
34. $f(x) = 3\sqrt{x^2 + 4} - x$ on $(-\infty, \infty)$
35. $f(x) = 3\sqrt{x^2 + 4} - x$ on $[0, 2]$
36. $f(x) = xe^{-5x}$ on $(-\infty, \infty)$
37. $f(x) = x^3 - \ln(x)$ on $\left[\frac{1}{2}, 2\right]$
38. (a) Find two numbers whose sum is 22 and whose product is as large as possible. (Suggestion: call the numbers x and $22 - x$).
    (b) Find two numbers whose sum is $A > 0$ and whose product is as large as possible.

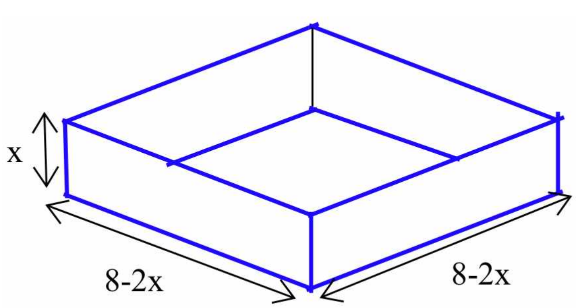
39. Find the coordinates of the point in the first quadrant on the circle $x^2 + y^2 = 1$ so that the rectangle in the figure below has the largest possible area.



40. Find the coordinates of the point in the first quadrant on the ellipse $9x^2 + 16y^2 = 144$ so that the rectangle in the figure below has:
   (a) the largest possible area.
   (b) The smallest possible area.



41. Find the value for x so the box shown below has:
   (a) the largest possible volume.
   (b) The smallest possible volume.



42. Find the radius and height of the cylinder that has the largest volume ($V = \pi r^2 h$) if the sum of the radius and height is 9.

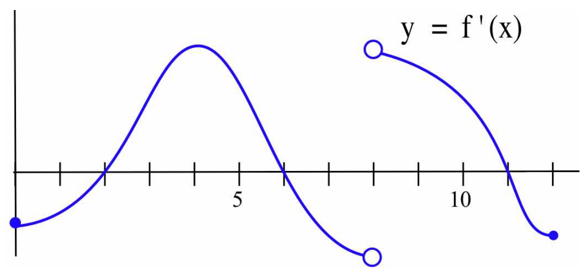
43. Suppose you are working with a polynomial of degree 3 on a closed interval.
   (a) What is the largest number of critical points the function can have on the interval?
   (b) What is the smallest number of critical points it can have?
   (c) What are the patterns for the most and fewest critical points a polynomial of degree n on a closed interval can have?

44. Suppose you have a polynomial of degree 3 divided by a polynomial of degree 2 on a closed interval.
   (a) What is the largest number of critical points the function can have on the interval?
   (b) What is the smallest number of critical points it can have?

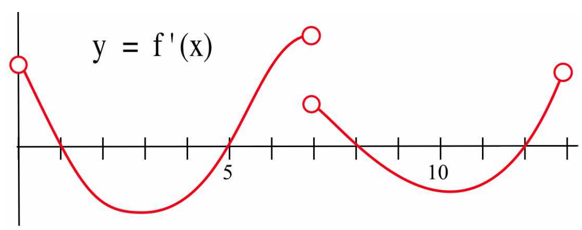
45. Suppose $f(1) = 5$ and $f'(1) = 0$. What can we conclude about the point $(1,5)$ if:
   (a) $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$?
   (b) $f'(x) < 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$?
   (c) $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$?
   (d) $f'(x) > 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$?

46. Suppose $f(2) = 3$ and f is continuous but not differentiable at $x = 2$. What can we conclude about the point $(2,3)$ if:
   (a) $f'(x) < 0$ for $x < 2$ and $f'(x) > 0$ for $x > 2$?
   (b) $f'(x) < 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$?
   (c) $f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$?
   (d) $f'(x) > 0$ for $x < 2$ and $f'(x) > 0$ for $x > 2$?

47. The figure below shows the graph of $f'(x)$, which is continuous on $(0, 12)$ except at $x = 8$.
   (a) Which values of x are critical points of $f(x)$?
   (b) At which values of x does f attain a local maximum?
   (c) At which values of x does f attain a local minimum?



48. The figure below shows the graph of $f'(x)$, which is continuous on $(0, 13)$ except at $x = 7$.
   (a) Which values of x are critical points?
   (b) At which values of x does f attain a local maximum?
   (c) At which values of x does f attain a local minimum?



49. State the contrapositive form of the Extreme Value Theorem.

50. Imagine the graph of $f(x) = 1 - x$. Does f have a **maximum** value for x in the given interval?

   (a) $[0, 2]$ (b) $[0, 2)$ (c) $(0, 2]$
   (d) $(0, 2)$ (e) $(1, \pi]$

51. Imagine the graph of $f(x) = 1 - x$. Does f have a **minimum** value for x in the given interval?

   (a) $[0, 2]$ (b) $[0, 2)$ (c) $(0, 2]$
   (d) $(0, 2)$ (e) $(1, \pi]$

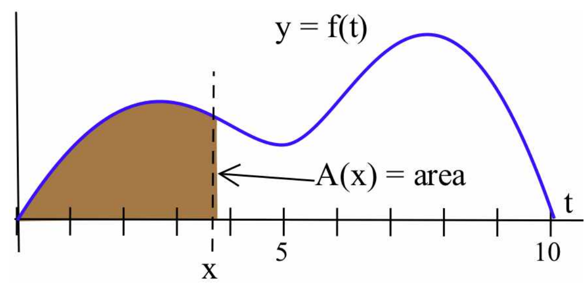
52. Imagine the graph of $f(x) = x^2$. Does f have a **maximum** value for x in the given interval?

   (a) $[-2, 3]$ (b) $[-2, 3)$ (c) $(-2, 3]$
   (d) $[-2, 1)$ (e) $(-2, 1]$

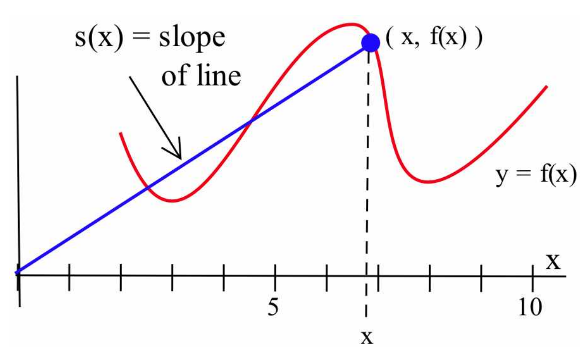
53. Imagine the graph of $f(x) = x^2$. Does f have a **minimum** value for x in the interval I?

   (a) $[-2, 3]$ (b) $[-2, 3)$ (c) $(-2, 3]$
   (d) $[-2, 1)$ (e) $(-2, 1]$

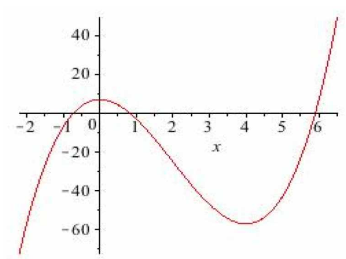
54. Define $A(x)$ to be the **area** bounded between the t-axis, the graph of $y = f(t)$ and a vertical line at $t = x$ (see figure below).
   (a) At what value of x is $A(x)$ minimum?
   (b) At what value of x is $A(x)$ maximum?



55. Define $S(x)$ to be the **slope** of the line through the points $(0, 0)$ and $(x, f(x))$ in the figure below.
   (a) At what value of x is $S(x)$ minimum?
   (b) At what value of x is $S(x)$ maximum?

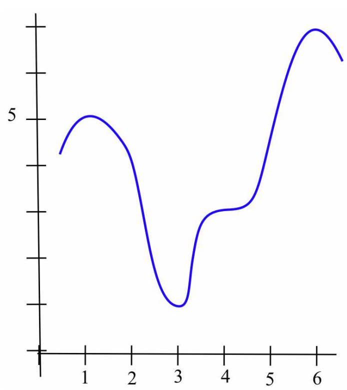


3.1 Practice Answers



1. $f(x) = x^2 + 4x - 5$ is a polynomial so f is differentiable for all x and $f'(x) = 2x + 4$; $f'(x) = 0$ when $x = -2$ so the only candidate for a local extreme is $x = -2$. Because the graph of f is a parabola opening up, the point $(-2, f(-2)) = (-2, -9)$ is a local minimum.

   $g(x) = 2x^3 - 12x^2 + 7$ is a polynomial so g is differentiable for all x and $g'(x) = 6x^2 - 24x = 6x(x-4)$ so $g'(x) = 0$ when $x = 0$ or 4, so the only candidates for a local extreme are $x = 0$ and $x = 4$. The graph of g (see margin) indicates that g has a local maximum at $(0, 7)$ and a local minimum at $(4, -57)$.



2. See the margin figure.

x	$f(x)$	$f'(x)$	max/min
1	5	0	local max
3	1	0	local min
4	3	0	neither
6	7	0	local max

3. $(1, f(1))$ is a global minimum; $x = 1$ is an endpoint
   $(2, f(2))$ is a local maximum; $f'(2) = 0$
   $(3, f(3))$ is a local/global minimum; f is not differentiable at $x = 3$
   $(4, f(4))$ is a global maximum; $x = 4$ is an endpoint

4. This is the same function used in Example 4, but now the interval is $[0, 5]$ instead of $[0, 10]$. See the Example for the calculations.

   Critical points:

   - endpoints: $x = 0$ and $x = 5$
   - f is differentiable for all $0 < x < 5$: none
   - $f'(x) = 0$: none in $[0, 5]$

   $f(0) \approx 4.67$ is the maximum of f on $[0, 5]$;
   $f(5) \approx 4.14$ is the minimum of f on $[0, 5]$.

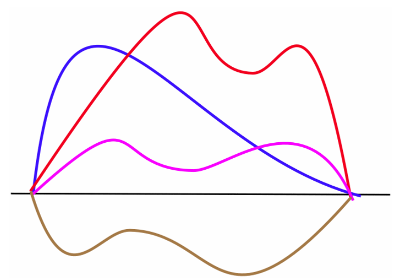
3.2 *Mean Value Theorem*

If you averaged 30 miles per hour during a trip, then at some instant during the trip you were traveling exactly 30 miles per hour.

That relatively obvious statement is the Mean Value Theorem as it applies to a particular trip. It may seem strange that such a simple statement would be important or useful to anyone, but the Mean Value Theorem is important and some of its consequences are very useful in a variety of areas. Many of the results in the rest of this chapter depend on the Mean Value Theorem, and one of the corollaries of the Mean Value Theorem will be used every time we calculate an "integral" in later chapters. A truly delightful aspect of mathematics is that an idea as simple and obvious as the Mean Value Theorem can be so powerful.

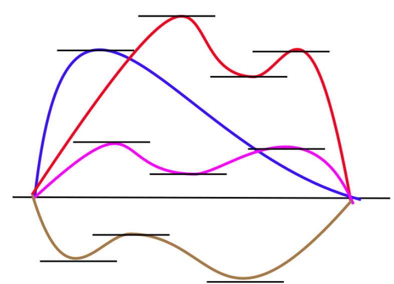
Before we state and prove the Mean Value Theorem and examine some of its consequences, we will consider a simplified version called Rolle's Theorem.

Rolle's Theorem



Pick any two points on the x-axis and think about all of the differentiable functions that pass through those two points. Because our functions are differentiable, they must be continuous and their graphs cannot have any holes or breaks. Also, since these functions are differentiable, their derivatives are defined everywhere between our two points and their graphs can not have any "corners" or vertical tangents.

The graphs of the functions in the margin figure can still have all sorts of shapes, and it may seem unlikely that they have any common properties other than the ones we have stated, but Michel Rolle (1652–1719) found one. He noticed that every one of these functions has one or more points where the tangent line is horizontal (see margin), and this result is named after him.



Rolle's Theorem:

If $f(a) = f(b)$
and $f(x)$ is continuous for $a \le x \le b$
and differentiable for $a < x < b$
then there is at least one number c between a and b so that $f'(c) = 0$.

Proof. We consider three cases: when $f(x) = f(a)$ for all x in (a, b), when $f(x) > f(a)$ for some x in (a, b), and when $f(x) < f(a)$ for some x in (a, b).

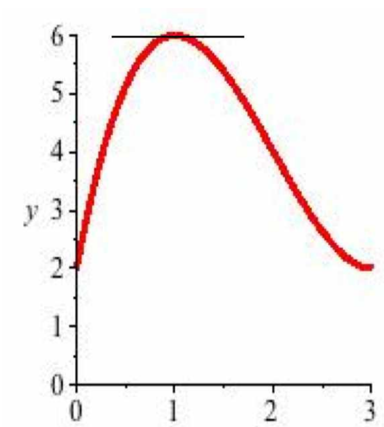
Case I: If $f(x) = f(a)$ for all x between a and b, then the graph of f is a horizontal line segment and $f'(c) = 0$ for *all* values of c strictly between a and b.

Case II: Suppose $f(x) > f(a)$ for some x in (a, b). Because f is continuous on the closed interval $[a, b]$, we know from the Extreme Value Theorem that f must attain a maximum value on the closed interval $[a, b]$. Because $f(x) > f(a)$ for some value of x in $[a, b]$, then the maximum of f must occur at some value c strictly between a and b: $a < c < b$. (Why can't the maximum be at a or b?) Because $f(c)$ is a local maximum of f, c is a critical number of f, meaning $f'(c) = 0$ or $f'(c)$ is undefined. But f is differentiable at all x between a and b, so the only possibility is that $f'(c) = 0$.

Notice that Rolle's Theorem tells us that (at least one) number c with the required properties exists, but does not tell us how to find c.

Case III: Suppose $f(x) < f(a)$ for some x in (a, b). Then, arguing as we did in Case II, f attains a minimum at some value $x = c$ strictly between a and b, and so $f'(c) = 0$.

In each case, there is *at least one* value of c between a and b so that $f'(c) = 0$. □



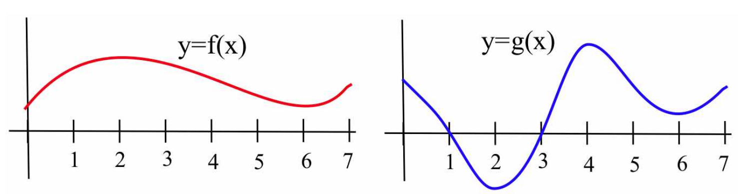
Example 1. Show that $f(x) = x^3 - 6x^2 + 9x + 2$ satisfies the hypotheses of Rolle's Theorem on the interval $[0, 3]$ and find a value of c that the theorem tells you must exist.

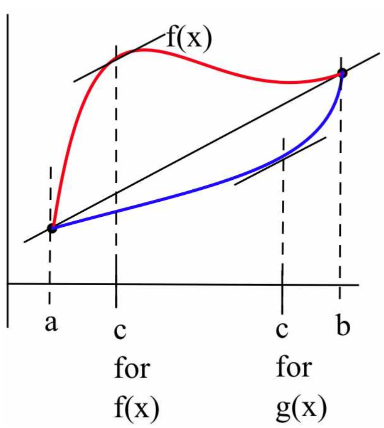
Solution. Because f is a polynomial, it is continuous and differentiable everywhere. Furthermore, $f(0) = 2 = f(3)$, so Rolle's Theorem applies. Differentiating:

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

so $f'(x) = 0$ when $x = 1$ and when $x = 3$. The value $c = 1$ is between 0 and 3. The margin figure shows a graph of f. ◀

Practice 1. Find the value(s) of c for Rolle's Theorem for the functions graphed below.





The Mean Value Theorem

Geometrically, the Mean Value Theorem is a "tilted" version of Rolle's Theorem (see margin). In each theorem we conclude that there is a number c so that the slope of the tangent line to f at $x = c$ is the same as the slope of the line connecting the two ends of the graph of f on the interval $[a, b]$. In Rolle's Theorem, the two ends of the graph of f are at the same height, $f(a) = f(b)$, so the slope of the line connecting the ends is zero. In the Mean Value Theorem, the two ends of the graph of f do not have to be at the same height, so the line through the two ends does not have to have a slope of zero.

Mean Value Theorem:

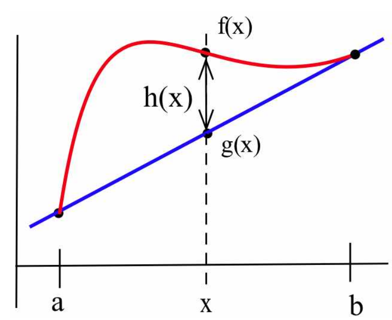
If $f(x)$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$

then there is at least one number c between a and b so the line tangent to the graph of f at $x = c$ is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. The proof of the Mean Value Theorem uses a tactic common in mathematics: introduce a new function that satisfies the hypotheses of some theorem we already know and then use the conclusion of that previously proven theorem. For the Mean Value Theorem we introduce a new function, $h(x)$, which satisfies the hypotheses of Rolle's Theorem. Then we can be certain that the conclusion of Rolle's Theorem is true for $h(x)$ and the Mean Value Theorem for f will follow from the conclusion of Rolle's Theorem for h.

First, let $g(x)$ be the linear function passing through the points $(a, f(a))$ and $(b, f(b))$ of the graph of f. The function g goes through the point $(a, f(a))$ so $g(a) = f(a)$. Similarly, $g(b) = f(b)$. The slope of the linear function $g(x)$ is $\dfrac{f(b) - f(a)}{b - a}$ so $g'(x) = \dfrac{f(b) - f(a)}{b - a}$ for all x between a and b, and g is continuous and differentiable. (The formula for g is $g(x) = f(a) + m(x - a)$ with $m = \frac{f(b)-f(a)}{b-a}$.)



Define $h(x) = f(x) - g(x)$ for $a \le x \le b$ (see margin). The function h satisfies the hypotheses of Rolle's theorem:

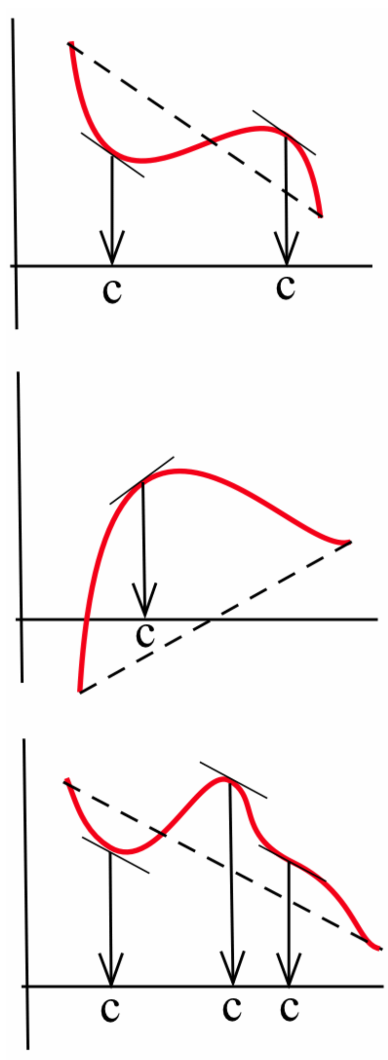
- $h(a) = f(a) - g(a) = 0$ and $h(b) = f(b) - g(b) = 0$
- $h(x)$ is continuous for $a \le x \le b$ because both f and g are continuous there
- $h(x)$ is differentiable for $a < x < b$ because both f and g are differentiable there

so the conclusion of Rolle's Theorem applies to h: there is a c between a and b so that $h'(c) = 0$.

The derivative of $h(x) = f(x) - g(x)$ is $h'(x) = f'(x) - g'(x)$ so we know that there is a number c between a and b with $h'(c) = 0$. But:

$$0 = h'(c) = f'(c) - g'(c) \Rightarrow f'(c) = g'(c) = \frac{f(b) - f(a)}{b - a}$$

which is exactly what we needed to prove. □



Graphically, the Mean Value Theorem says that there is at least one point c where the slope of the tangent line, $f'(c)$, equals the slope of the line through the end points of the graph segment, $(a, f(a))$ and $(b, f(b))$. The margin figure shows the locations of the parallel tangent lines for several functions and intervals.

The Mean Value Theorem also has a very natural interpretation if $f(x)$ represents the position of an object at time x: $f'(x)$ represents the velocity of the object at the **instant** x and $\frac{f(b)-f(a)}{b-a} = \frac{\text{change in position}}{\text{change in time}}$ represents the **average** (mean) velocity of the object during the time interval from time a to time b. The Mean Value Theorem says that there is a time c (between a and b) when the **instantaneous** velocity, $f'(c)$, is equal to the **average** velocity for the entire trip, $\frac{f(b)-f(a)}{b-a}$. If your average velocity during a trip is 30 miles per hour, then at some instant during the trip you were traveling exactly 30 miles per hour.

Practice 2. For $f(x) = 5x^2 - 4x + 3$ on the interval $[1,3]$, calculate $m = \frac{f(b)-f(a)}{b-a}$ and find the value(s) of c so that $f'(c) = m$.

Some Consequences of the Mean Value Theorem

If the Mean Value Theorem was just an isolated result about the existence of a particular point c, it would not be very important or useful. However, the Mean Value Theorem is the basis of several results about the behavior of functions over entire intervals, and it is these consequences that give it an important place in calculus for both theoretical and applied uses.

The next two corollaries are just the first of many results that follow from the Mean Value Theorem.

We already know, from the Main Differentiation Theorem, that the derivative of a constant function $f(x) = K$ is always 0, but can a nonconstant function have a derivative that is always 0? The first corollary says no.

Corollary 1:

If $\quad f'(x) = 0$ for all x in an interval I
then $\quad f(x) = K$, a constant, for all x in I.

Proof. Assume $f'(x) = 0$ for all x in an interval I. Pick any two points a and b (with $a \neq b$) in the interval. Then, by the Mean Value Theorem, there is a number c between a and b so that $f'(c) = \frac{f(b)-f(a)}{b-a}$. By our assumption, $f'(x) = 0$ for all x in I, so we know that $0 = f'(c) =$

$\dfrac{f(b)-f(a)}{b-a}$ and thus $f(b)-f(a)=0 \Rightarrow f(b)=f(a)$. But a and b were two arbitrary points in I, so the value of $f(x)$ is the same for any two values of x in I, and f is a constant function on the interval I. □

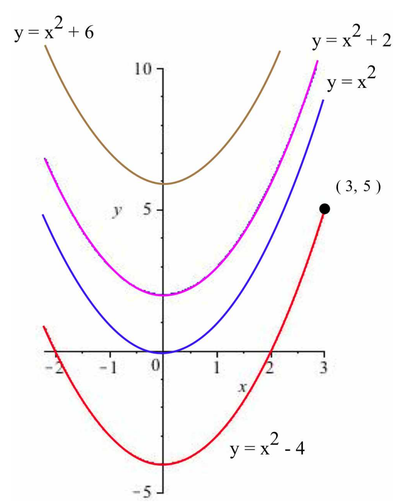
We already know that if two functions are "parallel" (differ by a constant), then their derivatives are equal, but can two non-parallel functions have the same derivative? The second corollary says no.

Corollary 2:

If	$f'(x)=g'(x)$ for all x in an interval I
then	$f(x)-g(x)=K$, a constant, for all x in I, so the graphs of f and g are "parallel" on the interval I.

Proof. This corollary involves two functions instead of just one, but we can imitate the proof of the Mean Value Theorem and introduce a new function $h(x)=f(x)-g(x)$. The function h is differentiable and $h'(x)=f'(x)-g'(x)=0$ for all x in I so, by Corollary 1, $h(x)$ is a constant function and $K=h(x)=f(x)-g(x)$ for all x in the interval. Thus $f(x)=g(x)+K$. □

We will use Corollary 2 hundreds of times in Chapters 4 and 5 when we work with "integrals." Typically you will be given the derivative of a function, $f'(x)$, and be asked to find **all** functions f that have that derivative. Corollary 2 tells us that if we can find **one** function f that has the derivative we want, then the only other functions that have the same derivative are of the form $f(x)+K$ where K is a constant: once you find one function with the right derivative, you have essentially found all of them.



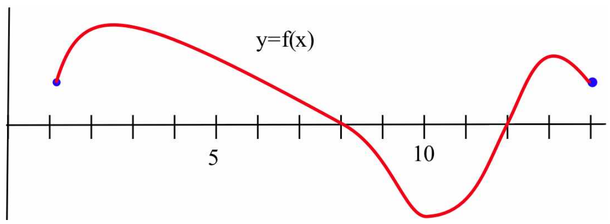
Example 2. (a) Find **all** functions whose derivatives equal $2x$. (b) Find a function $g(x)$ with $g'(x)=2x$ and $g(3)=5$.

Solution. (a) Observe that $f(x)=x^2 \Rightarrow f'(x)=2x$, so one function with the derivative we want is $f(x)=x^2$. Corollary 2 guarantees that every function g whose derivative is $2x$ has the form $g(x)=f(x)+K=x^2+K$. (b) Because $g'(x)=2x$, we know that g must have the form $g(x)=x^2+K$, but this gives a whole "family" of functions (see margin) and we want to find one member of that family. We also know that $g(3)=5$ so we want to find the member of the family that passes through the point $(3,5)$. Replacing $g(x)$ with 5 and x with 3 in the formula $g(x)=x^2+K$, we can solve for the value of K: $5=g(3)=(3)^2+K \Rightarrow K=-4$. The function we want is $g(x)=x^2-4$. ◀

Practice 3. Restate Corollary 2 as a statement about the positions and velocities of two cars.

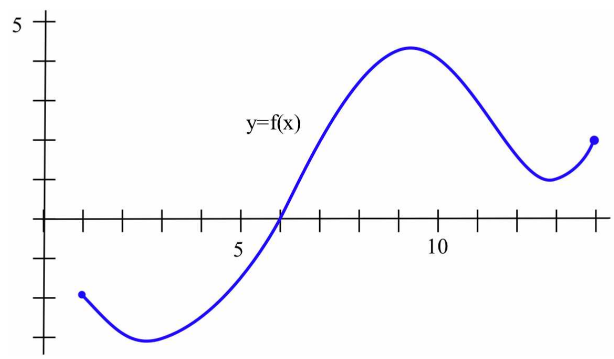
3.2 Problems

1. In the figure below, find the number(s) "c" that Rolle's Theorem promises (guarantees).



For Problems 2–4, verify that the hypotheses of Rolle's Theorem are satisfied for each of the functions on the given intervals, and find the value of the number(s) "c" that Rolle's Theorem promises.

2. (a) $f(x) = x^2$ on $[-2, 2]$
   (b) $f(x) = x^2 - 5x + 8$ on $[0, 5]$
3. (a) $f(x) = \sin(x)$ on $[0, \pi]$
   (b) $f(x) = \sin(x)$ on $[\pi, 5\pi]$
4. (a) $f(x) = x^3 - x + 3$ on $[-1, 1]$
   (b) $f(x) = x \cdot \cos(x)$ on $[0, \frac{\pi}{2}]$
5. Suppose you toss a ball straight up and catch it when it comes down. If $h(t)$ is the height of the ball t seconds after you toss it, what does Rolle's Theorem say about the velocity of the ball? Why is it easier to catch a ball that someone on the ground tosses up to you on a balcony, than for you to be on the ground and catch a ball that someone on a balcony tosses down to you?
6. If $f(x) = \dfrac{1}{x^2}$, then $f(-1) = 1$ and $f(1) = 1$ but $f'(x) = -\dfrac{2}{x^3}$ is never equal to 0. Why doesn't this function violate Rolle's Theorem?
7. If $f(x) = |x|$, then $f(-1) = 1$ and $f(1) = 1$ but $f'(x)$ is never equal to 0. Why doesn't this function violate Rolle's Theorem?
8. If $f(x) = x^2$, then $f'(x) = 2x$ is never 0 on the interval $[1, 3]$. Why doesn't this function violate Rolle's Theorem?
9. If I take off in an airplane, fly around for a while and land at the same place I took off from, then my starting and stopping heights are the same but the airplane is always moving. Why doesn't this violate Rolle's theorem, which says there is an instant when my velocity is 0?
10. Prove the following corollary of Rolle's Theorem: If $P(x)$ is a polynomial, then between any two roots of P there is a root of P'.
11. Use the corollary in Problem 10 to justify the conclusion that the only root of $f(x) = x^3 + 5x - 18$ is 2. (Suggestion: What could you conclude about f' if f had another root?)
12. In the figure below, find the location(s) of the "c" that the Mean Value Theorem promises.



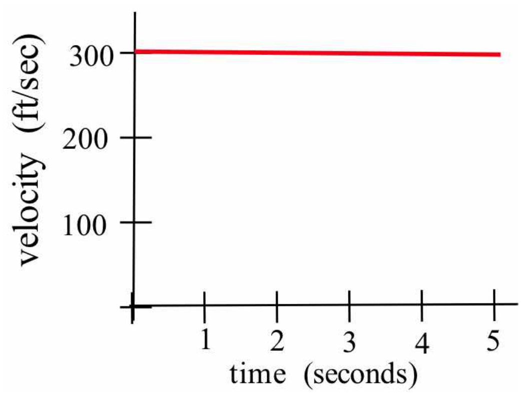
In Problems 13–15, verify that the hypotheses of the Mean Value Theorem are satisfied for each of the functions on the given intervals, and find the number(s) "c" that the Mean Value Theorem guarantees.

13. (a) $f(x) = x^2$ on $[0, 2]$
    (b) $f(x) = x^2 - 5x + 8$ on $[1, 5]$
14. (a) $f(x) = \sin(x)$ on $[0, \frac{\pi}{2}]$
    (b) $f(x) = x^3$ on $[-1, 3]$
15. (a) $f(x) = 5 - \sqrt{x}$ on $[1, 9]$
    (b) $f(x) = 2x + 1$ on $[1, 7]$

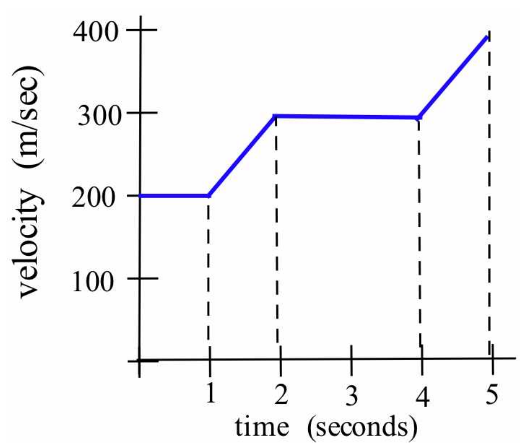
16. For the quadratic functions in parts (a) and (b) of Problem 13, the number c turned out to be the midpoint of the interval: $c = \frac{a+b}{2}$.
   (a) For $f(x) = 3x^2 + x - 7$ on $[1,3]$, show that $f'(2) = \dfrac{f(3) - f(1)}{3 - 1}$.
   (b) For $f(x) = x^2 - 5x + 3$ on $[2,5]$, show that $f'\left(\dfrac{7}{2}\right) = \dfrac{f(5) - f(2)}{5 - 2}$.
   (c) For $f(x) = Ax^2 + Bx + C$ on $[a,b]$, show that $f'\left(\dfrac{a+b}{2}\right) = \dfrac{f(b) - f(a)}{b - a}$.
17. If $f(x) = |x|$, then $f(-1) = 1$ and $f(3) = 3$ but $f'(x)$ is never equal to $\dfrac{f(3) - f(-1)}{3 - (-1)} = \dfrac{1}{2}$. Why doesn't this violate the Mean Value Theorem?

In Problems 18–19, you are a traffic-court judge. In each case, a driver has challenged a speeding ticket and you need to decide if the ticket is appropriate.

18. A tolltaker says, "Your Honor, based on the elapsed time from when the car entered the toll road until the car stopped at my booth, I know the average speed of the car was 83 miles per hour. I did not actually see the car speeding, but I know it was and I gave the driver a ticket."
19. The driver in the next case says, "Your Honor, my average velocity on that portion of the toll road was only 17 miles per hour, so I could not have been speeding."
20. Find three different functions (f, g and h) so that $f'(x) = g'(x) = h'(x) = \cos(x)$.
21. Find a function f so that $f'(x) = 3x^2 + 2x + 5$ and $f(1) = 10$.
22. Find $g(x)$ so that $g'(x) = x^2 + 3$ and $g(0) = 2$.
23. Find values for A and B so that the graph of the parabola $f(x) = Ax^2 + B$ is:
   (a) tangent to $y = 4x + 5$ at the point $(1,9)$.
   (b) tangent to $y = 7 - 2x$ at the point $(2,3)$.
   (c) tangent to $y = x^2 + 3x - 2$ at the point $(0,2)$.
24. Sketch the graphs of several members of the "family" of functions whose derivatives always equal 3. Give a formula that defines every function in this family.
25. Sketch the graphs of several members of the "family" of functions whose derivatives always equal $3x^2$. Give a formula that defines every function in this family.
26. At t seconds after takeoff, the upward velocity of a helicopter was $v(t) = 3t^2 + 2t$ feet/second. Two seconds after takeoff, the helicopter was 80 feet above sea level. Find a formula for the height of the helicopter at every time t.
27. Assume that a rocket is fired from the ground and has the upward velocity shown in the figure below. Estimate the height of the rocket when $t = 1$, 2 and 5 seconds.



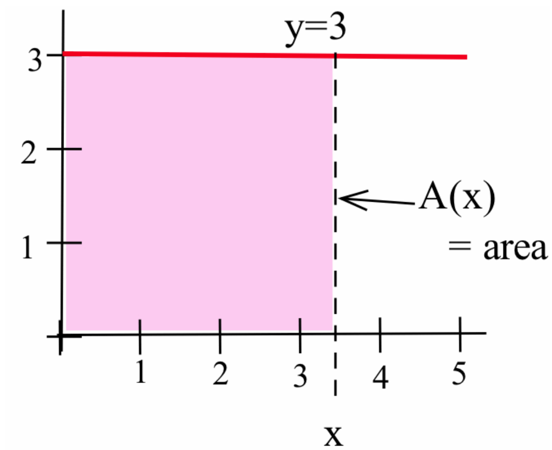
28. The figure below shows the upward velocity of a rocket. Use the information in the graph to estimate the height of the rocket when $t = 1$, 2 and 5 seconds.



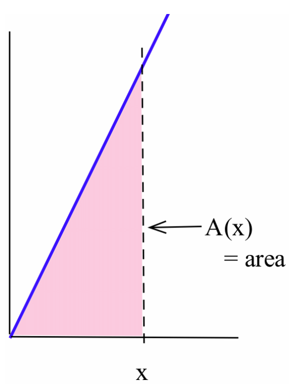
29. Determine a formula for $f(x)$ if you know: $f''(x) = 6$, $f'(0) = 4$ and $f(0) = -5$.

30. Determine a formula for $g(x)$ if you know: $g''(x) = 12x$, $g'(1) = 9$ and $g(2) = 30$.

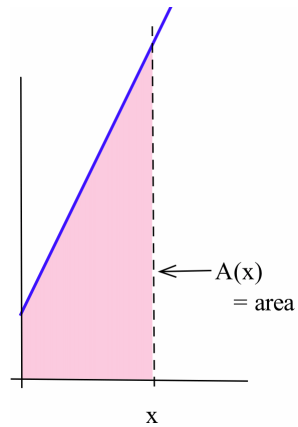
31. Define $A(x)$ to be the **area** bounded by the t-axis, the line $y = 3$ and a vertical line at $t = x$.
    (a) Find a formula for $A(x)$.
    (b) Determine $A'(x)$.



32. Define $A(x)$ to be the **area** bounded by the t-axis, the line $y = 2t$ and a vertical line at $t = x$.
    (a) Find a formula for $A(x)$.
    (b) Determine $A'(x)$.



33. Define $A(x)$ to be the **area** bounded by the t-axis, the line $y = 2t + 1$ and a vertical line at $t = x$.
    (a) Find a formula for $A(x)$.
    (b) Determine $A'(x)$.



In Problems 34–36, given a list of numbers a_1, a_2, a_3, a_4, $\ldots$, the **consecutive differences** between numbers in the list are: $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$, $\ldots$

34. If $a_1 = 5$ and the consecutive difference is always 0, what can you conclude about the numbers in the list?

35. If $a_1 = 5$ and the consecutive difference is always 3, find a formula for a_n.

36. Suppose the "a" list starts with 3, 4, 7, 8, 6, 10, 13,$\ldots$, and there is a "b" list that has the same consecutive differences as the "a" list.
    (a) If $b_1 = 5$, find the next six numbers in the "b" list. How is b_n related to a_n?
    (b) If $b_1 = 2$, find the next six numbers in the "b" list. How is b_n related to a_n?
    (c) If $b_1 = B$, find the next six numbers in the "b" list. How is b_n related to a_n?

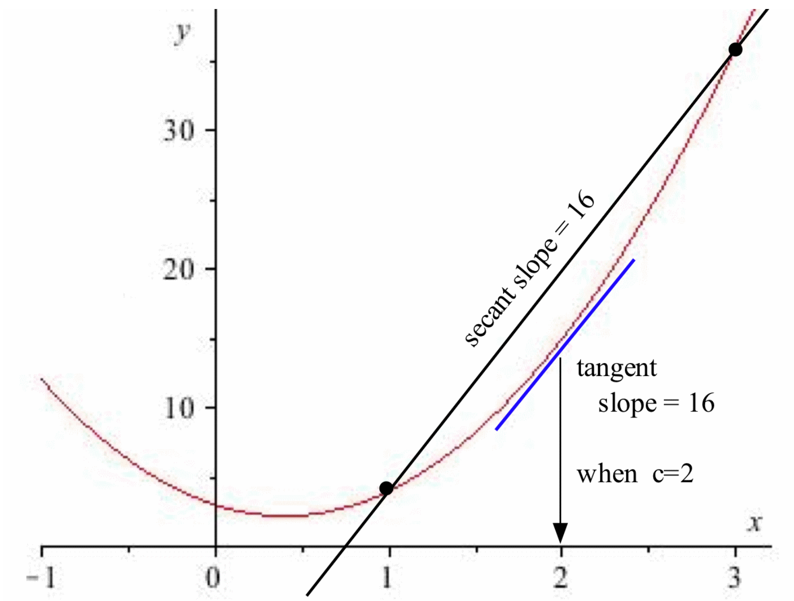
3.2 Practice Answers

1. $f'(x) = 0$ when $x = 2$ and 6, so $c = 2$ and $c = 6$.
   $g'(x) = 0$ when $x = 2$, 4 and 6, so $c = 2$, $c = 4$ and $c = 6$.

2. With $f(x) = 5x^2 - 4x + 3$ on $[1,3]$, $f(1) = 4$ and $f(3) = 36$ so:

$$m = \frac{f(b) - f(a)}{b - a} = \frac{36 - 4}{3 - 1} = 16$$

   $f'(x) = 10x - 4$ so $f'(c) = 10c - 4 = 16 \Rightarrow 10c = 20 \Rightarrow c = 2$. The graph of f showing the location of c appears below.



3. If two cars have the same velocities during an interval of time (so that $f'(t) = g'(t)$ for t in I) then the cars are always a constant distance apart during that time interval. (Note: "Same velocity" means **same speed** and **same direction**. If two cars are traveling at the same speed but in different directions, then the distance between them changes and is not constant.)

3.3 *The First Derivative and the Shape of f*

This section examines some of the interplay between the shape of the graph of a function f and the behavior of its derivative, f'. If we have a graph of f, we will investigate what we can conclude about the values of f'. And if we know values of f', we will investigate what we can conclude about the graph of f.

In this definition, I can be of the form (a,b), $[a,b)$, $(a,b]$, $[a,b]$, $(-\infty,b)$, $(-\infty,b]$, (a,∞), $[a,\infty)$ or $(-\infty,\infty)$, where $a<b$.

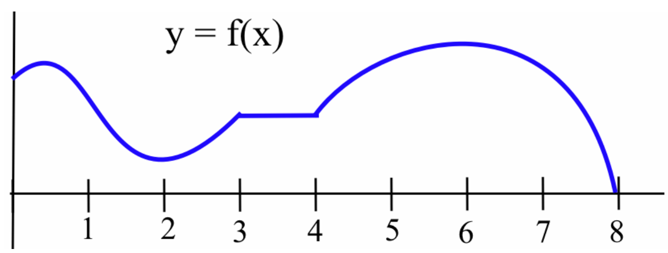
Definitions: Given any interval I, a function f is...

increasing on I if, for all x_1 and x_2 in I, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
decreasing on I if, for all x_1 and x_2 in I, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
monotonic on I if f is increasing or decreasing on I

Graphically, f is increasing (decreasing) if, as we move from left to right along the graph of f, the height of the graph increases (decreases).

These same ideas make sense if we consider $h(t)$ to be the height (in feet) of a rocket at time t seconds. We naturally say that the rocket is rising or that its height is increasing if the height $h(t)$ increases over a period of time, as t increases.

Example 1. List the intervals on which the function graphed below is increasing or decreasing.



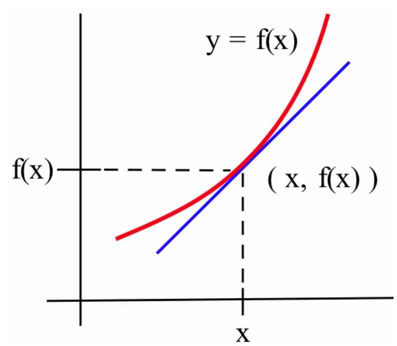
Solution. f is increasing on the intervals $[0,0.3]$ (approximately), $[2,3]$ and $[4,6]$. f is decreasing on (approximately) $[0.3,2]$ and $[6,8]$. On the interval $[3,4]$ the function is not increasing or decreasing—it is **constant**. It is also valid to say that f is increasing on the intervals $[0.5,0.8]$ and $(0.5,0.8)$ as well as many others, but we usually talk about the longest intervals on which f is monotonic. ◀

Practice 1. List the intervals on which the function graphed below is increasing or decreasing.

If we have an accurate graph of a function, then it is relatively easy to determine where f is monotonic, but if the function is defined by a formula, then a little more work is required. The next two theorems relate the values of the derivative of f to the monotonicity of f. The first theorem says that if we know where f is monotonic, then we also know something about the values of f'. The second theorem says that if we know about the values of f' then we can draw conclusions about where f is monotonic.

First Shape Theorem:
For a function f that is differentiable on an interval (a, b):

- if f is increasing on (a, b) then $f'(x) \geq 0$ for all x in (a, b)
- if f is decreasing on (a, b) then $f'(x) \leq 0$ for all x in (a, b)
- if f is constant on (a, b), then $f'(x) = 0$ for all x in (a, b)



Proof. Most people find a picture such as the one in the margin to be a convincing justification of this theorem: if the graph of f increases near a point $(x, f(x))$, then the tangent line is also increasing, and the slope of the tangent line is positive (or perhaps zero at a few places). A more precise proof, however, requires that we use the definitions of the derivative of f and of "increasing" (given above).

Case I: Assume that f is increasing on (a, b). We know that f is differentiable, so if x is any number in the interval (a, b) then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

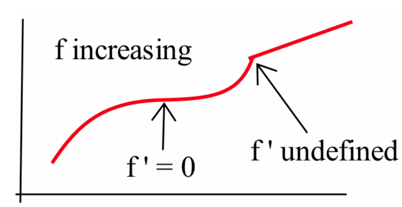
and this limit exists and is a finite value. If h is any small enough **positive** number so that $x + h$ is also in the interval (a, b), then $x < x + h \Rightarrow f(x) < f(x+h)$ (by the definition of "increasing"). We know that the numerator, $f(x+h) - f(x)$, and the denominator, h, are both positive, so the limiting value, $f'(x)$, must be positive or zero: $f'(x) \geq 0$.

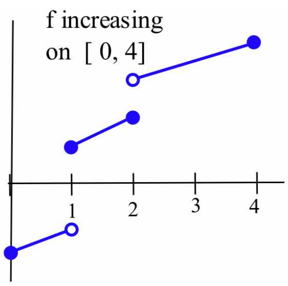
Case II: Assume that f is decreasing on (a, b). If $x < x + h$, then $f(x) > f(x+h)$ (by the definition of "decreasing"). So the numerator of the limit, $f(x+h) - f(x)$, will be negative but the denominator, h, will still be positive, so the limiting value, $f'(x)$, must be negative or zero: $f'(x) \leq 0$.

The proof of this part is very similar to the "increasing" proof.

Case III: The derivative of a constant is 0, so if f is constant on (a, b) then $f'(x) = 0$ for all x in (a, b). □

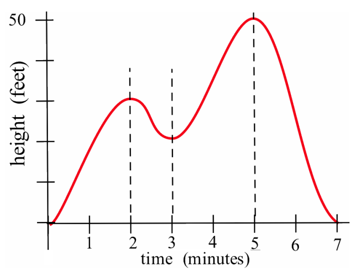
The previous theorem is easy to understand, but you need to pay attention to exactly what it says and what it does **not** say. It is possible for a differentiable function that is increasing on an interval to have horizontal tangent lines at some places in the interval (see margin). It is



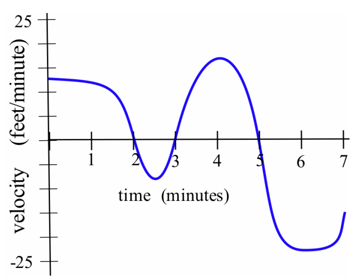


also possible for a continuous function that is increasing on an interval to have an undefined derivative at some places in the interval. Finally, it is possible for a function that is increasing on an interval to fail to be continuous at some places in the interval (see margin).

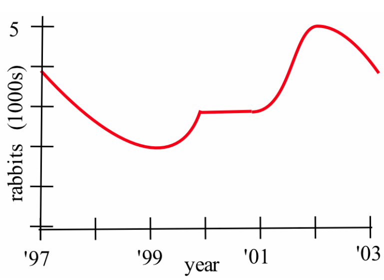
The First Shape Theorem has a natural interpretation in terms of the height $h(t)$ and upward velocity $h'(t)$ of a helicopter at time t. If the height of the helicopter is increasing ($h(t)$ is an increasing function), then the helicopter has a positive or zero upward velocity: $h'(t) \geq 0$. If the height of the helicopter is not changing, then its upward velocity is 0: $h'(t) = 0$.



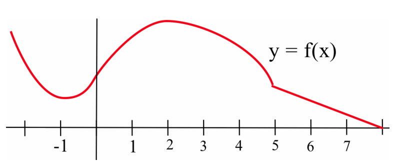
Example 2. A figure in the margin shows the height of a helicopter during a period of time. Sketch the graph of the upward velocity of the helicopter, $\frac{dh}{dt}$.



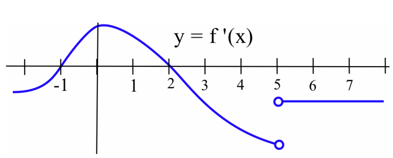
Solution. The graph of $v(t) = \frac{dh}{dt}$ appears in the margin. Notice that $h(t)$ has a local maximum when $t = 2$ and $t = 5$, and that $v(2) = 0$ and $v(5) = 0$. Similarly, $h(t)$ has a local minimum when $t = 3$, and $v(3) = 0$. When h is increasing, v is positive. When h is decreasing, v is negative. ◀



Practice 2. A figure in the margin shows the population of rabbits on an island during a 6-year period. Sketch the graph of the rate of population change, $\frac{dR}{dt}$, during those years.



Example 3. A graph of f appears in the margin; sketch a graph of f'.

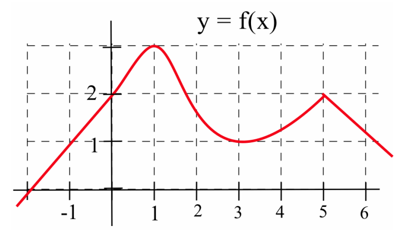


Solution. It is a good idea to look first for the points where $f'(x) = 0$ or where f is not differentiable (the critical points of f). These locations are usually easy to spot, and they naturally break the problem into several smaller pieces. The only numbers at which $f'(x) = 0$ are $x = -1$ and $x = 2$, so the only places the graph of $f'(x)$ will cross the x-axis are at $x = -1$ and $x = 2$: we can therefore plot the points $(-1, 0)$ and $(2, 0)$ on the graph of f'. The only place where f is not differentiable is at the "corner" above $x = 5$, so the graph of f' will not be defined for $x = 5$. The rest of the graph of f is relatively easy to sketch:

- if $x < -1$ then $f(x)$ is decreasing so $f'(x)$ is negative
- if $-1 < x < 2$ then $f(x)$ is increasing so $f'(x)$ is positive
- if $2 < x < 5$ then $f(x)$ is decreasing so $f'(x)$ is negative
- if $5 < x$ then $f(x)$ is decreasing so $f'(x)$ is negative

A graph of f' appears on the previous page: $f(x)$ is continuous at $x = 5$, but not differentiable at $x = 5$ (indicated by the "hole"). ◀

Practice 3. A graph of f appears in the margin. Sketch a graph of f'. (The graph of f has a "corner" at $x = 5$.)



The next theorem is almost the converse of the First Shape Theorem and explains the relationship between the values of the derivative and the graph of a function from a different perspective. It says that if we know something about the values of f', then we can draw some conclusions about the shape of the graph of f.

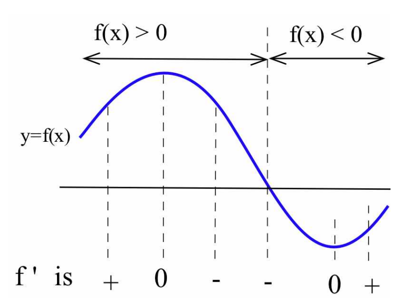
> **Second Shape Theorem**:
> For a function f that is differentiable on an interval I:
>
> - if $f'(x) > 0$ for all x in the interval I, then f is increasing on I
> - if $f'(x) < 0$ for all x in the interval I, then f is decreasing on I
> - if $f'(x) = 0$ for all x in the interval I, then f is constant on I

Proof. This theorem follows directly from the Mean Value Theorem, and the last part is just a restatement of the First Corollary of the Mean Value Theorem.

Case I: Assume that $f'(x) > 0$ for all x in I and pick any points a and b in I with $a < b$. Then, by the Mean Value Theorem, there is a point c between a and b so that $\frac{f(b) - f(a)}{b - a} = f'(c) > 0$ and we can conclude that $f(b) - f(a) > 0$, which means that $f(b) > f(a)$. Because $a < b \Rightarrow f(a) < f(b)$, we know that f is increasing on I.

Case II: Assume that $f'(x) < 0$ for all x in I and pick any points a and b in I with $a < b$. Then there is a point c between a and b so that $\frac{f(b) - f(a)}{b - a} = f'(c) < 0$, and we can conclude that $f(b) - f(a) = (b - a)f'(c) < 0$ so $f(b) < f(a)$. Because $a < b \Rightarrow f(a) > f(b)$, we know f is decreasing on I. □

Practice 4. Rewrite the Second Shape Theorem as a statement about the height $h(t)$ and upward velocity $h'(t)$ of a helicopter at time t seconds.



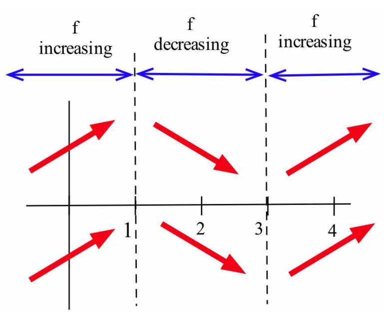
The value of a function f at a number x tells us the height of the graph of f above or below the point $(x, 0)$ on the x-axis. The value of f' at a number x tells us whether the graph of f is increasing or decreasing (or neither) as the graph passes through the point $(x, f(x))$ on the graph of f. If $f(x)$ is positive, it is possible for $f'(x)$ to be positive, negative, zero or undefined: the value of $f(x)$ has absolutely nothing to do with the value of f'. The margin figure illustrates some of the possible combinations of values for f and f'.

Practice 5. Graph a continuous function that satisfies the conditions on f and f' given below:

x	-2	-1	0	1	2	3
$f(x)$	1	-1	-2	-1	0	2
$f'(x)$	-1	0	1	2	-1	1

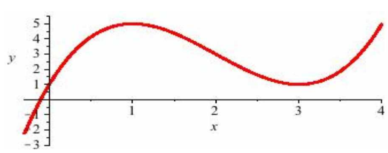
The Second Shape Theorem can be particularly useful if we need to graph a function f defined by a formula. Between any two consecutive critical numbers of f, the graph of f is monotonic (why?). If we can find all of the critical numbers of f, then the domain of f will be broken naturally into a number of pieces on which f will be monotonic.

Example 4. Use information about the values of f' to help graph $f(x) = x^3 - 6x^2 + 9x + 1$.

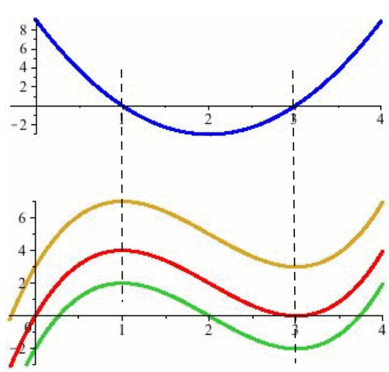


Solution. $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$ so $f'(x) = 0$ only when $x = 1$ or $x = 3$; f' is a polynomial, so it is always defined. The only critical numbers, $x = 1$ and $x = 3$, break the real number line into three pieces on which f is monotonic: $(-\infty, 1)$, $(1, 3)$ and $(3, \infty)$.

- $x < 1 \Rightarrow f'(x) = 3(\text{negative})(\text{negative}) > 0 \Rightarrow f$ increasing
- $1 < x < 3 \Rightarrow f'(x) = 3(\text{positive})(\text{negative}) < 0 \Rightarrow f$ is decreasing
- $3 < x \;\Rightarrow f'(x) = 3(\text{positive})(\text{positive}) > 0 \Rightarrow f$ is increasing



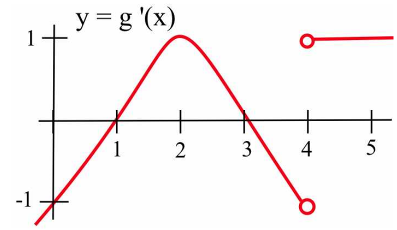
Although we don't yet know the value of f anywhere, we do know a lot about the shape of the graph of f: as we move from left to right along the x-axis, the graph of f increases until $x = 1$, then decreases until $x = 3$, after which the graph increases again (see margin). The graph of f "turns" when $x = 1$ and $x = 3$. To plot the graph of f, we still need to evaluate f at a few values of x, but only at a **very** few values: $f(1) = 5$, and $(1, 5)$ is a local maximum of f; $f(3) = 1$, and $(3, 1)$ is a local minimum of f. A graph of f appears in the margin. ◀



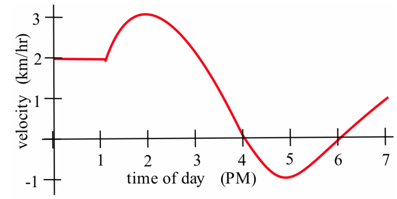
Practice 6. Use information about the values of f' to help graph the function $f(x) = x^3 - 3x^2 - 24x + 5$.

Example 5. Use the graph of f' in the margin to sketch the shape of the graph of f. Why isn't the graph of f' enough to completely determine the graph of f?

Solution. Several functions that have the derivative we want appear in the margin, and each provides a correct answer. By the Second Corollary to the Mean Value Theorem, we know there is a whole family of "parallel" functions that share the derivative we want, and each of these functions provides a correct answer. If we had additional information about the function—such as a point it passes through—then only one member of the family would satisfy the extra condition and there would be only one correct answer. ◀



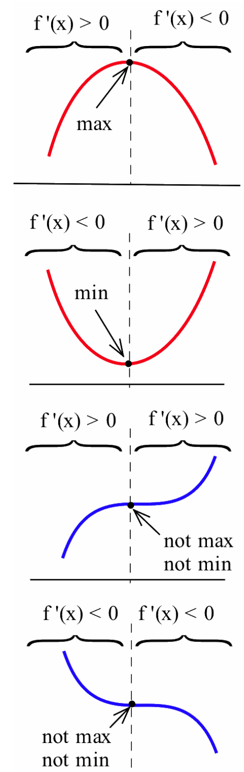
Practice 7. Use the graph of g' provided in the margin to sketch the shape of a graph of g.



Practice 8. A weather balloon is released from the ground and sends back its upward velocity measurements (see margin). Sketch a graph of the height of the balloon. When was the balloon highest?

Using the Derivative to Test for Extremes

The first derivative of a function tells about the general shape of the function, and we can use that shape information to determine whether an extreme point is a (local) maximum or minimum or neither.



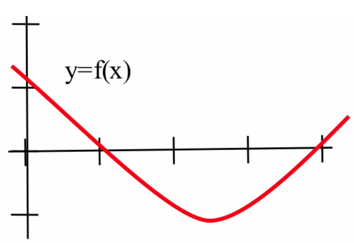
First Derivative Test for Local Extremes:
Let f be a continuous function with $f'(c) = 0$ or $f'(c)$ undefined.

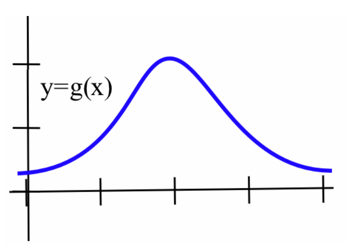
- If $f'(\text{left of } c) > 0$ and $f'(\text{right of } c) < 0$ then $(c, f(c))$ is a local maximum.
- If $f'(\text{left of } c) < 0$ and $f'(\text{right of } c) > 0$ then $(c, f(c))$ is a local minimum.
- If $f'(\text{left of } c) > 0$ and $f'(\text{right of } c) > 0$ then $(c, f(c))$ is **not** a local extreme.
- If $f'(\text{left of } c) < 0$ and $f'(\text{right of } c) < 0$ then $(c, f(c))$ is **not** a local extreme.

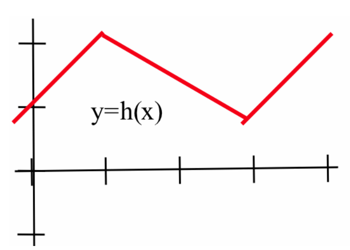
Practice 9. Find all extremes of $f(x) = 3x^2 - 12x + 7$ and use the First Derivative Test to classify them as maximums, minimums or neither.

3.3 Problems

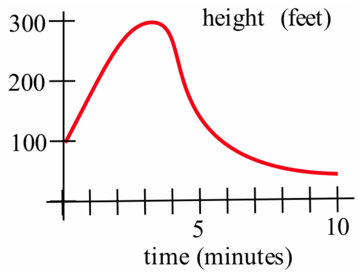
In Problems 1–3, sketch the graph of the **derivative** of each function.

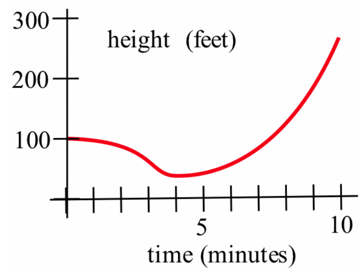
1. 

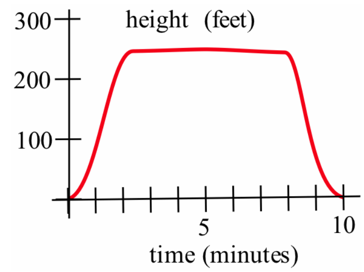
2. 

3. 

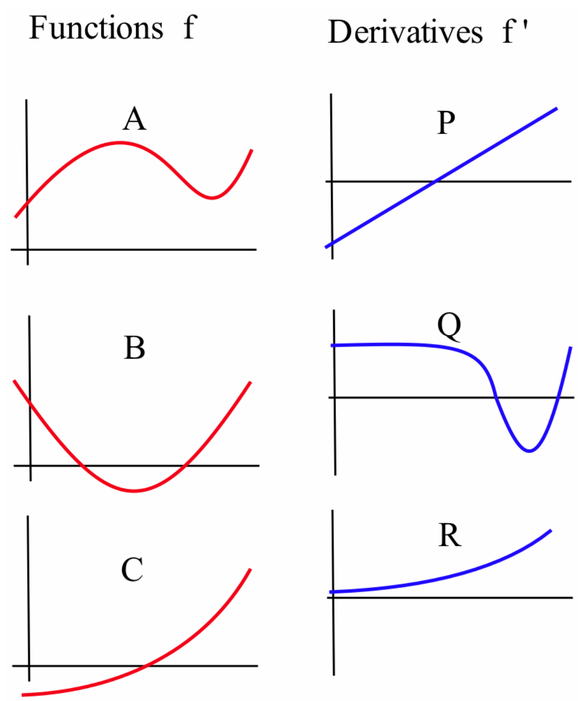
Problems 4–6 show the graph of the height of a helicopter; sketch a graph of its upward velocity.

4. 

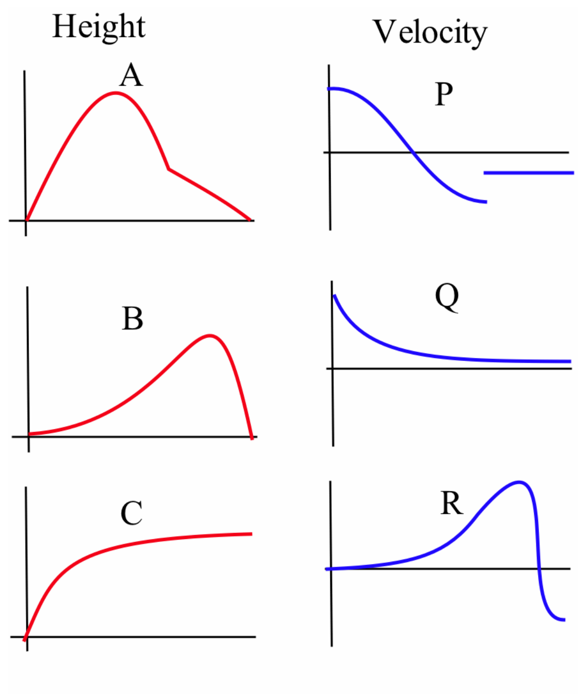
5. 

6. 

7. In the figure below, match the graphs of the functions with those of their derivatives.



8. Match the graphs showing the heights of rockets with those showing their velocities.

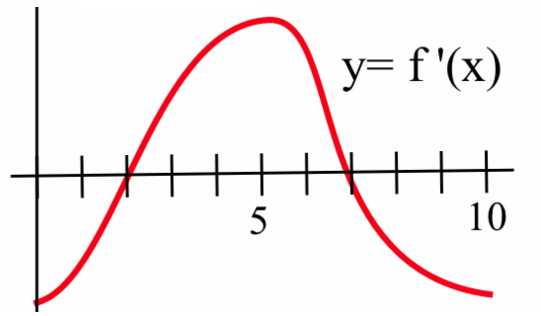


9. Use the Second Shape Theorem to show that $f(x) = \ln(x)$ is monotonic increasing on the interval $(0, \infty)$.

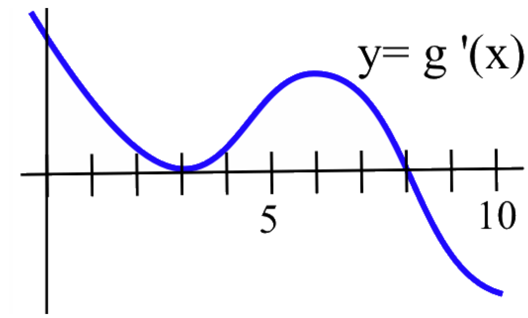
10. Use the Second Shape Theorem to show that $g(x) = e^x$ is monotonic increasing on the entire real number line.

11. A student is working with a complicated function f and has shown that the derivative of f is always positive. A minute later the student also claims that $f(x) = 2$ when $x = 1$ and when $x = \pi$. Without checking the student's work, how can you be certain that it contains an error?

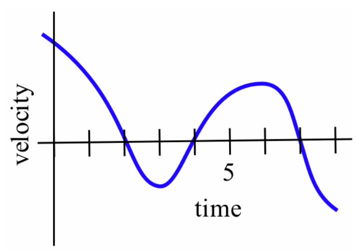
12. The figure below shows the graph of the **derivative** of a continuous function f.
    (a) List the critical numbers of f.
    (b) What values of x result in a local maximum?
    (c) What values of x result in a local minimum?

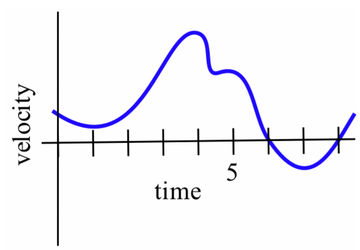


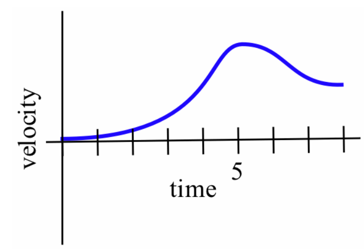
13. The figure below shows the graph of the **derivative** of a continuous function g.
    (a) List the critical numbers of g.
    (b) What values of x result in a local maximum?
    (c) What values of x result in a local minimum?



Problems 14–16 show the graphs of the upward velocities of three helicopters. Use the graphs to determine when each helicopter was at a (relative) maximum or minimum height.

14.


15.


16.


In 17–22, use information from the derivative of each function to help you graph the function. Find all local maximums and minimums of each function.

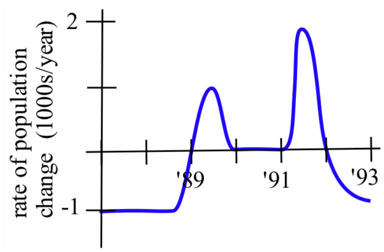
17. $f(x) = x^3 - 3x^2 - 9x - 5$
18. $g(x) = 2x^3 - 15x^2 + 6$
19. $h(x) = x^4 - 8x^2 + 3$
20. $s(t) = t + \sin(t)$
21. $r(t) = \dfrac{2}{t^2+1}$
22. $f(x) = \dfrac{x^2+3}{x}$
23. $f(x) = 2x + \cos(x)$ so $f(0) = 1$. Without graphing the function, you can be certain that f has how many **positive** roots?
24. $g(x) = 2x - \cos(x)$ so $g(0) = -1$. Without graphing the function, you can be certain that g has how many **positive** roots?
25. $h(x) = x^3 + 9x - 10$ has a root at $x = 1$. Without graphing h, show that h has no other roots.
26. Sketch the graphs of monotonic decreasing functions that have exactly (a) no roots (b) one root and (c) two roots.
27. Each of the following statements is false. Give (or sketch) a counterexample for each statement.
    (a) If f is increasing on an interval I, then $f'(x) > 0$ for all x in I.
    (b) If f is increasing and differentiable on I, then $f'(x) > 0$ for all x in I.
    (c) If cars A and B always have the same speed, then they will always be the same distance apart.

28. (a) Find several different functions f that all have the same derivative $f'(x) = 2$.
    (b) Determine a function f with derivative $f'(x) = 2$ that also satisfies $f(1) = 5$.
    (c) Determine a function g with $g'(x) = 2$ for which the graph of g goes through $(2, 1)$.

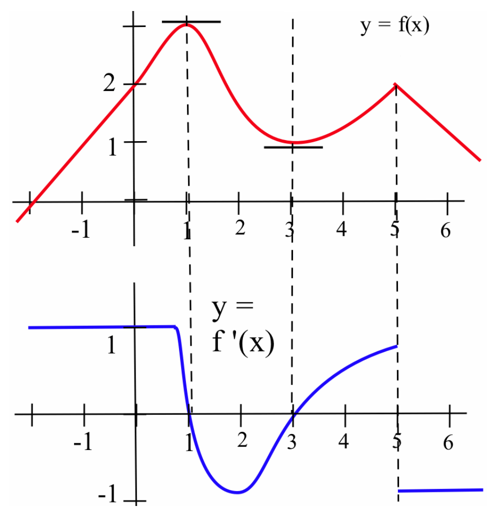
29. (a) Find several different functions h that all have the same derivative $h'(x) = 2x$.
    (b) Determine a function f with derivative $f'(x) = 2x$ that also satisfies $f(3) = 20$.
    (c) Determine a function g with $g'(x) = 2x$ for which the graph of g goes through $(2, 7)$.

30. Sketch functions with the given properties to help determine whether each statement is true or false.
    (a) If $f'(7) > 0$ and $f'(x) > 0$ for all x near 7, then $f(7)$ is a local maximum of f on $[1, 7]$.
    (b) If $g'(7) < 0$ and $g'(x) < 0$ for all x near 7, then $g(7)$ is a local minimum of g on $[1, 7]$.
    (c) If $h'(1) > 0$ and $h'(x) > 0$ for all x near 1, then $h(1)$ is a local minimum of h on $[1, 7]$.
    (d) If $r'(1) < 0$ and $r'(x) < 0$ for all x near 1, then $r(1)$ is a local maximum of r on $[1, 7]$.
    (e) If $s'(7) = 0$, then $s(7)$ is a local maximum of s on $[1, 7]$.

3.3 *Practice Answers*



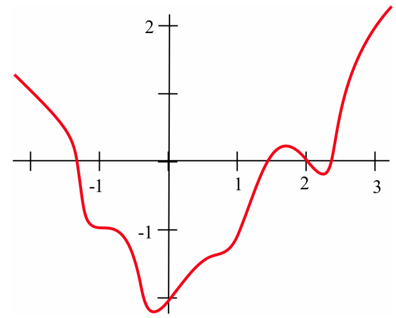
1. g is increasing on $[2, 4]$ and $[6, 8]$; g is decreasing on $[0, 2]$ and $[4, 5]$; g is constant on $[5, 6]$.
2. The graph in the margin shows the rate of population change, $\frac{dR}{dt}$.
3. A graph of f' appears below. Notice how the graph of f' is 0 where f has a maximum or minimum.



4. The Second Shape Theorem for helicopters:
   - If the upward velocity h' is positive during time interval I then the height h is increasing during time interval I.
   - If the upward velocity h' is negative during time interval I then the height h is decreasing during time interval I.
   - If the upward velocity h' is zero during time interval I then the height h is constant during time interval I.

5. A graph satisfying the conditions in the table appears in the margin.

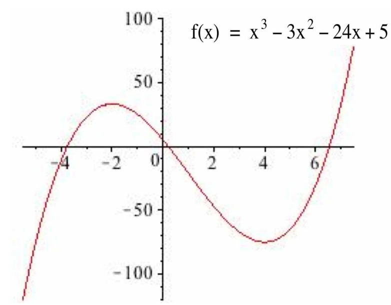
x	-2	-1	0	1	2	3
$f(x)$	1	-1	-2	-1	0	2
$f'(x)$	-1	0	1	2	-1	1



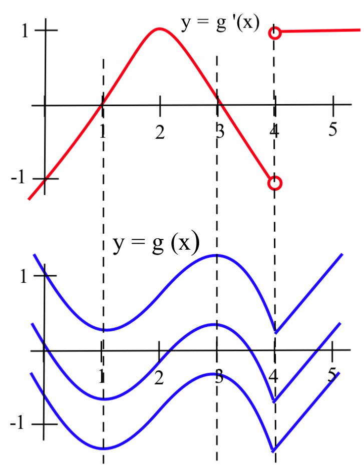
6. $f'(x) = 3x^2 - 6x - 24 = 3(x-4)(x+2)$ so $f'(x) = 0$ if $x = -2$ or $x = 4$.

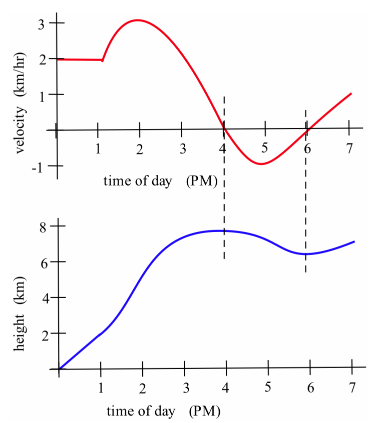
- $x < -2 \Rightarrow f'(x) = 3(\text{negative})(\text{negative}) > 0 \Rightarrow f$ increasing
- $-2 < x < 4 \Rightarrow f'(x) = 3(\text{negative})(\text{positive}) < 0 \Rightarrow f$ decreasing
- $x > 4 \Rightarrow f'(x) = 3(\text{positive})(\text{positive}) > 0 \Rightarrow f$ increasing

Thus f has a relative maximum at $x = -2$ and a relative minimum at $x = 4$. A graph of f appears in the margin.



7. The figure below left shows several possible graphs for g. Each has the correct shape to give the graph of g'. Notice that the graphs of g are "parallel" (differ by a constant).





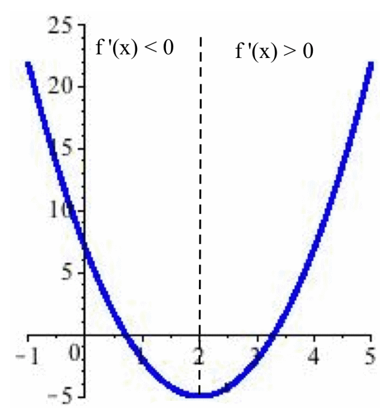
8. The figure above right shows the height graph for the balloon. The balloon was highest at 4 p.m. and had a local minimum at 6 p.m.

9. $f'(x) = 6x - 12$ so $f'(x) = 0$ only if $x = 2$.

- $x < 2 \Rightarrow f'(x) < 0 \Rightarrow f$ decreasing
- $x > 2 \Rightarrow f'(x) > 0 \Rightarrow f$ increasing

From this we can conclude that f has a minimum when $x = 2$ and has a shape similar to graph provided in the margin.

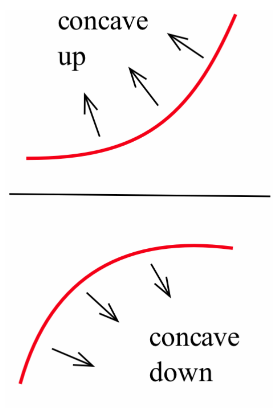
We could also have noticed that the graph of the quadratic function $f(x) = 3x^2 - 12x + 7$ must be an upward-opening parabola.



3.4 *The Second Derivative and the Shape of f*

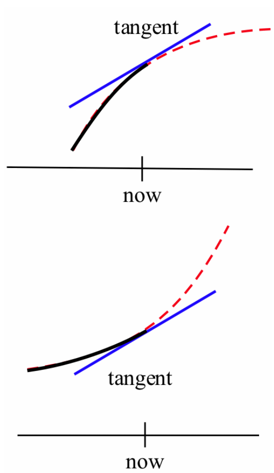
The first derivative of a function provides information about the shape of the function, so the second derivative of a function provides information about the shape of the first derivative, which in turn will provide additional information about the shape of the original function f.

In this section we investigate how to use the second derivative (and the shape of the first derivative) to reach conclusions about the shape of the original function. The first derivative tells us whether the graph of f is increasing or decreasing. The second derivative will tell us about the "concavity" of f: whether f is curving upward or downward.



Concavity

Graphically, a function is **concave up** if its graph is curved with the opening upward (see margin); similarly, a function is **concave down** if its graph opens downward. The concavity of a function can be important in applied problems and can even affect billion-dollar decisions.



An Epidemic: Suppose you, as an official at the CDC, must decide whether current methods are effectively fighting the spread of a disease — or whether more drastic measures are required. In the margin figure, $f(x)$ represents the number of people infected with the disease at time x in two different situations. In both cases the number of people with the disease, $f(\text{now})$, and the rate at which new people are getting sick, $f'(\text{now})$, are the same. The difference is the concavity of f, and that difference might have a big effect on your decision. In (a), f is concave down at "now," and it appears that the current methods are starting to bring the epidemic under control; in (b), f is concave up, and it appears that the epidemic is growing out of control.

Usually it is easy to determine the concavity of a function by examining its graph, but we also need a definition that does not require a graph of the function, a definition we can apply to a function described by a formula alone.

Definition: Let f be a differentiable function.

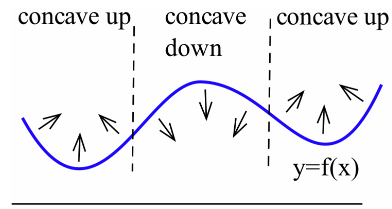
- f is **concave up** at a if the graph of f is above the tangent line L to f for all x close to (but not equal to) a:

$$f(x) > L(x) = f(a) + f'(a)(x - a)$$

- f is **concave down** at a if the graph of f is below the tangent line L to f for all x close to (but not equal to) a:

$$f(x) < L(x) = f(a) + f'(a)(x - a)$$

The margin figure shows the concavity of a function at several points. The next theorem provides an easily applied test for the concavity of a function given by a formula.

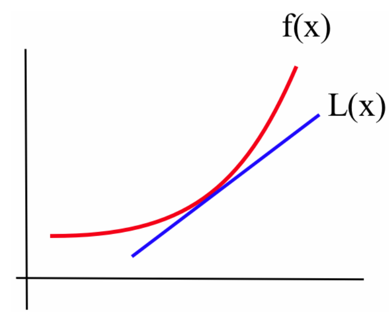


The Second Derivative Condition for Concavity:
Let f be a twice differentiable function on an interval I.

(a) $f''(x) > 0$ on $I \Rightarrow f'(x)$ increasing on $I \Rightarrow f$ concave up on I

(b) $f''(x) < 0$ on $I \Rightarrow f'(x)$ decreasing on $I \Rightarrow f$ concave down on I

(c) $f''(a) = 0 \Rightarrow$ no information
($f(x)$ may be concave up or concave down or neither at a)



Proof. (a) Assume that $f''(x) > 0$ for all x in I, and let a be any point in I. We want to show that f is concave up at a, so we need to prove that the graph of f (see margin) is above the tangent line to f at a: $f(x) > L(x) = f(a) + f'(a)(x-a)$ for x close to a. Assume that x is in I and apply the Mean Value Theorem to f on the interval with endpoints a and x: there is a number c between a and x so that

$$f'(c) = \frac{f(x) - f(a)}{x - a} \Rightarrow f(x) = f(a) + f'(c)(x-a)$$

Because $f'' > 0$ on I, we know that $f'' > 0$ between a and x, so the Second Shape Theorem tells us that f' is increasing between a and x. We will consider two cases: $x > a$ and $x < a$.

- If $x > a$ then $x - a > 0$ and c is in the interval $[a, x]$ so $a < c$. Because f' is increasing, $a < c \Rightarrow f'(a) < f'(c)$. Multiplying each side of this last inequality by the positive quantity $x - a$ yields $f'(a)(x-a) < f'(c)(x-a)$. Adding $f(a)$ to each side of this last inequality, we have:

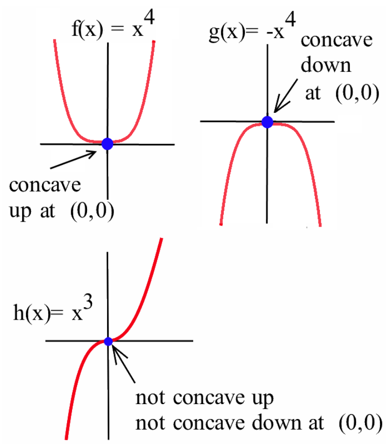
$$L(x) = f(a) + f'(a)(x-a) < f(a) + f'(c)(x-a) = f(x)$$

- If $x < a$ then $x - a < 0$ and c is in the interval $[x, a]$ so $c < a$. Because f' is increasing, $c < a \Rightarrow f'(c) < f'(a)$. Multiplying each side of this last inequality by the negative quantity $x - a$ yields $f'(c)(x-a) > f'(a)(x-a)$ so:

$$f(x) = f(a) + f'(c)(x-a) > f(a) + f'(a)(x-a) = L(x)$$

In each case we see that $f(x)$ is above the tangent line $L(x)$.

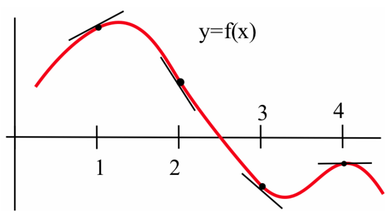
(b) The proof of this part is similar.



(c) Let $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$ (see margin). The second derivative of each of these functions is zero at $a = 0$, and at $(0,0)$ they all have the same tangent line: $L(x) = 0$ (the x-axis). However, at $(0,0)$ they all have different concavity: f is concave up, while g is concave down and h is neither concave up nor concave down. ☐

Practice 1. Use the graph of f in the lower margin figure to finish filling in the table with "+" for positive, "−" for negative or "0."

x	$f(x)$	$f'(x)$	$f''(x)$	concavity
1	+	+	−	down
2	+			
3	−			
4				



"Feeling" the Second Derivative

Earlier we saw that if a function $f(t)$ represents the position of a car at time t, then $f'(t)$ gives the velocity and $f''(t)$ the acceleration of the car at the instant t.

If we are driving along a straight, smooth road, then what we *feel* is the acceleration of the car:

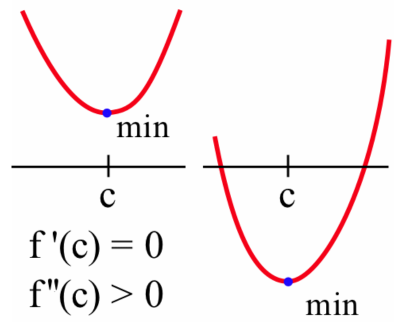
- a large positive acceleration feels like a "push" toward the back of the car
- a large negative acceleration (a deceleration) feels like a "push" toward the front of the car
- an acceleration of 0 for a period of time means the velocity is constant and we do not feel pushed in either direction

In a moving vehicle it is possible to measure these "pushes," the acceleration, and from that information to determine the velocity of the vehicle, and from the velocity information to determine the position. Inertial guidance systems in airplanes use this tactic: they measure front–back, left–right and up–down acceleration several times a second and then calculate the position of the plane. They also use computers to keep track of time and the rotation of the earth under the plane. After all, in six hours the Earth has made a quarter of a revolution, and Dallas has rotated more than 5,000 miles!

Example 1. The upward acceleration of a rocket was $a(t) = 30$ m/sec^2 during the first six seconds of flight, $0 \le t \le 6$. The velocity of the rocket at $t = 0$ was 0 m/sec and the height of the rocket above the ground at $t = 0$ was 25 m. Find a formula for the height of the rocket at time t and determine the height at $t = 6$ seconds.

Solution. $v'(t) = a(t) = 30 \Rightarrow v(t) = 30t + K$ for some constant K. We also know $v(0) = 0$ so $30(0) + K = 0 \Rightarrow K = 0$ and this $v(t) = 30t$.

Similarly, $h'(t) = v(t) = 30t \Rightarrow h(t) = 15t^2 + C$ for some constant C. We know that $h(0) = 25$ so $15(0)^2 + C = 25 \Rightarrow C = 25$. Thus $h(t) = 15t^2 + 25$ so $h(6) = 15(6)^2 + 25 = 565$ m. ◀



f'' and Extreme Values of f

The concavity of a function can also help us determine whether a critical point is a maximum or minimum or neither. For example, if a point is at the bottom of a concave-up function then that point is a minimum.

> **The Second Derivative Test for Extremes**:
> Let f be a twice differentiable function.
>
> (a) If $f'(c) = 0$ and $f''(c) < 0$ then f is concave down and has a local maximum at $x = c$.
>
> (b) If $f'(c) = 0$ and $f''(c) > 0$ then f is concave up and has a local minimum at $x = c$.
>
> (c) If $f'(c) = 0$ and $f''(c) = 0$ then f may have a local maximum, a local minimum or neither at $x = c$.

Proof. (a) Assume that $f'(c) = 0$. If $f''(c) < 0$ then f is concave down at $x = c$ so the graph of f will be below the tangent line $L(x)$ for values of x near c. The tangent line, however, is given by $L(x) = f(c) + f'(c)(x - c) = f(c) + 0(x - c) = f(c)$, so if x is close to c then $f(x) < L(x) = f(c)$ and f has a local maximum at $x = c$.

(b) The proof for a local minimum of f is similar.

(c) If $f'(c) = 0$ and $f''(c) = 0$, then we cannot immediately conclude anything about local maximums or minimums of f at $x = c$. The functions $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$ all have their first and second derivatives equal to zero at $x = 0$, but f has a local minimum at 0, g has a local maximum at 0, and h has neither a local maximum nor a local minimum at $x = 0$. □

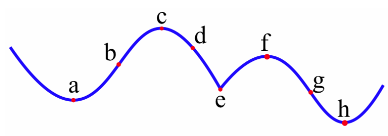
The Second Derivative Test for Extremes is very useful when f'' is easy to calculate and evaluate. Sometimes, however, the First Derivative Test—or simply a graph of the function—provides an easier way to determine if the function has a local maximum or a local minimum: it depends on the function and on which tools you have available.

Practice 2. $f(x) = 2x^3 - 15x^2 + 24x - 7$ has critical numbers $x = 1$ and $x = 4$. Use the Second Derivative Test for Extremes to determine whether $f(1)$ and $f(4)$ are maximums or minimums or neither.

Inflection Points

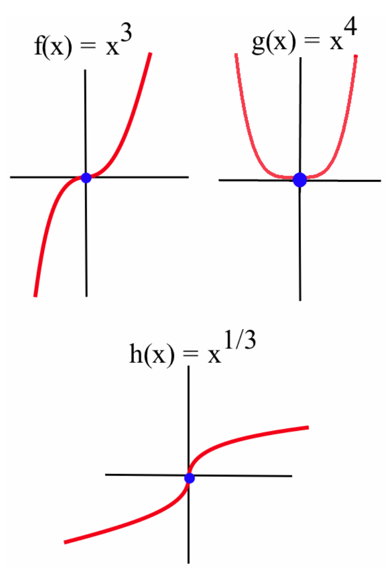
Maximums and minimums typically occur at places where the second derivative of a function is positive or negative, but the places where the second derivative is 0 are also of interest.

> **Definition**:
>
> An **inflection point** is a point on the graph of a function where the concavity of the function changes, from concave up to concave down or from concave down to concave up.



Practice 3. Which of the labeled points in the margin figure are inflection points?

To find the inflection points of a function we can use the second derivative of the function. If $f''(x) > 0$, then the graph of f is concave up at the point $(x, f(x))$ so $(x, f(x))$ is not an inflection point. Similarly, if $f''(x) < 0$ then the graph of f is concave down at the point $(x, f(x))$ and the point is not an inflection point. The only points left that can possibly be inflection points are the places where $f''(x) = 0$ or where $f''(x)$ does not exist (in other words, where f' is not differentiable). To find the inflection points of a function we need only check the points where $f''(x)$ is 0 or undefined. If $f''(c) = 0$ or is undefined, then the point $(c, f(c))$ **may or may not** be an inflection point—we need to check the concavity of f on each side of $x = c$. The functions in the next example illustrate what can happen at such a point.



Example 2. Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = \sqrt[3]{x}$ (see margin). For which of these functions is the point $(0, 0)$ an inflection point?

Solution. Graphically, it is clear that the concavity of $f(x) = x^3$ and $h(x) = \sqrt[3]{x}$ changes at $(0, 0)$, so $(0, 0)$ is an inflection point for f and h. The function $g(x) = x^4$ is concave up everywhere, so $(0, 0)$ is not an inflection point of g.

$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$ so the only point at which $f''(x) = 0$ or is undefined (f' is not differentiable) is at $x = 0$. If $x < 0$ then $f''(x) < 0$ so f is concave down; if $x > 0$ then $f''(x) > 0$ so f is concave up. Thus at $x = 0$ the concavity of f changes so the point $(0, f(0)) = (0, 0)$ is an inflection point of $f(x) = x^3$.

$g(x) = x^4 \Rightarrow g'(x) = 4x^3 \Rightarrow g''(x) = 12x^2$ so the only point at which $g''(x) = 0$ or is undefined is at $x = 0$. But $g''(x) > 0$ (so g is concave up) for any $x \neq 0$. Thus the concavity of g never changes, so the point $(0, g(0)) = (0, 0)$ is not an inflection point of $g(x) = x^4$.

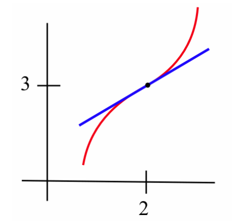
$h(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow h'(x) = \dfrac{1}{3}x^{-\frac{2}{3}} \Rightarrow h''(x) = -\dfrac{2}{9}x^{-\frac{5}{3}}$ so h'' is not defined if $x = 0$ (and $h''(x) \neq 0$ elsewhere); $h''(\text{negative number}) > 0$

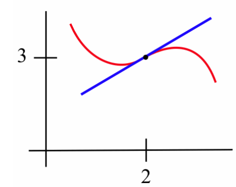
and h''(positive number) < 0, so h changes concavity at $(0,0)$ and $(0,0)$ is an inflection point of $h(x) = \sqrt[3]{x}$. ◀

Practice 4. Find all inflection points of $f(x) = x^4 - 12x^3 + 30x^2 + 5x - 7$.

Example 3. Sketch a graph of a function with $f(2) = 3$, $f'(2) = 1$ and an inflection point at $(2,3)$.

Solution. Two solutions appear in the margin. ◀

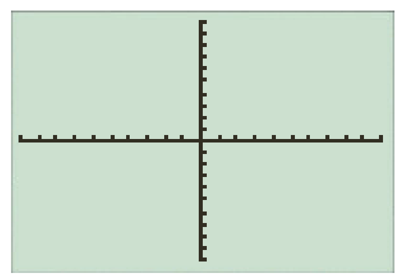




Using f' and f'' to Graph f

Today you can easily graph most functions of interest using a graphing calculator—and create even nicer graphs using an app on your phone or a Web-based graphing utility. Earlier generations of calculus students did not have these tools, so they relied on calculus to help them draw graphs of unfamiliar functions by hand. While you can create a graph in seconds that your predecessors may have labored over for half an hour or longer, you can still use calculus to help you select an appropriate graphing "window," and to be confident that your window has not missed any points of interest on the graph of a function.

Example 4. Create a graph of $f(x) = xe^{-9x^2}$ that shows all local and global extrema and all inflection points.



Solution. If you graph $f(x)$ on a calculator using the standard window ($-10 \le x \le 10$ and $-10 \le y \le 10$) you will likely see nothing other than the coordinate axes (see margin). You might consult a table of values for the function to help adjust the window, but this trial-and-error technique will still not guarantee that you have displayed all points of interest. Computing the first derivative of f, we get:

$$f'(x) = x\left[-18xe^{-9x^2}\right] + e^{-9x^2} \cdot 1 = \left[1 - 18x^2\right] e^{-9x^2}$$

which is defined for all values of x; $f'(x) = 0 \Rightarrow 1 - 18x^2 = 0 \Rightarrow x^2 = \frac{1}{18} \Rightarrow x = \pm\frac{1}{3\sqrt{2}}$, so the only critical numbers are $x = -\frac{1}{3\sqrt{2}}$ and $x = \frac{1}{3\sqrt{2}}$. Computing the second derivative of f, we get:

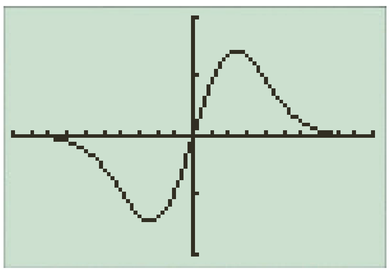
$$\begin{aligned} f''(x) &= \left[1 - 18x^2\right] \cdot \left[-18xe^{-9x^2}\right] + e^{-9x^2} \cdot [-36x] \\ &= \left[324x^3 - 54x\right] e^{-9x^2} = 54x\left[6x^2 - 1\right] e^{-9x^2} \end{aligned}$$

We can check that $f''\left(-\frac{1}{3\sqrt{2}}\right) = 12\sqrt{\frac{2}{e}} > 0$, so f must have a local minimum at $x = -\frac{1}{3\sqrt{2}}$; similarly, $f''\left(\frac{1}{3\sqrt{2}}\right) = -12\sqrt{\frac{2}{e}} < 0$, so f must have a local maximum at $x = \frac{1}{3\sqrt{2}}$.

Furthermore, $f''(x) = 0$ only when $x = 0$ or when $6x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{6}}$, so these three values are candidates for locations of inflection points of f. Noting that:

$$-1 < -\frac{1}{\sqrt{6}} < -\frac{1}{3\sqrt{2}} < 0 < \frac{1}{3\sqrt{2}} < \frac{1}{\sqrt{6}} < 1$$

and that $f''(-1) = -270e^{-9} < 0$ and $f''\left(-\frac{1}{3\sqrt{2}}\right) = 12\sqrt{\frac{2}{e}} > 0$, we observe that f is concave down to the left of $x = -\frac{1}{\sqrt{6}}$ and concave up to the right of $x = -\frac{1}{\sqrt{6}}$, so f does in fact have an inflection point at $x = -\frac{1}{\sqrt{6}}$. Likewise, $f''\left(\frac{1}{3\sqrt{2}}\right) = -12\sqrt{\frac{2}{e}} < 0$ and $f''(1) = 270e^{-9} > 0$, so $f''(x)$ switches sign at $x = 0$ and at $x = \frac{1}{\sqrt{6}}$, and therefore $f(x)$ changes concavity at those points as well.



We have now identified two local extrema of f and three inflection points of f. Equally important, we have used calculus to show that these five points of interest are the *only* places where extrema or inflection points can occur. If we create a graph of f that includes these five points, our graph is guaranteed to include all "interesting" features of the graph of f. A window with $-1 \le x \le 1$ and $-0.2 < y < 0.2$ (because the local extreme values are $f\left(\pm\frac{1}{2\sqrt{3}}\right) \approx \pm 0.14$) should provide a graph (see margin) that includes all five points of interest. ◀

Practice 5. Compute the first and second derivatives of the function $g(x) = x^4 + 4x^3 - 90x^2 + 13$, locate all extrema and inflection points of $g(x)$, and create a graph of $g(x)$ that shows these points of interest.

Most problems in calculus textbooks are set up to make solving these equations relatively straightforward, but in general this will not be the case.

Even with calculus, we will typically need calculators or computers to help solve the equations $f'(x) = 0$ and $f''(x) = 0$ that we use to find critical numbers and candidates for inflection points.

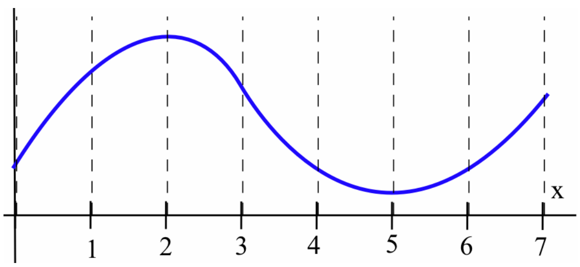
3.4 Problems

In Problems 1–2, each statement describes a quantity $f(t)$ changing over time. For each statement, tell what f represents and whether the first and second derivatives of f are positive or negative.

1. (a) "Unemployment rose again, but the rate of increase is smaller than last month."
   (b) "Our profits declined again, but at a slower rate than last month."
   (c) "The population is still rising and at a faster rate than last year."
2. (a) "The child's temperature is still rising, but more slowly than it was a few hours ago."
   (b) "The number of whales is decreasing, but at a slower rate than last year."
   (c) "The number of people with the flu is rising and at a faster rate than last month."
3. Sketch the graphs of functions that are defined and concave up everywhere and have exactly: (a) no roots. (b) 1 root. (c) 2 roots. (d) 3 roots.

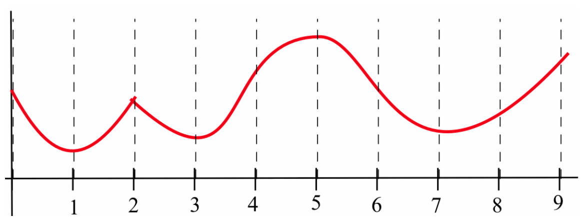
4. On which intervals is the function graphed below:

   (a) concave up? (b) concave down?



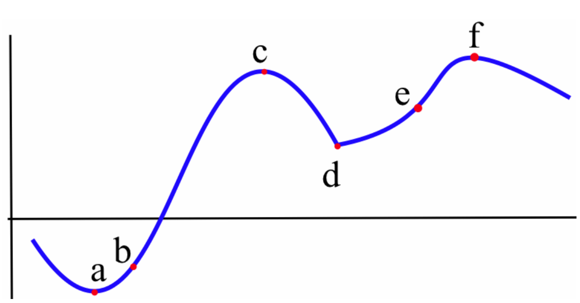
5. On which intervals is the function graphed below:

   (a) concave up? (b) concave down?

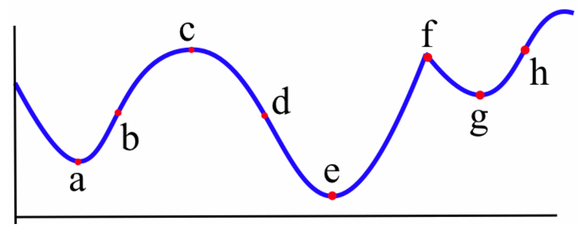


Problems 6–10 give a function and values of x so that $f'(x) = 0$. Use the Second Derivative Test to determine whether each point $(x, f(x))$ is a local maximum, a local minimum or neither.

6. $f(x) = 2x^3 - 15x^2 + 6$; $x = 0, 5$
7. $g(x) = x^3 - 3x^2 - 9x + 7$; $x = -1, 3$
8. $h(x) = x^4 - 8x^2 - 2$; $x = -2, 0, 2$
9. $f(x) = \sin^5(x)$; $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$
10. $f(x) = x \cdot \ln(x)$; $x = \frac{1}{e}$
11. At which values of x labeled in the figure below is the point $(x, f(x))$ an inflection point?

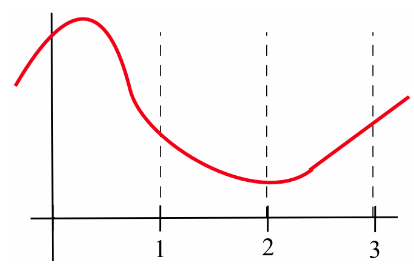


12. At which values of x labeled in the figure below is the point $(x, g(x))$ an inflection point?



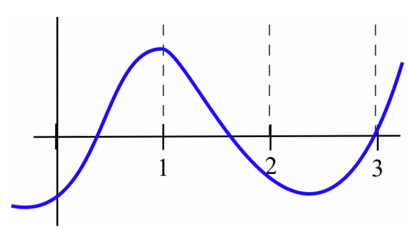
13. How many inflection points can a:
    (a) quadratic polynomial have?
    (b) cubic polynomial have?
    (c) polynomial of degree n have?
14. Fill in the table with "+," "−,"or "0" for the function graphed below.

x	$f(x)$	$f'(x)$	$f''(x)$
0			
1			
2			
3			

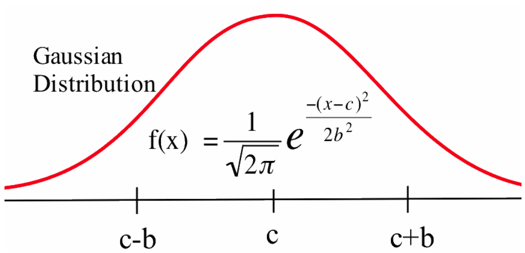


15. Fill in the table with "+," "−,"or "0" for the function graphed below.

x	$g(x)$	$g'(x)$	$g''(x)$
0			
1			
2			
3			



16. Sketch functions f for x-values near 1 so that $f(1) = 2$ and:
   (a) $f'(1) > 0$, $f''(1) > 0$
   (b) $f'(1) > 0$, $f''(1) < 0$
   (c) $f'(1) < 0$, $f''(1) > 0$
   (d) $f'(1) > 0$, $f''(1) = 0$, $f''(1^-) < 0$, $f''(1^+) > 0$
   (e) $f'(1) > 0$, $f''(1) = 0$, $f''(1^-) > 0$, $f''(1^+) < 0$
17. Some people like to think of a concave-up graph as one that will "hold water" and of a concave-down graph as one which will "spill water." That description is accurate for a concave-down graph, but it can fail for a concave-up graph. Sketch the graph of a function that is concave up on an interval but will not "hold water."
18. The function $f(x) = \frac{1}{2\pi} e^{-\frac{(x-c)^2}{2b^2}}$ defines the **Gaussian distribution** used extensively in statistics and probability; its graph (see below) is a "bell-shaped" curve.



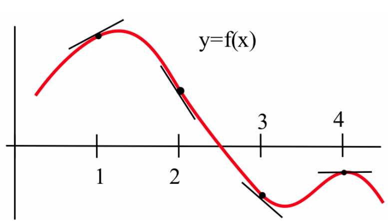
   (a) Show that f has a maximum at $x = c$. (The value c is called the **mean** of this distribution.)
   (b) Show that f has inflection points where $x = c + b$ and $x = c - b$. (The value b is called the **standard deviation** of this distribution.)

In Problems 19–36, locate all critical numbers, local extrema and inflection points of the given function, and use these results to sketch a graph of the function showing all points of interest.

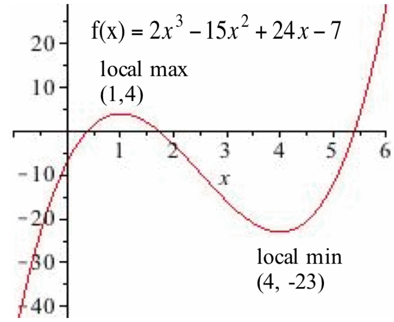
19. $f(x) = x^3 - 21x^2 + 144x - 350$
20. $g(x) = \frac{1}{6}x^3 + x^2 - \frac{45}{2}x + 100$
21. $f(x) = e^{7x} - 5x$
22. $g(x) = e^{7x} - 5x$
23. $f(x) = e^{-3x} + x$
24. $g(x) = e^{-3x} - x$
25. $f(x) = xe^{-3x}$
26. $g(x) = xe^{5x}$
27. $f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$
28. $g(x) = 6x^{\frac{4}{3}} + 3x^{\frac{1}{3}}$
29. $f(x) = \ln\left(1 + x^2\right)$
30. $g(x) = \ln\left(x^2 - 6x + 10\right)$
31. $f(x) = \sqrt[3]{x^2 + 2x + 2}$
32. $g(x) = \sqrt{x^2 + 2x + 2}$
33. $f(x) = x^{\frac{2}{3}}\left(1 - x\right)^{\frac{1}{3}}$
34. $g(x) = x^{\frac{1}{3}}\left(1 - x\right)^{\frac{2}{3}}$
35. $f(\theta) = \sin(\theta) + \sin^2(\theta)$
36. $g(\theta) = \cos(\theta) - \sin^2(\theta)$

3.4 Practice Answers

1. See the margin figure for reference.

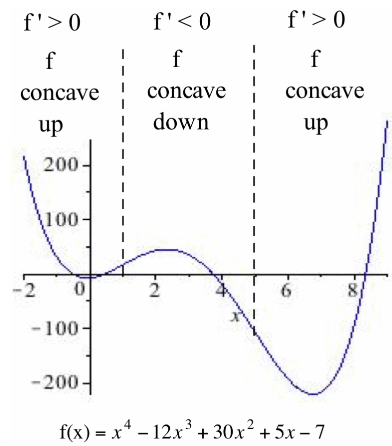


x	$f(x)$	$f'(x)$	$f''(x)$	concavity
1	+	+	−	down
2	+	−	−	down
3	−	−	+	up
4	−	0	−	down



2. $f'(x) = 6x^2 - 30x + 24$, which is defined for all x. $f'(x) = 0$ if $x = 1$ or $x = 4$ (critical values). $f''(x) = 12x - 30$ so $f''(1) = -18 < 0$ tells us that f is concave down at the critical value $x = 1$, so $(1, f(1)) = (1, 4)$ is a relative maximum; and $f''(4) = 18 > 0$ tells us that f is concave up at the critical value $x = 4$, so $(4, f(4)) = (4, -23)$ is a relative minimum. A graph of f appears in the margin.

3. The points labeled b and g are inflection points.



4. $f'(x) = 4x^3 - 36x^2 + 60x + 5 \Rightarrow f''(x) = 12x^2 - 72x + 60 = 12(x^2 - 6x + 5) = 12(x-1)(x-5)$ so the only candidates to be inflection points are $x = 1$ and $x = 5$.

   - If $x < 1$ then $f''(x) = 12(\text{neg})(\text{neg}) > 0$
   - If $1 < x < 5$ then $f''(x) = 12(\text{pos})(\text{neg}) < 0$
   - If $5 < x$ then $f''(x) = 12(\text{pos})(\text{pos}) > 0$

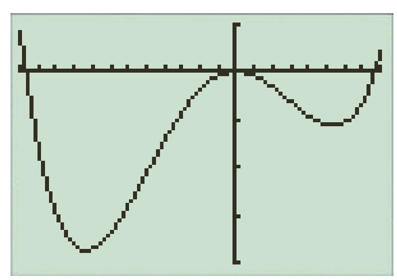
   f changes concavity at $x = 1$ and $x = 5$, so $x = 1$ and $x = 5$ are both inflection points. A graph of f appears in the margin.

5. $g(x) = x^4 + 4x^3 - 90x^2 + 13 \Rightarrow g'(x) = 4x^3 + 12x^2 - 180x \Rightarrow g''(x) = 12x^2 + 24x - 180$; because $g'(x)$ and $g''(x)$ are polynomials, they exist everywhere. The critical numbers for $g(x)$ occur where $g'(x) = 0 \Rightarrow 4x^3 + 12x^2 - 180x = 4x(x^2 + 3x - 45) = 4x(x+9)(x-5) = 0 \Rightarrow x = -9$, $x = 0$ or $x = 5$. Using the Second Derivative Test: $g''(-9) = 576 > 0$, so $g(x)$ has a local minimum at $x = -9$; $g''(0) = -180 < 0$, so $g(x)$ has a local maximum at $x = 0$; and $g''(5) = 240 > 0$, so $g(x)$ has a local minimum at $x = 5$.

   Candidates for inflection points occur where $g''(x) = 0$:

   $$12x^2 + 24x - 180 = 12(x^2 + 2x - 15) = 12(x-3)(x+5) = 0$$
   $$\Rightarrow x = -5 \text{ or } x = 3$$

   Observing that $g''(x) > 0$ for $x < -5$, $g''(x) < 0$ for $-5 < x < 3$ and $g''(x) > 0$ for $x > 3$ confirms that both candidates are in fact inflection points. A graphing window with $-12 \le x \le 8$ (this is only one reasonable possibility) should include all points of interest. Checking that $g(-9) = -3632$, $g(0) = 13$ and $g(5) = -1112$ suggests that a graphing window with $-4000 \le y \le 1000$ should work (see margin).



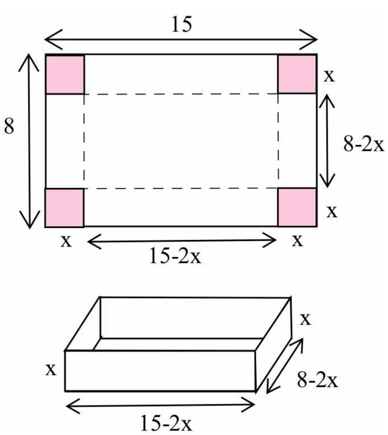
3.5 *Applied Maximum and Minimum Problems*

We have used derivatives to find maximums and minimums of functions given by formulas, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme value(s). Typically, someone will describe a problem and ask your help to maximize or minimize a quantity: "What is the largest volume of a package that the post office will accept?"; "What is the quickest way to get from here to there?"; or "What is the least expensive way to accomplish some task?" These problems often involve restrictions — or **constraints** — and sometimes neither the problem nor the constraints are clearly stated.

Before we can use calculus or other mathematical techniques to solve these "**max/min**" problems, we need to understand the situation at hand and translate the problem into mathematical form. After solving the problem using calculus (or other mathematical techniques) we need to check that our mathematical solution really solves the original problem. Often, the most challenging part of this procedure is understanding the problem and translating it into mathematical form.

In this section we examine some problems that require understanding, translation, solution and checking. Most will not be as complicated as those a working scientist, engineer or economist needs to solve, but they represent a step toward developing the required skills.

Example 1. The company you own has a large supply of 8-inch by 15-inch rectangular pieces of tin, and you decide to use them to make boxes by cutting a square from each corner and folding up the sides (see margin). For example, if you cut a 1-inch square from each corner, the resulting 6-inch by 13-inch by 1-inch box has a volume of 78 cubic inches. The amount of money you can charge for a box depends on how much the box holds, so you want to make boxes with the largest possible volume. What size square should you cut from each corner?



Solution. To help understand the problem, first drawing a diagram can be very helpful. Then we need to translate it into a mathematical problem:

- identify the variables
- label the variable and constant parts of the diagram
- write the quantity to be maximized as a function of the variables

If we label the side of the square to be removed as x inches, then the box is x inches high, $8-2x$ inches wide and $15-2x$ inches long, so the volume is:

$$\begin{aligned}(\text{length})(\text{width})(\text{height}) &= (15-2x)(8-2x)\cdot x \\ &= 4x^3 - 46x^2 + 120x \text{ cubic inches}\end{aligned}$$

Now we have a mathematical problem, to maximize the function $V(x) = 4x^3 - 46x^2 + 120x$, so we use existing calculus techniques, computing $V'(x) = 12x^2 - 92x + 120$ to find the critical points.

- Set $V'(x) = 0$ and solve by factoring or using the quadratic formula:

  $$V'(x) = 12x^2 - 92x + 120 = 4(3x-5)(x-6) = 0 \Rightarrow x = \frac{5}{3} \text{ or } x = 6$$

  so $x = \frac{5}{3}$ and $x = 6$ are critical points of V.

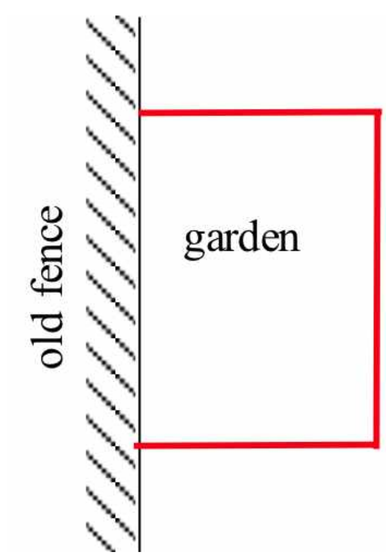
- $V'(x)$ is a polynomial so it is defined everywhere and there are no critical points resulting from an undefined derivative.

- What are the endpoints for x in this problem? A square cannot have a negative length, so $x \geq 0$. We cannot remove more than half of the width, so $8 - 2x \geq 0 \Rightarrow x \leq 4$. Together, these two inequalities say that $0 \leq x \leq 4$, so the endpoints are $x = 0$ and $x = 4$. (Note that the value $x = 6$ is not in this interval, so $x = 6$ cannot maximize the volume and we do not consider it further.)

The maximum volume must occur at the critical point $x = \frac{5}{3}$ or at one of the endpoints ($x = 0$ and $x = 4$): $V(0) = 0$, $V(\frac{5}{3}) = \frac{2450}{27} \approx 90.74$ cubic inches, and $V(4) = 0$, so the maximum volume of the box occurs when we remove a $\frac{5}{3}$-inch by $\frac{5}{3}$-inch square from each corner, resulting in a box $\frac{5}{3}$ inches high, $8 - 2(\frac{5}{3}) = \frac{14}{3}$ inches wide and $15 - 2(\frac{5}{3}) = \frac{35}{3}$ inches long. ◀

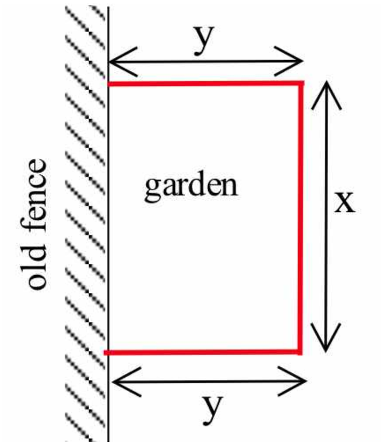
Practice 1. If you start with 7-inch by 15-inch pieces of tin, what size square should you remove from each corner so the box will have as large a volume as possible? [Hint: $12x^2 - 88x + 105 = (2x-3)(6x-35)$]

We were fortunate in the previous Example and Practice problem because the functions we created to describe the volume were functions of only one variable. In other situations, the function we get will have more than one variable, and we will need to use additional information to rewrite our function as a function of a single variable. Typically, the constraints will contain the additional information we need.



Example 2. We want to fence a rectangular area in our backyard for a garden. One side of the garden is along the edge of the yard, which is already fenced, so we only need to build a new fence along the other three sides of the rectangle (see margin). If a neighbor gives us 80 feet of fencing left over from a home-improvement project, what dimensions should the garden have in order to enclose the largest possible area using all of the available material?

Solution. As a first step toward understanding the problem, we draw a diagram or picture of the situation. Next, we identify the variables:



in this case, the length (call it x) and width (call it y) of the garden. The margin figure shows a labeled diagram, which we can use to write a formula for the function that we want to maximize:

$$A = \text{area of the rectangle} = (\text{length})(\text{width}) = x \cdot y$$

Unfortunately, our function A involves two variables, x and y, so we need to find a relationship between them (an equation containing both x and y) that we can solve for wither x or y. The constraint says that we have 80 feet of fencing available, so $x + 2y = 80 \Rightarrow y = 40 - \frac{x}{2}$. Then:

$$A = x \cdot y = x\left(40 - \frac{x}{2}\right) = 40x - \frac{x^2}{2}$$

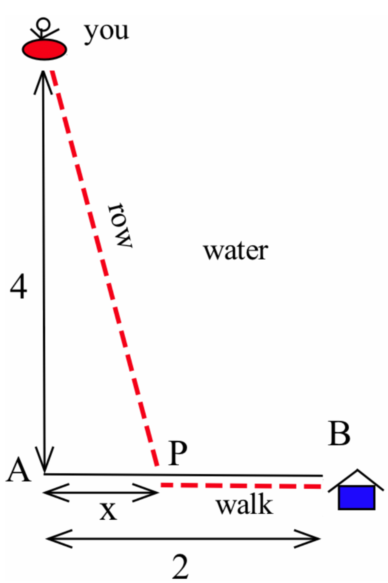
which is a function of a single variable (x). We want to maximize A.

$A'(x) = 40 - x$ so the only way $A'(x) = 0$ is to have $x = 40$, and $A(x)$ is differentiable for all x so the only critical number (other than the endpoints) is $x = 40$. Finally, $0 \leq x \leq 80$ (why?) so we also need to check $x = 0$ and $x = 80$: the maximum area must occur at $x = 0$, $x = 40$ or $x = 80$.

$$A(0) = 40(0) - \frac{0^2}{2} = 0 \text{ square feet}$$
$$A(40) = 40(40) - \frac{40^2}{2} = 800 \text{ square feet}$$
$$A(80) = 40(80) - \frac{80^2}{2} = 0 \text{ square feet}$$

so the largest rectangular garden has an area of 800 square feet, with dimensions $x = 40$ feet by $y = 40 - \frac{40}{2} = 20$ feet. ◀

Practice 2. Suppose you decide to create the rectangular garden in a **corner** of your yard. Then two sides of the garden are bounded by the existing fence, so you only need to use the available 80 feet of fencing to enclose the other two sides. What are the dimensions of the new garden of largest area? What are the dimensions if you have F feet of new fencing available?



Example 3. You need to reach home as quickly as possible, but you are in a rowboat on a lake 4 miles from shore and your home is 2 miles up the shore (see margin). If you can row at 3 miles per hour and walk at 5 miles per hour, toward which point on the shore should you row? What if your home is 7 miles up the coast?

Solution. The margin figure shows a labeled diagram with the variable x representing the distance along the shore from point A, the nearest point on the shore to your boat, to point P, the point you row toward.

The total time — rowing and walking — is:

$$\begin{aligned} T &= \text{total time} \\ &= (\text{rowing time from boat to } P) + (\text{walking time from } P \text{ to } B) \\ &= \frac{\text{distance from boat to } P}{\text{rate rowing boat}} + \frac{\text{distance from } P \text{ to } B}{\text{rate walking along shore}} \\ &= \frac{\sqrt{x^2+4^2}}{3} + \frac{2-x}{5} = \frac{\sqrt{x^2+16}}{3} + \frac{2-x}{5} \end{aligned}$$

It is not reasonable to row to a point below A and then walk home, so $x \geq 0$. Similarly, we can conclude that $x \leq 2$, so our interval is $0 \leq x \leq 2$ and the endpoints are $x = 0$ and $x = 2$.

To find the other critical numbers of T between $x = 0$ and $x = 2$, we need the derivative of T:

$$T'(x) = \frac{1}{3} \cdot \frac{1}{2} \left(x^2+16\right)^{-\frac{1}{2}} (2x) - \frac{1}{5} = \frac{x}{3\sqrt{x^2+16}} - \frac{1}{5}$$

This derivative is defined for all values of x (and in particular for all values in the interval $0 \leq x \leq 2$). To find where $T'(x) = 0$ we solve:

$$\begin{aligned} \frac{x}{3\sqrt{x^2+16}} - \frac{1}{5} = 0 \;&\Rightarrow\; 5x = 3\sqrt{x^2+16} \\ &\Rightarrow\; 25x^2 = 9x^2 + 144 \\ &\Rightarrow 16x^2 = 144 \;\Rightarrow x^2 = 9 \;\Rightarrow x = \pm 3 \end{aligned}$$

Neither of these numbers, however, is in our interval $0 \leq x \leq 2$, so neither of them gives a minimum time. The only critical numbers for T on this interval are the endpoints, $x = 0$ and $x = 2$:

$$\begin{aligned} T(0) &= \frac{\sqrt{0+16}}{3} + \frac{2-0}{5} = \frac{4}{3} + \frac{2}{5} \approx 1.73 \text{ hours} \\ T(2) &= \frac{\sqrt{2^2+16}}{3} + \frac{2-2}{5} = \frac{\sqrt{20}}{3} \approx 1.49 \text{ hours} \end{aligned}$$

The quickest route has P 2 miles down the coast: you should row directly toward home.

If your home is 7 miles down the coast, then the interval for x is $0 \leq x \leq 7$, which has endpoints $x = 0$ and $x = 7$. Our function for the travel time is now:

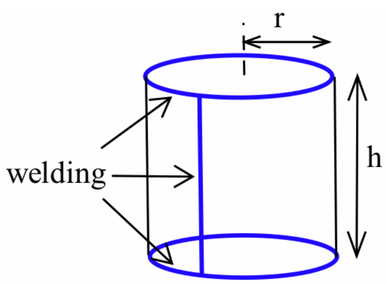
$$T(x) = \frac{\sqrt{x^2+16}}{3} + \frac{7-x}{5} \;\Rightarrow\; T'(x) = \frac{x}{3\sqrt{x^2+16}} - \frac{1}{5}$$

so the only point in our interval where $T'(x) = 0$ is at $x = 3$ and the derivative is defined for all values in this interval. So the only critical numbers for T are $x = 0$, $x = 3$ and $x = 7$:

$$T(0) = \frac{\sqrt{0+16}}{3} + \frac{7-0}{5} = \frac{4}{3} + \frac{7}{5} \qquad \approx 2.73 \text{ hours}$$
$$T(3) = \frac{\sqrt{3^2+16}}{3} + \frac{7-3}{5} = \frac{\sqrt{65}}{3} + \frac{4}{5} \qquad \approx 2.47 \text{ hours}$$
$$T(7) = \frac{\sqrt{7^2+16}}{3} + \frac{7-7}{5} = \frac{5}{3} \qquad \approx 2.68 \text{ hours}$$

The quickest way home is to aim for a point P that is 3 miles down the shore, row directly to P, and then walk along the shore to home. ◀

One challenge of max/min problems is that they may require geometry, trigonometry or other mathematical facts and relationships.



Example 4. Find the height and radius of the least expensive closed cylinder that has a volume of 1,000 cubic inches. Assume that the materials needed to construct the cylinder are free, but that it costs 80¢ per inch to weld the top and bottom onto the cylinder and to weld the seam up the side of the cylinder (see margin).

Solution. If we let r be the radius of the cylinder and h be its height, then the volume is $V = \pi r^2 h = 1000$. The quantity we want to minimize is cost, and

$$\begin{aligned} C &= (\text{top seam cost}) + (\text{bottom seam cost}) + (\text{side seam cost}) \\ &= (\text{total seam length}) \left(80 \frac{¢}{\text{inch}}\right) \\ &= (2\pi r + 2\pi r + h)(80) = 320\pi r + 80h \end{aligned}$$

Unfortunately, C is a function of two variables, r and h, but we can use the information in the constraint ($V = \pi r^2 h = 1000$) to solve for h and then substitute this expression for h into the formula for C:

$$1000 = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2} \Rightarrow C = 320\pi r + 80h = 320\pi r + 80\left(\frac{1000}{\pi r^2}\right)$$

which is a function of a single variable. Differentiating:

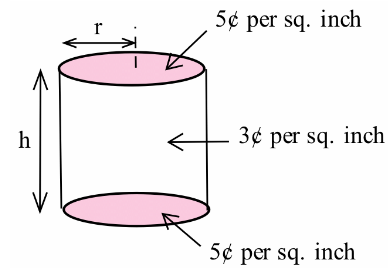
$$C'(r) = 320\pi - \frac{160000}{\pi r^3}$$

which is defined except when $r = 0$ (a value that does not make sense in the original problem) and there are no restrictions on r (other than $r > 0$) so there are no endpoints to check. Thus C will be at a minimum when $C'(r) = 0$:

$$320\pi - \frac{160000}{\pi r^3} = 0 \Rightarrow r^3 = \frac{500}{\pi^2} \Rightarrow r = \sqrt[3]{\frac{500}{\pi^2}}$$

so $r \approx 3.7$ inches and $h = \dfrac{1000}{\pi r^2} = \dfrac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi^2}}\right)^2} \approx 23.3$ inches. ◀

Practice 3. Find the height and radius of the least expensive closed cylinder that has a volume of 1,000 cubic inches, assuming that the only cost for this cylinder is the price of the materials: the material for the top and bottom costs 5¢ per square inch, while the material for the sides costs 3¢ per square inch (see margin).



Example 5. Find the dimensions of the least expensive rectangular box that is three times as long as it is wide and which holds 100 cubic centimeters of water. The material for the bottom costs 7¢ per cm^2, the sides cost 5¢ per cm^2 and the top costs 2¢ per cm^2.

Solution. Label the box so that w = width, l = length and h = height. Then our cost function C is:

$$\begin{aligned} C &= (\text{bottom cost}) + (\text{cost of front and back}) + (\text{cost of ends}) + (\text{top cost}) \\ &= (\text{bottom area})(7) + (\text{front and back area})(5) + (\text{ends area})(5) + (\text{top area})(2) \\ &= (wl)(7) + (2lh)(5) + (2wh)(5) + (wl)(2) \\ &= 7wl + 10lh + 10wh + 2wl \\ &= 9wl + 10lh + 10wh \end{aligned}$$

Unfortunately, C is a function of three variables (w, l and h) but we can use the information from the constraints to eliminate some of the variables: the box is "three times as long as it is wide" so $l = 3w$ and

$$C = 9wl + 10lh + 10wh = 9w(3w) + 10(3w)h + 10wh = 27w^2 + 40wh$$

We also know the volume V is 100 in^3 and $V = lwh = 3w^2h$ (because $l = 3w$), so $h = \dfrac{100}{3w^2}$. Then:

$$C = 27w^2 + 40wh = 27w^2 + 40w\left(\frac{100}{3w^2}\right) = 27w^2 + \frac{4000}{3w}$$

which is a function of a single variable. Differentiating:

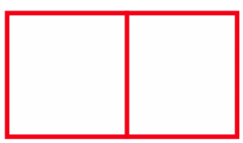
$$C'(w) = 54w - \frac{4000}{3w^2}$$

which is defined everywhere except $w = 0$ (yielding a box of volume 0) and there is no constraint interval, so C is minimized when $C'(w) = 0 \Rightarrow w = \sqrt[3]{\dfrac{4000}{162}} \approx 2.91$ inches $\Rightarrow l = 3w \approx 8.73$ inches $\Rightarrow h = \frac{100}{3w^2} \approx 3.94$ inches. The minimum cost is approximately \$6.87. ◀

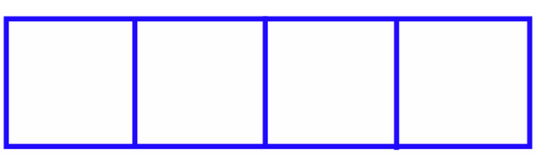
Problems described in words are usually more difficult to solve because we first need to understand and "translate" a real-life problem into a mathematical problem. Unfortunately, those skills only seem to come with practice. With practice, however, you will start to recognize patterns for understanding, translating and solving these problems, and you will develop the skills you need. So read carefully, draw pictures, think hard—and do the best you can.

3.5 Problems

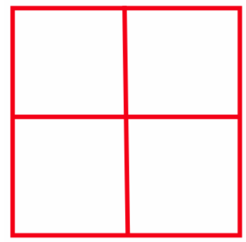
1. (a) You have 200 feet of fencing to enclose a rectangular vegetable garden. What should the dimensions of your garden be in order to enclose the largest area?
   (b) Show that if you have P feet of fencing available, the garden of greatest area is a square.
   (c) What are the dimensions of the largest rectangular garden you can enclose with P feet of fencing if one edge of the garden borders a straight river and does not need to be fenced?
   (d) Just thinking—calculus will not help: What do you think is the shape of the largest garden that can be enclosed with P feet of fencing if we do not require the garden to be rectangular? What if one edge of the garden borders a (straight) river?
2. (a) You have 200 feet of fencing available to construct a rectangular pen with a fence divider down the middle (see below). What dimensions of the pen enclose the largest total area?



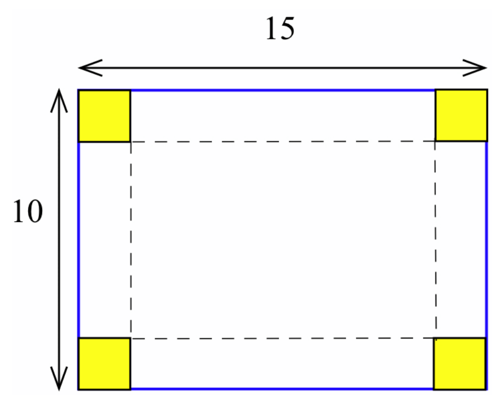
   (b) If you need two dividers, what dimensions of the pen enclose the largest area?
   (c) What are the dimensions in parts (a) and (b) if one edge of the pen borders on a river and does not require any fencing?
3. You have 120 feet of fencing to construct a pen with four equal-sized stalls.
   (a) If the pen is rectangular and shaped like the one shown below, what are the dimensions of the pen of largest area and what is that area?



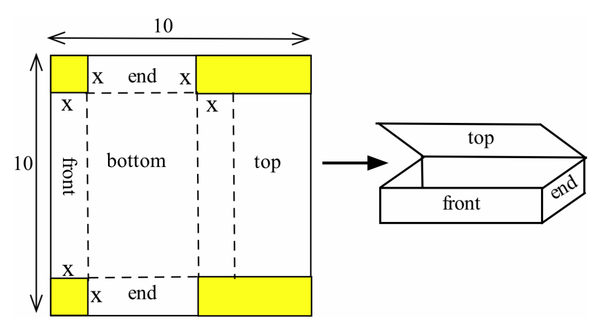
   (b) The square pen below uses 120 feet of fencing but encloses a larger area (400 ft^2) than the best design in part (a). Design a pen that uses only 120 feet of fencing and has four equal-sized stalls but encloses more than 400 ft^2. (Hint: Don't use rectangles and squares.)



4. (a) You need to form a 10-inch by 15-inch piece of tin into a box (with no top) by cutting a square from each corner and folding up the sides. How much should you cut so the resulting box has the greatest volume?



   (b) If the piece of tin is A inches by B inches, how much should you cut from each corner so the resulting box has the greatest volume?
5. Find the dimensions of a box with largest volume formed from a 10-inch by 10-inch piece of cardboard cut and folded as shown below.



6. (a) You must construct a square-bottomed box with no top that will hold 100 cubic inches of water. If the bottom and sides are made from the same material, what are the dimensions of the box that uses the least material? (Assume that no material is wasted.)

   (b) Suppose the box in part (a) uses different materials for the bottom and the sides. If the bottom material costs 5¢ per square inch and the side material costs 3¢ per square inch, what are the dimensions of the least expensive box that will hold 100 cubic inches of water?

   (This is a "classic" problem with many variations. We could require that the box be twice as long as it is wide, or that the box have a top, or that the ends cost a different amount than the front and back, or even that it costs a certain amount to weld each edge. You should be able to set up the cost equations for these variations.)

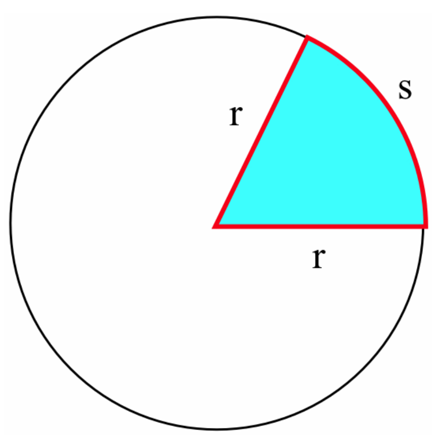
7. (a) Determine the dimensions of the least expensive cylindrical can that will hold 100 cubic inches if the materials cost 2¢, 5¢ and 3¢ per square inch, respectively, for the top, bottom and sides.

   (b) How do the dimensions of the least expensive can change if the bottom material costs more than 5¢ per square inch?

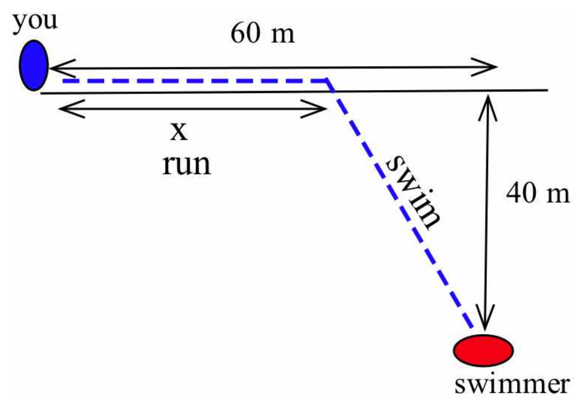
8. You have 100 feet of fencing to build a pen in the shape of a circular sector, the "pie slice" shown below. The area of such a sector is $\frac{rs}{2}$.

   (a) What value of r maximizes the enclosed area?

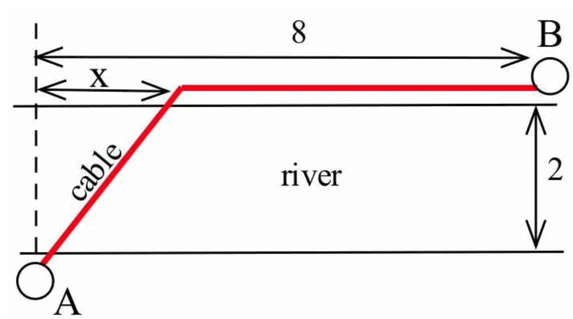
   (b) What central angle maximizes the area?



9. You are a lifeguard standing at the edge of the water when you notice a swimmer in trouble (see figure below) 40 m out in the water from a point 60 m down the beach. Assuming you can run at a speed of 8 meters per second and swim at a rate of 2 meters per second, how far along the shore should you run before diving into the water in order to reach the swimmer as quickly as possible?



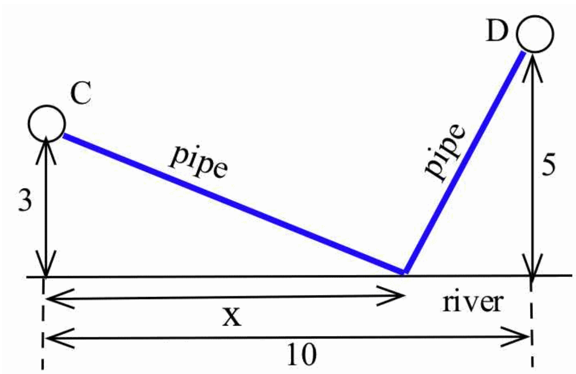
10. You have been asked to determine the least expensive route for a telephone cable that connects Andersonville with Beantown (see figure below).



   (a) If it costs \$5000 per mile to lay the cable on land and \$8000 per mile to lay the cable across the river (with the cost of the cable included), find the least expensive route.

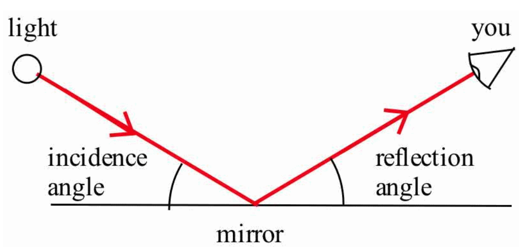
   (b) What is the least expensive route if the cable costs \$7000 per mile in addition to the cost to lay it?

11. You have been asked to determine where a water works should be built along a river between Chesterville and Denton (see below) to minimize the total cost of the pipe to the towns.

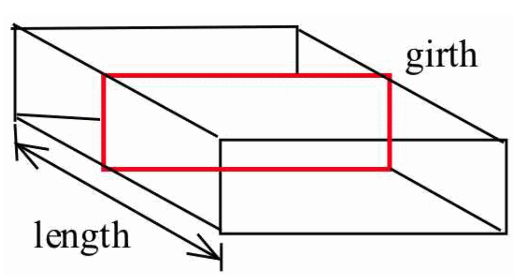


(a) Assume that the same size (and cost) pipe is used to each town. (This part can be done quickly without using calculus.)
(b) Assume instead that the pipe to Chesterville costs \$3000 per mile and to Denton it costs \$7000 per mile.

12. Light from a bulb at A is reflected off a flat mirror to your eye at point B (see below). If the time (and length of the path) from A to the mirror and then to your eye is a minimum, show that the angle of incidence equals the angle of reflection. (Hint: This is similar to the previous problem.)

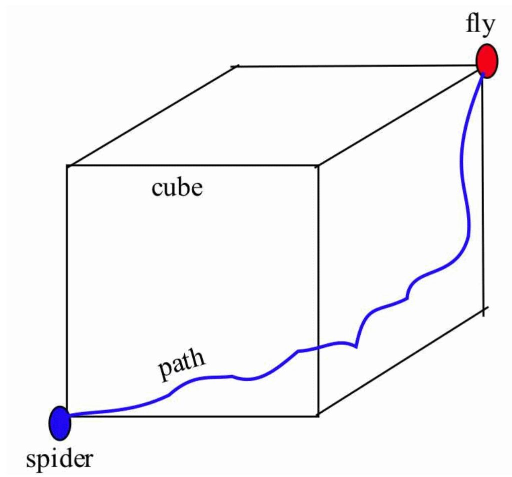


13. U.S. postal regulations state that the sum of the length and girth (distance around) of a package must be no more than 108 inches (see below).



(a) Find the dimensions of the acceptable box with a square end that has the largest volume.
(b) Find the dimensions of the acceptable box that has the largest volume if its end is a rectangle twice as long as it is wide.
(c) Find the dimensions of the acceptable box with a circular end that has the largest volume.

14. Just thinking—you don't need calculus for this problem: A spider and a fly are located on opposite corners of a cube (see below). What is the shortest path along the surface of the cube from the spider to the fly?



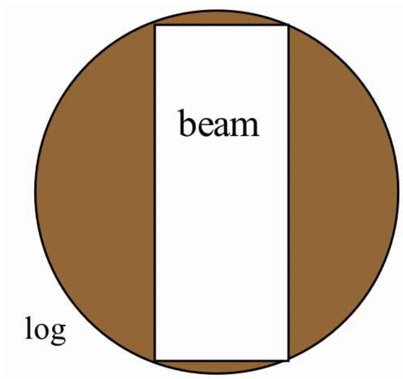
15. Two sides of a triangle are 7 and 10 inches long. What is the length of the third side so the area of the triangle will be greatest? (This problem can be done without using calculus. How? If you do use calculus, consider the angle θ between the two sides.)

16. Find the shortest distance from the point $(2, 0)$ to the curve:

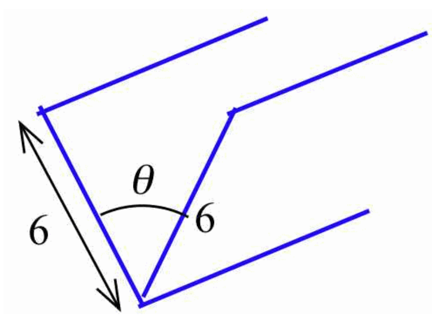
(a) $y = 3x - 1$ (b) $y = x^2$
(c) $x^2 + y^2 = 1$ (d) $y = \sin(x)$

17. Find the dimensions of the rectangle with the largest area if the base must be on the x-axis and its other two corners are on the graph of:
(a) $y = 16 - x^2$, $-4 \le x \le 4$
(b) $x^2 + y^2 = 1$
(c) $|x| + |y| = 1$
(d) $y = \cos(x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

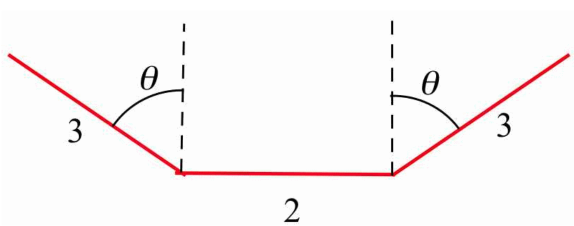
18. The strength of a wooden beam is proportional to the product of its width and the square of its height (see figure below). What are the dimensions of the strongest beam that can be cut from a log with diameter:
    (a) 12 inches?
    (b) d inches?



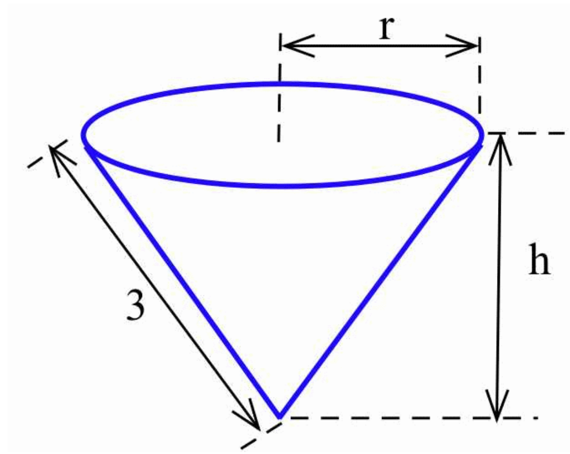
19. You have a long piece of 12-inch-wide metal that you plan to fold along the center line to form a V-shaped gutter (see below). What angle θ will yield a gutter that holds the most water (that is, has the largest cross-sectional area)?



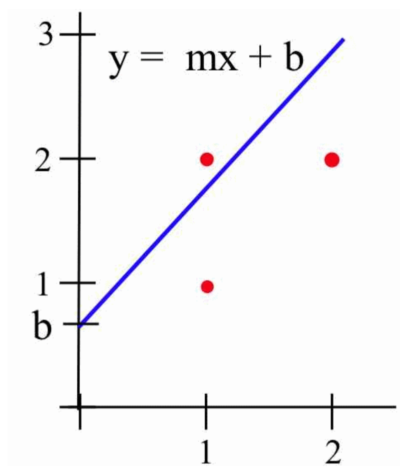
20. You have a long piece of 8-inch-wide metal that you plan to make into a gutter by bending up 3 inches on each side (see below). What angle θ will yield a gutter that holds the most water?



21. You have a 6-inch-diameter paper disk that you want to form into a drinking cup by removing a pie-shaped wedge (sector) and then forming the remaining paper into a cone (see below). Find the height and top radius of the cone so the that the volume of the cup is as large as possible.



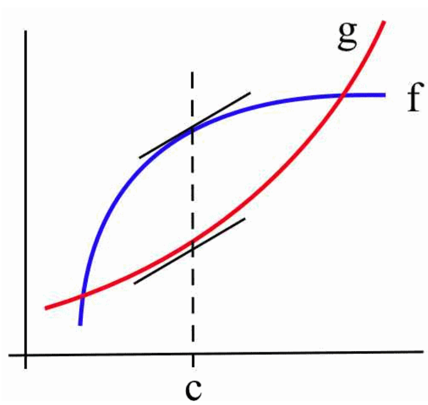
22. (a) What value of b minimizes the sum of the squares of the vertical distances from $y = 2x + b$ to the points $(1, 1)$, $(1, 2)$ and $(2, 2)$?



    (b) What slope m minimizes the sum of the squares of the vertical distances from the line $y = mx$ to the points $(1, 1)$, $(1, 2)$ and $(2, 2)$?
    (c) What slope m minimizes the sum of the squares of the vertical distances from the line $y = mx$ to the points $(2, 1)$, $(4, 3)$, $(-2, -2)$ and $(-4, -2)$?

23. You own a small airplane that holds a maximum of 20 passengers. It costs you \$100 per flight from St. Thomas to St. Croix for gas and wages plus an additional \$6 per passenger for the extra gas required by the extra weight. The charge per passenger is \$30 each if 10 people charter your plane (10 is the minimum number you will fly), and this charge is reduced by \$1 per passenger for each passenger over 10 who travels (that is, if 11 fly they each pay \$29, if 12 fly they each pay \$28, etc.). What number of passengers on a flight will maximize your profit?

24. Prove: If f and g are differentiable functions and if the vertical distance between f and g is greatest at $x = c$, then $f'(c) = g'(c)$ and the tangent lines to f and g are parallel when $x = c$.

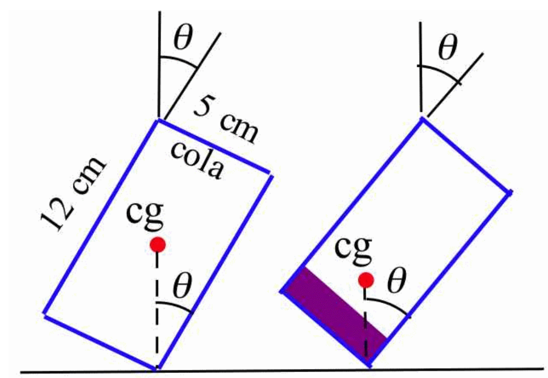


25. Profit = revenue − expenses. Assuming revenue and expenses are differentiable functions, show that when profit is maximized, then marginal revenue $\left(\frac{dR}{dx}\right)$ equals marginal expense $\left(\frac{dE}{dx}\right)$.

26. Dean Simonton claims the "productivity levels" of people in various fields can be described as a function of their "career age" t by $p(t) = e^{-at} - e^{-bt}$ where a and b are constants depending on the field, and career age is approximately 20 less than the actual age of the individual.
   (a) Based on this model, at what ages do mathematicians ($a = 0.03$, $b = 0.05$), geologists ($a = 0.02$, $b = 0.04$) and historians ($a = 0.02$, $b = 0.03$) reach their maximum productivity?
   (b) Simonton says, "With a little calculus we can show that the curve $(p(t))$ maximizes at $t = \frac{1}{b-a}\ln\left(\frac{b}{a}\right)$." Use calculus to show that Simonton is correct.

   Note: Models of this type have uses for describing the behavior of groups, but it is dangerous—and usually invalid—to apply group descriptions or comparisons to individuals in a group. (*Scientific Genius* by Dean Simonton, Cambridge University Press, 1988, pp. 69–73)

27. After the table was wiped and the potato chips dried off, the question remained: "Just how far could a can of cola be tipped before it fell over?"
   (a) For a full can or an empty can the answer was easy: the center of gravity (CG) of the can is at the middle of the can, half as high as the height of the can, and we can tilt the can until the CG is directly above the bottom rim (see below left). Find θ if the height of the can is 12 cm and the diameter is 5 cm.

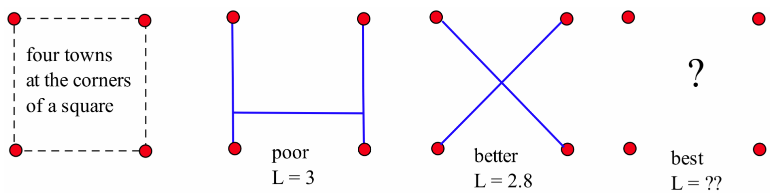


   (b) For a partly filled can, more thinking was needed. Some ideas you will see in Chapter 5 tell us that the CG of a can holding x cm of cola is $C(x) = \frac{360 + 9.6x^2}{60 + 19.2x}$ cm above the bottom of the can. Find the height x of cola that will make the CG as low as possible.
   (c) Assuming that the cola is frozen solid (so the top of the cola stays parallel to the bottom of the can), how far can we tilt a can containing x cm of cola? (See above right.)
   (d) If the can contained x cm of liquid cola, could we tilt it farther or less far than the frozen cola before it would fall over?

28. Just thinking—calculus will not help with this one.

(a) Four towns are located at the corners of a square. What is the shortest length of road we can construct so that it is possible to travel along the road from any town to any other town?

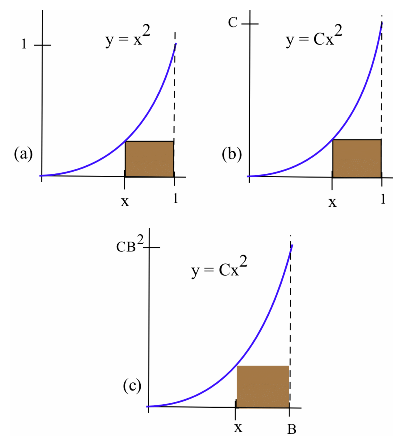
The problem of finding the shortest path connecting several points in the plane is called the "Steiner problem." It is important for designing computer chips and telephone networks to be as efficient as possible.



(b) What is the shortest connecting path for five towns located on the corners of a pentagon?

Generalized Max/Min Problems

The previous max/min problems were mostly numerical problems: the amount of fencing in Problem 2 was 200 feet, the lengths of the piece of tin in Problem 4 were 10 and 15, and the parabola in Problem 17(a) was $y = 16 - x^2$. In working those problems, you might have noticed some patterns among the numbers in the problem and the numbers in your answers, and you might have wondered if the pattern was a coincidence or if there really was a general pattern at work. Rather than trying several numerical examples to see if the "pattern" holds, mathematicians, engineers, scientists and others sometimes resort to generalizing the problem. We free the problem from the particular numbers by replacing the numbers with letters, and then we solve the generalized problem. In this way, relationships between the values in the problem and those in the solution can become more obvious. Solutions to these generalized problems are also useful if you want to program a computer to quickly provide numerical answers.



29. (a) Find the dimensions of the rectangle with the greatest area that can be built so the base of the rectangle is on the x-axis between 0 and 1 ($0 \le x \le 1$) and one corner of the rectangle is on the curve $y = x^2$ (see above right). What is the area of this rectangle?

(b) Generalize the problem in part (a) for the parabola $y = Cx^2$ with $C > 0$ and $0 \le x \le 1$.

(c) Generalize for the parabola $y = Cx^2$ with $C > 0$ and $0 \le x \le B$.

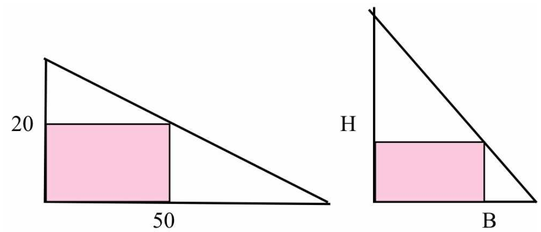
30. (a) Find the dimensions of the rectangle with the greatest area that can be built so the base of the rectangle is on the x-axis between 0 and 1 and one corner of the rectangle is on the curve $y = x^3$. What is the area of this rectangle?

(b) Generalize the problem in part (a) for the curve $y = Cx^3$ with $C > 0$ and $0 \le x \le 1$.

(c) Generalize for the curve $y = Cx^3$ with $C > 0$ and $0 \le x \le B$.

(d) Generalize for the curve $y = Cx^n$ with $C > 0$, n a positive integer, and $0 \le x \le B$.

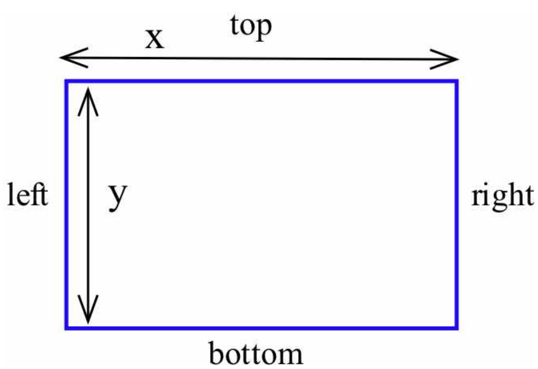
31. (a) The base of a right triangle is 50 and the height is 20. Find the dimensions and area of the rectangle with the greatest area that can be enclosed in the triangle if the base of the rectangle must lie on the base of the triangle.



(b) The base of a right triangle is B and the height is H. Find the dimensions and area of the rectangle with the greatest area that can be enclosed in the triangle if the base of the rectangle must lie on the base of the triangle.

(c) State your general conclusion from part (b) in words.

32. (a) You have T dollars to buy fencing material to enclose a rectangular plot of land. The fence for the top and bottom costs \$5 per foot and for the sides it costs \$3 per foot. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom, and for the sides?

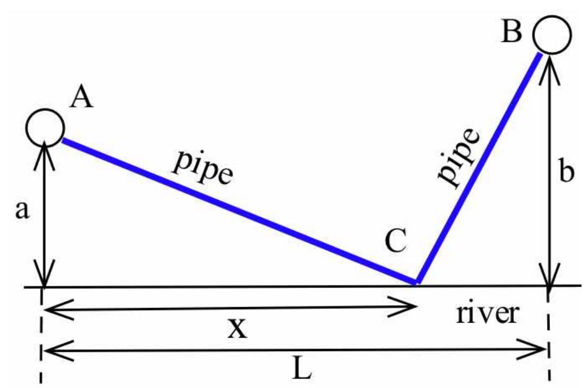


(b) You have T dollars to buy fencing material to enclose a rectangular plot of land. The fence for the top and bottom costs \$A per foot and for the sides it costs \$B per foot. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom (together), and for the sides (together)?

(c) You have T dollars to buy fencing material to enclose a rectangular plot of land. The fence costs \$A per foot for the top, \$B/foot for the bottom, \$C/ft for the left side and \$D/ft for the right side. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom (together), and for the sides (together)?

33. Determine the dimensions of the least expensive cylindrical can that will hold V cubic inches if the top material costs \$A per square inch, the bottom material costs \$B per square inch, and the side material costs \$C per square inch.

34. Find the location of C in the figure below so that the sum of the distances from A to C and from C to B is a minimum.



3.5 Practice Answers

1. $V(x) = x(15-2x)(7-2x) = 4x^3 - 44x^2 + 105x$ so:

$$V'(x) = 12x^2 - 88x + 105 = (2x-3)(6x-35)$$

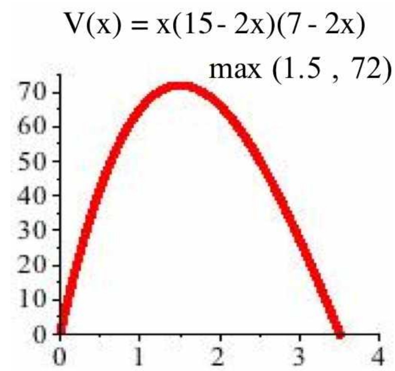
which is defined for all x: the only critical numbers are the endpoints $x = 0$ and $x = \frac{7}{2}$ and where $V'(x) = 0$: $x = \frac{3}{2}$ and $x = \frac{35}{6}$ (but $\frac{35}{6}$ is not in the interval $[0, \frac{7}{2}]$ so it is not practical). The maximum volume must occur when $x = 0$, $x = \frac{3}{2}$ or $x = \frac{7}{2}$:

$$V(0) = 0 \cdot (15 - 2 \cdot 0) \cdot (7 - 2 \cdot 0) = 0$$

$$V\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \left(15 - 2 \cdot \frac{3}{2}\right) \cdot \left(7 - 2 \cdot \frac{3}{2}\right) = \frac{3}{2}(12)(4) = 72$$

$$V\left(\frac{7}{2}\right) = \frac{7}{2}\left(15 - 2 \cdot \frac{7}{2}\right) \cdot \left(7 - 2 \cdot \frac{7}{2}\right) = \frac{7}{2}(8)(0) = 0$$

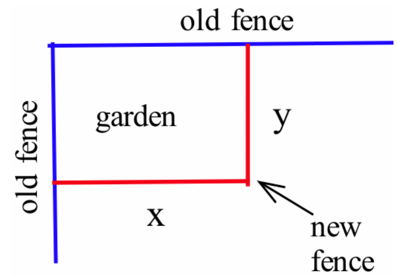
The maximum-volume box will result from cutting a 1.5-by-1.5 inch square from each corner. A graph of $V(x)$ appears in the margin.



2. (a) We have 80 feet of fencing (see margin). Our assignment is to maximize the area of the garden: $A = x \cdot y$ (two variables). Fortunately, we have the constraint that $x + y = 80$, so $y = 80 - x$ and our assignment reduces to maximizing a function of one variable:

$$A = x \cdot y = x \cdot (80 - x) = 80x - x^2 \Rightarrow A'(x) = 80 - 2x$$

so $A'(x) = 0 \Rightarrow x = 40$. Because $A''(x) = -2 < 0$, the graph of A is concave down, hence A has a maximum at $x = 40$. The maximum area is $A(40) = 40 \cdot 40 = 1600$ ft^2 when $x = 40$ feet and $y = 40$ feet. The maximum-area garden is a square.



(b) This is similar to part (a) except we have F feet of fencing instead of 80 feet: $x + y = F \Rightarrow y = F - x$ and we want to maximize $A = xy = x(F - x) = Fx - x^2$. Differentiating, $A'(x) = F - 2x$ so $A'(x) = 0 \Rightarrow x = \frac{F}{2} \Rightarrow y = \frac{F}{2}$. The maximum area is $A\left(\frac{F}{2}\right) = \frac{F^2}{4}$ square feet when the garden is a square with half of the new fence used on each of the two new sides.

3. The cost C is given by:

$$\begin{aligned} C &= 5(\text{area of top}) + 3(\text{area of sides}) + 5(\text{area of bottom}) \\ &= 5(\pi r^2) + 3(2\pi r h) + 5(\pi r^2) \end{aligned}$$

so our assignment is to minimize $C = 10\pi r^2 + 6\pi r h$, a function of two variables (r and h). Fortunately, we also have the constraint that volume $= 1000$ in^3 $= \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}$. So:

$$C = 10\pi r^2 + 6\pi r\left(\frac{1000}{\pi r^2}\right) = 10\pi r^2 + \frac{6000}{r} \Rightarrow C'(r) = 20\pi r - \frac{6000}{r^2}$$

which exists for $r \neq 0$ ($r = 0$ is not in the domain of $C(r)$).

$$C'(r) = 0 \Rightarrow 20\pi r - \frac{6000}{r^2} = 0 \Rightarrow 20\pi r^3 = 6000 \Rightarrow r = \sqrt[3]{\frac{6000}{20\pi}} \approx 4.57 \text{ inches}$$

When $r = 4.57$, $h = \frac{1000}{\pi(4.57)^2} \approx 15.24$ inches. Examining the second derivative, $C''(r) = 20\pi + \frac{12000}{r^3} > 0$ for all $r > 0$ so C is concave up and we have found the minimum cost.

3.6 *Asymptotic Behavior of Functions*

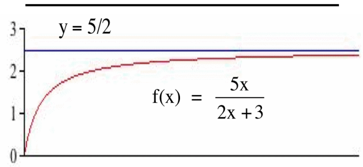
When you turn on an automobile or a light bulb or a computer, many things happen. Some of them are uniquely part of the start-up process of the system. These "transient" things occur only during start up, and then the system settles down to its steady-state operation. This start-up behavior can be very important, but sometimes we want to investigate the steady-state — or long-term — behavior: how does the system behave "after a long time?" In this section we investigate and describe the long-term behavior of functions and the systems they model: how does a function behave "when x (or $-x$) is arbitrarily large?"

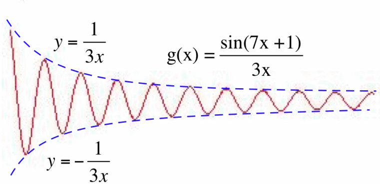
Limits as x Becomes Arbitrarily Large ("Approaches Infinity")

The same type of questions we considered about a function f as x approached a finite number can also be asked about f as x "becomes arbitrarily large" (or "increases without bound") — that is, eventually becomes larger than any fixed number.

Example 1. What happens to the values of $f(x) = \dfrac{5x}{2x+3}$ and $g(x) = \dfrac{\sin(7x+1)}{3x}$ as x becomes arbitrarily large (increases without bound)?

x	$\frac{5x}{2x+3}$	$\frac{\sin(7x+1)}{3x}$
10	2.17	0.031702
100	2.463	−0.001374
1000	2.4962	0.000333
10,000	2.4996	0.000001





Solution. One approach is numerical: evaluate $f(x)$ and $g(x)$ for some "large" values of x and see if there is a pattern to the values of $f(x)$ and $g(x)$. The margin table shows the values of $f(x)$ and $g(x)$ for several large values of x. When x is very large, it appears that the values of $f(x)$ are close to $2.5 = \frac{5}{2}$ and the values of $g(x)$ are close to 0. In fact, we can guarantee that the values of $f(x)$ are as close to $\frac{5}{2}$ as someone wants by taking x to be "big enough." The values of $f(x) = \dfrac{5x}{2x+3}$ may or may not ever equal $\frac{5}{2}$ (they never do), but if x is "large," then $f(x)$ is "very close to" $\frac{5}{2}$. Similarly, we can guarantee that the values of $g(x)$ are as close to 0 as someone wants by taking x to be "big enough." The graphs of f and g for "large" values of x appear in the margin. ◀

Practice 1. What happens to the values of $f(x) = \dfrac{3x+4}{x-2}$ and $g(x) = \dfrac{\cos(5x)}{2x+7}$ as x becomes arbitrarily large?

We can express the answers to Example 1 using limits. "As x becomes arbitrarily large, the values of $\dfrac{5x}{2x+3}$ approach $\frac{5}{2}$" can be written:

$$\lim_{x\to\infty} \frac{5x}{2x+3} = \frac{5}{2}$$

and "the values of $\frac{\sin(7x+1)}{3x}$ approach 0" can be written:

$$\lim_{x\to\infty} \frac{\sin(7x+1)}{3x} = 0$$

We read $\lim_{x\to\infty}$ as "the limit as x approaches infinity," meaning "the limit as x becomes arbitrarily large" or "as x increases without bound."

The notation "$x \to -\infty$," read as "x approaches negative infinity," means that the values of $-x$ become arbitrarily large.

During this discussion — and throughout this book — we do not treat "infinity" or "∞" as a number, but only as a useful notation. "Infinity" is not part of the real number system, and we use the common notation "$x \to \infty$" and the phrase "x approaches infinity" only to mean that "x becomes arbitrarily large."

Practice 2. Rewrite your answers to Practice 1 using limit notation.

The expression $\lim_{x\to\infty} f(x)$ asks about the behavior of $f(x)$ as the values of x get larger and larger without any bound. One way to determine this behavior is to look at the values of $f(x)$ for some values of x that are very "large." If the values of the function get arbitrarily close to a single number as x gets larger and larger, then we will say that number is the limit of the function as x approaches infinity.

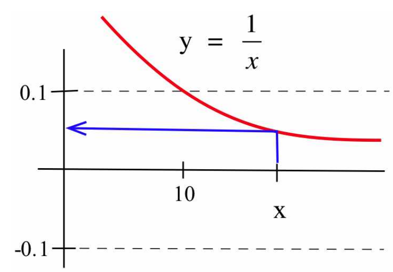
A more formal definition of the limit as "$x \to \infty$" appears at the end of this section.

Practice 3. Fill in the table for $f(x) = \frac{6x+7}{3-2x}$ and $g(x) = \frac{\sin(3x)}{x}$ and use those values to estimate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$.

x	$\frac{6x+7}{3-2x}$	$\frac{\sin(3x)}{x}$
10		
200		
500		
20,000		

Example 2. How large must x be to guarantee that $f(x) = \frac{1}{x} < 0.1$? That $f(x) < 0.001$? That $f(x) < E$ (with $E > 0$)?

Solution. If $x > 10$, then $\frac{1}{x} < \frac{1}{10} = 0.1$. If $x > 1000$, then $\frac{1}{x} < \frac{1}{1000} = 0.001$. In general, if E is any positive number, then we can guarantee that $|f(x)| < E$ by picking only values of $x > \frac{1}{E} > 0$: if $x > \frac{1}{E}$, then $\frac{1}{x} < E$. From this we can conclude that $\lim_{x\to\infty} \frac{1}{x} = 0$. ◀



Practice 4. How large must x be to guarantee that $f(x) = \frac{1}{x^2} < 0.1$? That $f(x) < 0.001$? That $f(x) < E$ (with $E > 0$)? Evaluate $\lim_{x\to\infty} \frac{1}{x^2}$.

The Main Limit Theorem (Section 1.2) about limits of combinations of functions still holds true if the limits as "$x \to a$" are replaced with limits as "$x \to \infty$" but we will not prove those results.

Polynomials arise regularly in applications, and we often need the limit, as "$x \to \infty$," of ratios of polynomials or functions containing powers of x. In these situations the following technique is often helpful:

- factor the highest power of x in the denominator from both the numerator and the denominator
- cancel the common factor from the numerator and denominator

The limit of the new denominator is a constant, so the limit of the resulting ratio is easier to determine.

Example 3. Determine $\lim_{x\to\infty} \frac{7x^2+3x-4}{3x^2-5}$ and $\lim_{x\to\infty} \frac{9x+2}{3x^2-5x+1}$.

Solution. Factoring x^2 out of the numerator and the denominator of the first rational function results in:

$$\lim_{x\to\infty} \frac{7x^2+3x-4}{3x^2-5} = \lim_{x\to\infty} \frac{x^2(7+\frac{3}{x}-\frac{4}{x^2})}{x^2(3-\frac{5}{x^2})} = \lim_{x\to\infty} \frac{7+\frac{3}{x}-\frac{4}{x^2}}{3-\frac{5}{x^2}} = \frac{7}{3}$$

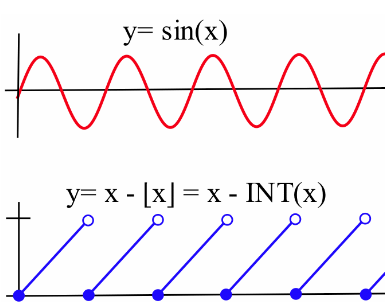
where we used the facts that $\frac{3}{x} \to 0$, $\frac{4}{x^2} \to 0$ and $\frac{5}{x^2} \to 0$ as $x \to \infty$. Similarly:

$$\lim_{x\to\infty} \frac{9x+2}{3x^2-5x+1} = \lim_{x\to\infty} \frac{x^2(\frac{9}{x}+\frac{2}{x^2})}{x^2(3-\frac{5}{x}+\frac{1}{x^2})} = \lim_{x\to\infty} \frac{\frac{9}{x}+\frac{2}{x^2}}{3-\frac{5}{x}+\frac{1}{x^2}} = \frac{0}{3} = 0$$

because $\frac{k}{x} \to 0$ and $\frac{c}{x^2} \to 0$ as $x \to \infty$ for any constants k and c. ◀

If we need to evaluate a more difficult limit as $x \to \infty$, it is often useful to algebraically manipulate the function into the form of a ratio and then use the previous technique.

If the values of the function oscillate and do not approach a single number as x becomes arbitrarily large, then the function does not have a limit as x approaches ∞: the limit **does not exist**.



Example 4. Evaluate $\lim_{x\to\infty} \sin(x)$ and $\lim_{x\to\infty} x - \lfloor x \rfloor$

Solution. As $x \to \infty$, $f(x) = \sin(x)$ and $g(x) = x - \lfloor x \rfloor$ do not have limits. As x grows without bound, the values of $f(x) = \sin(x)$ oscillate between -1 and $+1$ (see margin), and these values do not approach a single number. Similarly, $g(x) = x - \lfloor x \rfloor$ continues to take on all values between 0 and 1, and these values never approach a single number. ◀

Using Calculators to Help Find Limits as "$x \to \infty$" or "$x \to -\infty$"

Calculators only store a limited number of digits for each quantity. This becomes a severe limitation when we deal with extremely large quantities.

Example 5. The value of $f(x) = (x+1) - x$ is clearly equal to 1 for all values of x, and your calculator will give the right answer if you use it to evaluate $f(4)$ or $f(5)$. Now use it to evaluate f for a big value of x, say $x = 10^{40}$: $f(10^{40}) = (10^{40} + 1) - 10^{40} = 1$, but most calculators do not store 40 digits of a number, and they will respond that $f(10^{40}) = 0$, which is **wrong**. In this example the calculator's error is obvious, but similar errors can occur in less obvious ways when using calculators for computations involving very large numbers.

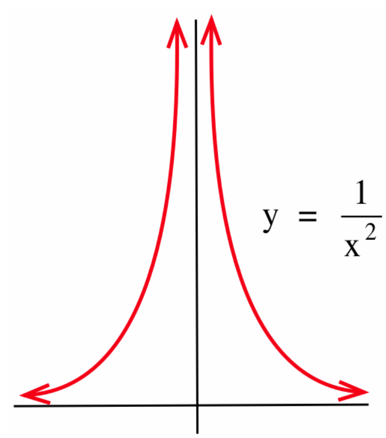
You should be careful with — and somewhat suspicious of — the answers your calculator gives you.

Calculators can still be helpful for examining limits as $x \to \infty$ and $x \to -\infty$ as long as we don't place too much faith in their responses.

Even if you have forgotten some of the properties of the natural logarithm function $\ln(x)$ and the cube root function $\sqrt[3]{x}$, a little experimentation on your calculator can help convince you that $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}} = 0$.

The Limit Is Infinite

The function $f(x) = \frac{1}{x^2}$ is undefined at $x = 0$, but we can still ask about the behavior of $f(x)$ for values of x "close to" 0. The margin figure indicates that if x is very small (close to 0) then $f(x)$ is very large. As the values of x get closer to 0, the values of $f(x)$ grow larger and can be made as large as we want by picking x to be close enough to 0. Even though the values of f are not approaching any one number, we use the "infinity" notation to indicate that the values of f are growing without bound, and write: $\lim_{x\to 0} \frac{1}{x^2} = \infty$.



The values of $\frac{1}{x^2}$ do not *equal* "infinity": the notation $\lim_{x\to 0} \frac{1}{x^2} = \infty$ means that the values of $\frac{1}{x^2}$ can be made arbitrarily large by picking values of x very close to 0.

The limit, as $x \to 0$, of $\frac{1}{x}$ is slightly more complicated. If x is close to 0, then the value of $f(x) = \frac{1}{x}$ can be a large positive number or a large negative number, depending on the sign of x. The function $f(x) = \frac{1}{x}$ does not have a (two-sided) limit as x approaches 0, but we can still investigate one-sided limits:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Example 6. Determine $\lim_{x\to 3^+} \frac{x-5}{x-3}$ and $\lim_{x\to 3^-} \frac{x-5}{x-3}$.

Solution. As $x \to 3^+$, $x - 5 \to -2$ and $x - 3 \to 0$. Because the denominator is approaching 0, we cannot use the Main Limit Theorem, and we need to examine the function more carefully. When $x \to 3^+$, we know that $x > 3$ so $x - 3 > 0$. So if x is close to 3 and slightly larger than 3, then the ratio of $x - 5$ to $x - 3$ is:

$$\frac{\text{a number close to } -2}{\text{small positive number}} = \text{large negative number}$$

As $x > 3$ gets closer to 3:

$$\frac{x-5}{x-3} = \frac{\text{a number closer to } -2}{\text{positive and closer to } 0} = \text{larger negative number}$$

By taking $x > 3$ even closer to 3, the denominator gets closer to 0 but remains positive, so the ratio gets arbitrarily large and negative: $\lim_{x \to 3^+} \frac{x-5}{x-3} = -\infty$.

As $x \to 3^-$, $x - 5 \to -2$ and $x - 3 \to 0$ as before, but now we know that $x < 3$ so $x - 3 < 0$. So if x is close to 3 and slightly smaller than 3, then the ratio of $x - 5$ to $x - 3$ is:

$$\frac{\text{a number close to } -2}{\text{small negative number}} = \text{large positive number}$$

so $\lim_{x \to 3^-} \frac{x-5}{x-3} = \infty$. ◀

Practice 5. Find: (a) $\lim_{x \to 2^+} \frac{7}{2-x}$ (b) $\lim_{x \to 2^+} \frac{3x}{2x-4}$ (c) $\lim_{x \to 2^+} \frac{3x^2 - 6x}{x-2}$.

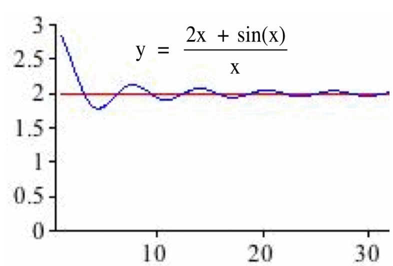
Horizontal Asymptotes

The limits of f, as "$x \to \infty$" and "$x \to -\infty$," provide information about horizontal asymptotes of f.

Definition: The line $y = K$ is a **horizontal asymptote** of f if:

$$\lim_{x \to \infty} f(x) = K \quad \text{or} \quad \lim_{x \to -\infty} f(x) = K$$

Example 7. Find any horizontal asymptotes of $f(x) = \frac{2x + \sin(x)}{x}$.



Solution. Computing the limit as $x \to \infty$:

$$\lim_{x \to \infty} \frac{2x + \sin(x)}{x} = \lim_{x \to \infty} \left[\frac{2x}{x} + \frac{\sin(x)}{x}\right] = \lim_{x \to \infty} \left[2 + \frac{\sin(x)}{x}\right]$$
$$= 2 + \lim_{x \to \infty} \frac{\sin(x)}{x} = 2 + 0 = 2$$

so the line $y = 2$ is a horizontal asymptote of f. The limit, as "$x \to -\infty$," is also 2 so $y = 2$ is the *only* horizontal asymptote of f. The graphs of f and $y = 2$ appear in the margin. A function may or may not cross its asymptote. ◀

You likely explored horizontal asymptotes in a previous course using terms like "end behavior" and investigating only rational functions. The tools of calculus allow us to make the the notion of "end behavior" more precise and investigate a wider variety of functions.

Vertical Asymptotes

As with horizontal asymptotes, you have likely studied vertical asymptotes before (at least for rational functions). We can now define vertical asymptotes using infinite limits.

> **Definition**: The vertical line $x = a$ is a **vertical asymptote** of the graph of f if either or both of the one-sided limits of f, as $x \to a^-$ or $x \to a^+$, is infinite.

If our function f is the ratio of a polynomial $P(x)$ and a polynomial $Q(x)$, $f(x) = \dfrac{P(x)}{Q(x)}$, then the only **candidates** for vertical asymptotes are the values of x where $Q(x) = 0$. However, the fact that $Q(a) = 0$ is **not** enough to guarantee that the line $x = a$ is a vertical asymptote of f; we also need to evaluate $P(a)$.

If $Q(a) = 0$ and $P(a) \neq 0$, then the line $x = a$ must be a vertical asymptote of f. If $Q(a) = 0$ and $P(a) = 0$, then the line $x = a$ may or may not be a vertical asymptote.

Example 8. Find the vertical asymptotes of $f(x) = \dfrac{x^2 - x - 6}{x^2 - x}$ and $g(x) = \dfrac{x^2 - 3x}{x^2 - x}$.

Solution. Factoring the numerator and denominator of $f(x)$ yields $f(x) = \dfrac{(x-3)(x+2)}{x(x-1)}$ so the only values of x that make the denominator 0 are $x = 0$ and $x = 1$, and these are the only candidates to be vertical asymptotes. Because $\lim\limits_{x \to 0^+} f(x) = +\infty$ and $\lim\limits_{x \to 1^+} f(x) = -\infty$, both $x = 0$ and $x = 1$ are vertical asymptotes of f.

Factoring the numerator and denominator of $g(x)$ yields $\dfrac{x(x-3)}{x(x-1)}$ so the only candidate to be vertical asymptotes are $x = 0$ and $x = 1$. Because $\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^+} \dfrac{x(x-3)}{x(x-1)} = \lim\limits_{x \to 1^+} \dfrac{x-3}{x-1} = -\infty$ the line $x = 1$ must be a vertical asymptote of g. But $\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} \dfrac{x-3}{x-1} = 3 \neq \pm\infty$ so $x = 0$ is **not** a vertical asymptote of g. ◀

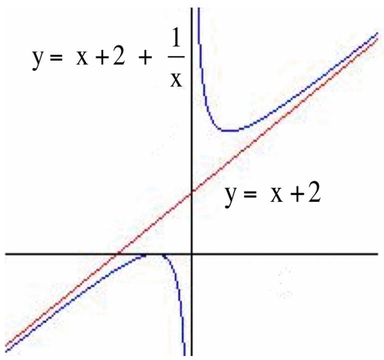
Practice 6. Find the vertical asymptotes of $f(x) = \dfrac{x^2 + x}{x^2 + x - 2}$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$.

Other Asymptotes as "$x \to \infty$" and "$x \to -\infty$"

If the limit of $f(x)$, as $x \to \infty$ or $x \to -\infty$, is a constant K, then the graph of f gets arbitrarily close to the horizontal line $y = K$, in which case we call $y = K$ a horizontal asymptote of f. Some functions, however, approach lines that are not horizontal.

Example 9. Find all asymptotes of $f(x) = \dfrac{x^2 + 2x + 1}{x} = x + 2 + \dfrac{1}{x}$.

Solution. If x is a large positive (or negative) number, then $\dfrac{1}{x}$ is very close to 0, and the graph of $f(x)$ is very close to the line $y = x + 2$ (see margin). The line $y = x + 2$ is an asymptote of the graph of f.



If x is a large positive number, then $\dfrac{1}{x}$ is positive, and the graph of f is slightly above the graph of $y = x + 2$. If x is a large negative number, then $\dfrac{1}{x}$ is negative, and the graph of f will be slightly below the graph of $y = x + 2$. The $\dfrac{1}{x}$ piece of f never equals 0, so the graph of f never crosses or touches the graph of the asymptote $y = x + 2$.

The graph of f also has a vertical asymptote at $x = 0$ because $\displaystyle\lim_{x \to 0^+} f(x) = \infty$ and $\displaystyle\lim_{x \to 0^-} f(x) = -\infty$. ◀

Practice 7. Find all asymptotes of $g(x) = \dfrac{2x^2 - x - 1}{x + 1} = 2x - 3 + \dfrac{2}{x + 1}$.

Some functions even have **nonlinear asymptotes**: asymptotes that are not straight lines. The graphs of these functions approach some nonlinear function when the values of x become arbitrarily large.

Example 10. Find all asymptotes of $f(x) = \dfrac{x^4 + 3x^3 + x^2 + 4x + 5}{x^2 + 1} = x^2 + 3x + \dfrac{x + 5}{x^2 + 1}$.

Solution. When x is very large, positive or negative, then $\dfrac{x + 5}{x^2 + 1}$ is very close to 0 and the graph of f is very close to the graph of $g(x) = x^2 + 3x$. The function $g(x) = x^2 + 3x$ is a nonlinear asymptote of f. The denominator of f is never 0 and f has no vertical asymptotes. ◀

Practice 8. Find all asymptotes of $f(x) = \dfrac{x^3 + 2\sin(x)}{x} = x^2 + \dfrac{2\sin(x)}{x}$.

If we can write $f(x)$ as a sum of two functions, $f(x) = g(x) + r(x)$, with $\displaystyle\lim_{x \to \pm\infty} r(x) = 0$, then the graph of f is asymptotic to the graph of g, and g is an asymptote of f. In this situation:

- if $g(x) = K$, then f has a horizontal asymptote $y = K$
- if $g(x) = ax + b$, then f has a linear asymptote $y = ax + b$
- otherwise f has a nonlinear asymptote $y = g(x)$

Formal Definition of $\lim_{x \to \infty} f(x) = K$

The following definition states precisely what we mean by the phrase "we can guarantee that the values of $f(x)$ are arbitrarily close to K by restricting the values of x to be sufficiently large."

> **Definition**: $\lim_{x \to \infty} f(x) = K$ means that, for every given $\epsilon > 0$, there is a number N so that:
>
> if x is larger than N
> then $f(x)$ is within ϵ units of K.
>
> Equivalently: $|f(x) - K| < \epsilon$ whenever $x > N$.

Example 11. Show that $\lim_{x \to \infty} \frac{x}{2x+1} = \frac{1}{2}$.

Solution. Typically, we need to do two things. First we need to find a value of N, often depending on ϵ. Then we need to show that the value of N we found satisfies the conditions of the definition.

Assume that $|f(x) - K|$ is less than ϵ and solve for $x > 0$:

$$\epsilon > \left| \frac{x}{2x+1} \right| = \left| \frac{2x - (2x+1)}{2(2x+1)} \right| = \left| \frac{-1}{4x+2} \right| = \frac{1}{4x+2}$$
$$\Rightarrow 4x + 2 > \frac{1}{\epsilon} \Rightarrow x > \frac{1}{4}\left(\frac{1}{\epsilon} - 2\right)$$

So, given any $\epsilon > 0$, take $N = \frac{1}{4}\left(\frac{1}{\epsilon} - 2\right)$.

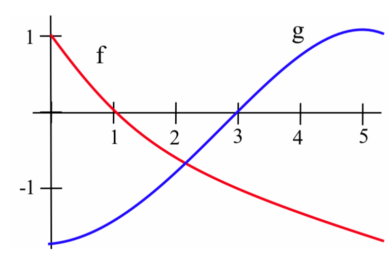
Now we can just reverse the order of the steps above to show that this N satisfies the limit definition. If $x > 0$ and $x > \frac{1}{4}\left(\frac{1}{\epsilon} - 2\right)$ then:

$$4x + 2 > \frac{1}{\epsilon} \Rightarrow \epsilon > \frac{1}{4x+2} = \left| \frac{x}{2x+1} - \frac{1}{2} \right| = |f(x) - K|$$

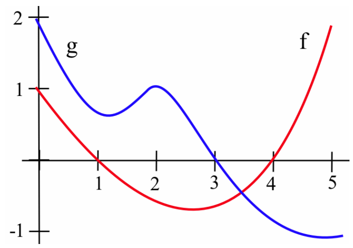
We have shown that "for every given ϵ, there is an N" that satisfies the definition. ◀

3.6 Problems

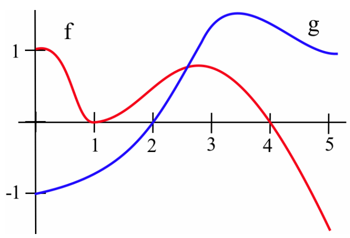
1. The margin figure shows $f(x)$ and $g(x)$ for $0 \le x \le 5$. Define a new function $h(x) = \frac{f(x)}{g(x)}$.
   (a) At what value of x does $h(x)$ have a root?
   (b) Determine the limits of $h(x)$ as $x \to 1^+$, $x \to 1^-$, $x \to 3^+$ and $x \to 3^-$.
   (c) Where does $h(x)$ have a vertical asymptote?



2. The figure below shows $f(x)$ and $g(x)$ on the interval $0 \le x \le 5$. Let $h(x) = \dfrac{f(x)}{g(x)}$.
   (a) At what value(s) of x does $h(x)$ have a root?
   (b) Where does $h(x)$ have vertical asymptotes?



3. The figure below shows $f(x)$ and $g(x)$ for $0 \le x \le 5$. Let $h(x) = \dfrac{f(x)}{g(x)}$. Determine the limits of $h(x)$ as $x \to 2^+$, $x \to 2^-$, $x \to 4^+$ and $x \to 4^-$.



For Problems 4–24, calculate the limit of each expression as "$x \to \infty$."

4. $\dfrac{6}{x+2}$

5. $\dfrac{28}{3x-5}$

6. $\dfrac{7x+12}{3x-2}$

7. $\dfrac{4-3x}{x+8}$

8. $\dfrac{5\sin(2x)}{2x}$

9. $\dfrac{\cos(3x)}{5x-1}$

10. $\dfrac{2x-3\sin(x)}{5x-1}$

11. $\dfrac{4+x\cdot\sin(x)}{2x-4}$

12. $\dfrac{x^2-5x+2}{x^2+8x-4}$

13. $\dfrac{2x^2-9}{3x^2+10x}$

14. $\dfrac{\sqrt{x+5}}{\sqrt{4x-2}}$

15. $\dfrac{5x^2-7x+2}{2x^3+4x}$

16. $\dfrac{x+\sin(x)}{x-\sin(x)}$

17. $\dfrac{7x^2+x\cdot\sin(x)}{3-x^2+\sin(7x^2)}$

18. $\dfrac{7x^{143}+734x-2}{x^{150}-99x^{83}+25}$

19. $\dfrac{\sqrt{9x^2+16}}{2+\sqrt{x^2+1}}$

20. $\sin\left(\dfrac{3x+5}{2x-1}\right)$

21. $\cos\left(\dfrac{7x+4}{x^2+x+1}\right)$

22. $\ln\left(\dfrac{3x^2+5x}{x^2-4}\right)$

23. $\ln(x+8)-\ln(x-5)$

24. $\ln(3x+8)-\ln(2x+5)$

25. Salt water with a concentration of 0.2 pounds of salt per gallon flows into a large tank that initially contains 50 gallons of pure water.
    (a) If the flow rate of salt water into the tank is 4 gallons per minute, what is the volume $V(t)$ of water and the amount $A(t)$ of salt in the tank t minutes after the flow begins?
    (b) Show that the salt concentration $C(t)$ at time t is $C(t) = \dfrac{0.8t}{4t+50}$.
    (c) What happens to the concentration $C(t)$ after a "long" time?
    (d) Redo parts (a)–(c) for a large tank that initially contains 200 gallons of pure water.

26. Under certain laboratory conditions, an agar plate contains $B(t) = 100\left(2-e^{-t}\right)$ bacteria t hours after the start of the experiment.
    (a) How many bacteria are on the plate at the start of the experiment ($t=0$)?
    (b) Show that the population is always increasing. (Show $B'(t) > 0$ for all $t > 0$.)
    (c) What happens to the population $B(t)$ after a "long" time?
    (d) Redo parts (a)–(c) for $B(t) = A(2-e^{-t})$.

For Problems 27–41 , calculate the limits.

27. $\displaystyle\lim_{x\to 0} \frac{x+5}{x^2}$

28. $\displaystyle\lim_{x\to 3} \frac{x-1}{(x-3)^2}$

29. $\displaystyle\lim_{x\to 5} \frac{x-7}{(x-5)^2}$

30. $\displaystyle\lim_{x\to 2^+} \frac{x-1}{x-2}$

31. $\displaystyle\lim_{x\to 2^-} \frac{x-1}{x-2}$

32. $\displaystyle\lim_{x\to 3^+} \frac{x-1}{x-2}$

33. $\displaystyle\lim_{x\to 4^+} \frac{x+3}{4-x}$

34. $\displaystyle\lim_{x\to 1^-} \frac{x^2+5}{1-x}$

35. $\displaystyle\lim_{x\to 3^+} \frac{x^2-4}{x^2-2x-3}$

36. $\displaystyle\lim_{x\to 2} \frac{x^2-x-2}{x^2-4}$

37. $\lim_{x\to 0} \frac{x-2}{1-\cos(x)}$

38. $\lim_{x\to\infty} \frac{x^3+7x-4}{x^2+11x}$

39. $\lim_{x\to 5} \frac{\sin(x-5)}{(x-5)}$

40. $\lim_{x\to 0} \frac{x+1}{\sin^2(x)}$

41. $\lim_{x\to 0^+} \frac{1+\cos(x)}{1-e^x}$

In Problems 42–59, write an **equation** of each asymptote for each function and state whether it is a vertical, horizontal or slant asymptote.

42. $f(x) = \frac{x+2}{x-1}$

43. $f(x) = \frac{x-3}{x^2}$

44. $f(x) = \frac{x-1}{x^2-x}$

45. $f(x) = \frac{x+5}{x^2-4x+3}$

46. $f(x) = \frac{x+\sin(x)}{3x-3}$

47. $f(x) = \frac{x^2-4}{x+1}$

48. $f(x) = \frac{\cos(x)}{x^2}$

49. $f(x) = 2 + \frac{3-x}{x-1}$

50. $f(x) = \frac{x\cdot\sin(x)}{x^2-x}$

51. $f(x) = \frac{2x^2+x+5}{x}$

52. $f(x) = \frac{x^2+x}{x+1}$

53. $f(x) = \frac{1}{x-2} + \sin(x)$

54. $f(x) = x + \frac{x}{x^2+1}$

55. $f(x) = x^2 + \frac{x}{x^2+1}$

56. $f(x) = x^2 + \frac{x}{x+1}$

57. $f(x) = \frac{x\cdot\cos(x)}{x-3}$

58. $f(x) = \frac{x^3-x^2+2x-1}{x-1}$

59. $f(x) = \sqrt{\frac{x^2+3x+2}{x+3}}$

3.6 Practice Answers

1. As x becomes arbitrarily large, the values of $f(x)$ approach 3 and the values of $g(x)$ approach 0.

2. $\lim_{x\to\infty} \frac{3x+4}{x-2} = 3$ and $\lim_{x\to\infty} \frac{\cos(5x)}{2x+7} = 0$

3. The completed table appears in the margin.

x	$\frac{6x+7}{3-2x}$	$\frac{\sin(3x)}{x}$
10	−3.94117647	−0.09880311
200	−3.04030227	0.00220912
500	−3.00160048	0.00017869
20,000	−3.00040003	0.00004787
	↓	↓
	−3	0

4. If $x > \sqrt{10} \approx 3.162$ then $f(x) = \frac{1}{x^2} < 0.1$.

   If $x > \sqrt{1000} \approx 31.62$ then $f(x) = \frac{1}{x^2} < 0.001$.

   If $x > \sqrt{\frac{1}{E}} = \frac{1}{\sqrt{E}}$ then $f(x) = \frac{1}{x^2} < E$.

5. (a) As $x \to 2^+$, $2-x \to 0$, and $x > 2$ so $2-x < 0$: $2-x$ takes on small negative values.

   $$\frac{7}{2-x} = \frac{7}{\text{small negative number}} = \text{large negative number}$$

   so we represent the limit as: $\lim_{x\to 2^+} \frac{7}{2-x} = -\infty$.

   (b) As $x \to 2^+$, $2x-4 \to 0$, and $x > 2$ so $2x-4 > 0$: $2x-4$ takes on small positive values. And as $x \to 2^+$, $3x \to 6$ so:

   $$\frac{3x}{2x-4} = \frac{\text{number near 6}}{\text{small positive number}} = \text{large positive number}$$

   so we represent the limit as: $\lim_{x\to 2^+} \frac{3x}{2x-4} = +\infty$.

(c) As $x \to 2^+$, $3x^2 - 6x \to 0$ and $x - 2 \to 0$ so we need to do more work. Factoring the numerator as $3x^2 - 6x = 3x(x-2)$:

$$\lim_{x \to 2^+} \frac{3x^2 - 6x}{x - 2} = \lim_{x \to 2^+} \frac{3x(x-2)}{x-2} = \lim_{x \to 2^+} 3x = 6$$

where we were able to cancel the $x - 2$ factor because the limit involves values of x close to (but not equal to) 2.

6. (a) $f(x) = \dfrac{x^2 + x}{x^2 + x - 2} = \dfrac{x(x+1)}{(x-1)(x+2)}$ so f has vertical asymptotes at $x = 1$ and $x = -2$.

(b) $g(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{x-1}$ so the value $x = 1$ is not in the domain of g. If $x \neq 1$, then $g(x) = x + 1$: g has a "hole" when $x = 1$ and no vertical asymptotes.

7. $g(x) = 2x - 3 + \dfrac{2}{x+1}$ has a vertical asymptote at $x = -1$ and no horizontal asymptotes, but $\displaystyle\lim_{x \to \infty} \frac{2}{x+1} = 0$ so g has the linear asymptote $y = 2x - 3$.

8. $f(x) = x^2 + \dfrac{2\sin(x)}{x}$ is not defined at $x = 0$, so f has a vertical asymptote or a "hole" there; $\displaystyle\lim_{x \to 0} x^2 + \frac{2\sin(x)}{x} = 0 + 2 = 2$ so f has a "hole" when $x = 0$. Because $\displaystyle\lim_{x \to \infty} \frac{2\sin(x)}{x} = 0$, f has the nonlinear asymptote $y = x^2$ (but no horizontal asymptotes).

3.7 *L'Hôpital's Rule*

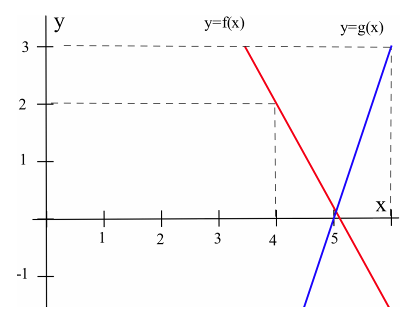
When taking limits of slopes of secant lines, $m_{\text{sec}} = \dfrac{f(x+h)-f(x)}{h}$ as $h \to 0$, we frequently encountered one difficulty: both the numerator and the denominator approached 0. And because the denominator approached 0, we could not apply the Main Limit Theorem. In many situations, however, we managed to get past this "$\frac{0}{0}$" difficulty by using algebra or geometry or trigonometry to rewrite the expression and then take the limit. But there was no common approach or pattern. The algebraic steps we used to evaluate $\lim\limits_{h\to 0} \dfrac{(2+h)^2-4}{h}$ seem quite different from the trigonometric steps needed for $\lim\limits_{h\to 0} \dfrac{\sin(2+h)-\sin(2)}{h}$.

In this section we consider a single technique, called l'Hôpital's Rule, that enables us to quickly and easily evaluate many limits of the form "$\frac{0}{0}$" as well as several other challenging indeterminate forms.

Although discovered by Johann Bernoulli, this rule was named for the Marquis de l'Hôpital (pronounced low-pee-TALL), who published it in his 1696 calculus textbook, *Analysis of the Infinitely Small for the Understanding of Curved Lines*.

A Linear Example

The graphs of two linear functions appear in the margin and we want to find $\lim\limits_{x\to 5} \dfrac{f(x)}{g(x)}$. Unfortunately, $\lim\limits_{x\to 5} f(x) = 0$ and $\lim\limits_{x\to 5} g(x) = 0$ so we cannot apply the Main Limit Theorem. We do know, however, that f and g are linear, so we can calculate their slopes, and we know that they both lines go through the point $(5,0)$ so we can find their equations: $f(x) = -2(x-5)$ and $g(x) = 3(x-5)$.



Now the limit is easier to compute:

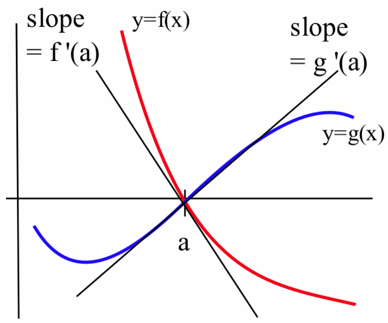
$$\lim_{x\to 5} \frac{f(x)}{g(x)} = \lim_{x\to 5} \frac{-2(x-5)}{3(x-5)} = \lim_{x\to 5} \frac{-2}{3} = -\frac{2}{3} = \frac{\text{slope of } f}{\text{slope of } g}$$

In fact, this pattern works for any two linear functions: If f and g are linear functions with slopes $m \neq 0$ and $n \neq 0$ and a common root at $x = a$, then $f(x) - f(a) = m(x-a)$ and $g(x) - g(a) = n(x-a)$ so $f(x) = m(x-a)$ and $g(x) = n(x-a)$. Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{m(x-a)}{n(x-a)} = \lim_{x\to a} \frac{m}{n} = \frac{m}{n} = \frac{\text{slope of } f}{\text{slope of } g}$$

A more powerful result—that the same pattern holds true for differentiable functions even if they are not linear—is called l'Hôpital's Rule.

> L'Hôpital's Rule ("$\frac{0}{0}$" Form)
>
> If f and g are differentiable at $x = a$,
> $f(a) = 0$, $g(a) = 0$ and $g'(a) \neq 0$
>
> then $\displaystyle \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \frac{\text{slope of } f \text{ at } a}{\text{slope of } g \text{ at } a}$



Unfortunately, we have ignored some subtle difficulties, such as $g(x)$ or $g'(x)$ possibly being 0 when x is close to, but not equal to, a. Because of these issues, a full-fledged proof of l'Hôpital's Rule is omitted.

Idea for a proof: Even though f and g may not be linear functions, they *are* differentiable. So at the point $x = a$ they are "almost linear" in the sense that we can approximate them quite well using their tangent lines at that point (see margin).

Because $f(a) = g(a) = 0$, $f(x) \approx f(a) + f'(a)(x-a) = f'(a)(x-a)$ and $g(x) \approx g(a) + g'(a)(x-a) = g'(a)(x-a)$. So:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \approx \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x\to a} \frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)}$$

Example 1. Determine $\displaystyle\lim_{x\to 0} \frac{x^2+\sin(5x)}{3x}$ and $\displaystyle\lim_{x\to 1} \frac{\ln(x)}{e^x - e}$.

Solution. We could evaluate the first limit without l'Hôpital's Rule, but let's use it anyway. We can match the pattern of l'Hôpital's Rule by letting $a = 0$, $f(x) = x^2 + \sin(5x)$ and $g(x) = 3x$. Then $f(0) = 0$, $g(0) = 0$, and f and g are differentiable with $f'(x) = 2x + 5\cos(5x)$ and $g'(x) = 3$, so:

$$\lim_{x\to 0} \frac{x^2+\sin(5x)}{3x} = \frac{f'(0)}{g'(0)} = \frac{2\cdot 0 + 5\cos(5\cdot 0)}{3} = \frac{5}{3}$$

For the second limit, let $a = 1$, $f(x) = \ln(x)$ and $g(x) = e^x - e$. Then $f(1) = 0$, $g(1) = 0$, f and g are differentiable for x near 1 (when $x > 0$), and $f'(x) = \dfrac{1}{x}$ and $g'(x) = e^x$. Then:

$$\lim_{x\to 1} \frac{\ln(x)}{e^x - e} = \frac{f'(1)}{g'(1)} = \frac{\frac{1}{1}}{e^1} = \frac{1}{e}$$

Here no simplification was possible, so we needed l'Hôpital's Rule. ◄

Practice 1. Evaluate $\displaystyle\lim_{x\to 0} \frac{1-\cos(5x)}{3x}$ and $\displaystyle\lim_{x\to 2} \frac{x^2+x-6}{x^2+2x-8}$.

Strong Version of l'Hôpital's Rule

We can strengthen L'Hôpital's Rule to include cases when $g'(a) = 0$, and the indeterminate form "$\frac{\infty}{\infty}$" when f and g increase without bound.

L'Hôpital's Rule (Strong "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" Forms)

If f and g are differentiable on an open interval I containing a, $g'(x) \neq 0$ on I except possibly at a, and $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"}$ or $\text{"}\frac{\infty}{\infty}\text{"}$

then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the limit on the right exists.

(Here "a" can represent a finite number or "∞.")

Example 2. Evaluate $\lim_{x\to\infty} \frac{e^{7x}}{5x}$.

Solution. As "$x \to \infty$," both e^{7x} and $5x$ increase without bound, so we have an "$\frac{\infty}{\infty}$" indeterminate form and can use the Strong Version of l'Hôpital's Rule: $\lim_{x\to\infty} \frac{e^{7x}}{5x} = \lim_{x\to\infty} \frac{7e^{7x}}{5} = \infty$. ◀

The limit of $\frac{f'}{g'}$ may also be an indeterminate form, in which case we can apply l'Hôpital's Rule again to the ratio $\frac{f'}{g'}$. We can continue using l'Hôpital's Rule at each stage as long as we have an indeterminate quotient.

Example 3. Compute $\lim_{x\to 0} \frac{x^3}{x-\sin(x)}$.

Solution. As $x \to 0$, $f(x) = x^3 \to 0$ and $g(x) = x - \sin(x) \to 0$ so:

$$\lim_{x\to 0} \frac{x^3}{x-\sin(x)} = \lim_{x\to 0} \frac{3x^2}{1-\cos(x)} = \lim_{x\to 0} \frac{6x}{\sin(x)} = \lim_{x\to 0} \frac{6}{\cos(x)} = 6$$

where we have used l'Hôpital's Rule three times in succession. (At each stage, you should verify the conditions for l'Hôpital's Rule hold.) ◀

Practice 2. Use l'Hôpital's Rule to find $\lim_{x\to\infty} \frac{x^2+e^x}{x^3+8x}$.

Which Function Grows Faster?

Sometimes we want to compare the asymptotic behavior of two systems or functions for large values of x. L'Hôpital's Rule can be useful in such situations. For example, if we have two algorithms for sorting names, and each algorithm takes longer and longer to sort larger collections of names, we may want to know which algorithm will accomplish the task more efficiently for really large collections of names.

Example 4. Algorithm A requires $n \cdot \ln(n)$ steps to sort n names and algorithm B requires $n^{1.5}$ steps. Which algorithm will be better for sorting very large collections of names?

Solution. We can compare the ratio of the number of steps each algorithm requires, $\frac{n \cdot \ln(n)}{n^{1.5}}$, and then take the limit of this ratio as n grows arbitrarily large: $\lim_{n\to\infty} \frac{n \cdot \ln(n)}{n^{1.5}}$.

If this limit is infinite, we say that $n \cdot \ln(n)$ "grows faster" than $n^{1.5}$. If the limit is 0, we say that $n^{1.5}$ grows faster than $n \cdot \ln(n)$.

Because $n \cdot \ln(n)$ and $n^{1.5}$ both grow arbitrarily large when n becomes large, we can simplify the ratio to $\dfrac{\ln(n)}{n^{0.5}}$ and then use l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\ln(n)}{n^{0.5}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{0.5n^{-0.5}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

We conclude that $n^{1.5}$ grows faster than $n \cdot \ln(n)$ so algorithm A requires fewer steps for really large sorts. ◀

Practice 3. Algorithm A requires e^n operations to find the shortest path connecting n towns, while algorithm B requires $100 \cdot \ln(n)$ operations for the same task and algorithm C requires n^5 operations. Which algorithm is best for finding the shortest path connecting a very large number of towns? The worst?

Other Indeterminate Forms

We call "$\frac{0}{0}$" an **indeterminate form** because knowing that f approaches 0 and g approaches 0 is not enough to determine the limit of $\frac{f}{g}$, even if that limit exists. The ratio of a "small" number divided by a "small" number can be almost anything as three simple "$\frac{0}{0}$" examples show:

$$\lim_{x\to 0} \frac{3x}{x} = 3 \quad \text{while} \quad \lim_{x\to 0} \frac{x^2}{x} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{5x}{x^3} = \infty$$

Similarly, "$\frac{\infty}{\infty}$" is an indeterminate form because knowing that f and g both grow arbitrarily large is not enough to determine the value of the limit of $\frac{f}{g}$ or even if the limit exists:

$$\lim_{x\to\infty} \frac{3x}{x} = 3 \quad \text{while} \quad \lim_{x\to\infty} \frac{x^2}{x} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{5x}{x^3} = 0$$

In addition to the indeterminate quotient forms "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" there are several other "indeterminate forms." In each case, the resulting limit depends not only on each function's limit but also on how quickly each function approaches its limit.

- **Product**: If f approaches 0 and g grows arbitrarily large, the product $f \cdot g$ has the indeterminate form "$0 \cdot \infty$."
- **Exponent**: If f and g both approach 0, the function f^g has the indeterminate form "0^0."
- **Exponent**: If f approaches 1 and g grows arbitrarily large, the function f^g has the indeterminate form "1^∞."
- **Exponent**: If f grows arbitrarily large and g approaches 0, the function f^g has the indeterminate form "∞^0."

- **Difference**: If f and g both grow arbitrarily large, the function $f - g$ has the indeterminate form "$\infty - \infty$."

Unfortunately, l'Hôpital's Rule can only be used directly with an indeterminate quotient ($\frac{0}{0}$ or "$\frac{\infty}{\infty}$"), but we can algebraically manipulate these other forms into quotients and *then* apply l'Hôpital's Rule.

Example 5. Evaluate $\displaystyle\lim_{x \to 0^+} x \cdot \ln(x)$.

Solution. This limit involves an indeterminate product (of the form "$0 \cdot -\infty$") but we need a quotient in order to apply l'Hôpital's Rule. If we rewrite the product $x \cdot \ln(x)$ as a quotient:

$$\lim_{x \to 0^+} x \cdot \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} -x = 0$$

results from applying the "$\frac{\infty}{\infty}$" version of l'Hôpital's Rule. ◀

To use l'Hôpital's Rule on a product $f \cdot g$ with indeterminate form "$0 \cdot \infty$," first rewrite $f \cdot g$ as a quotient: $\dfrac{f}{\frac{1}{g}}$ or $\dfrac{g}{\frac{1}{f}}$. Then apply l'Hôpital's Rule.

Example 6. Evaluate $\displaystyle\lim_{x \to 0^+} x^x$.

Solution. This limit involves the indeterminate form 0^0. We can convert it to a product by recalling a property of exponential and logarithmic functions: for any positive number a, $a = e^{\ln(a)}$ so:

$$f^g = e^{\ln(f^g)} = e^{g \cdot \ln(f)}$$

Applying this to x^x:

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{\ln(x^x)} = \lim_{x \to 0^+} e^{x \cdot \ln(x)}$$

This last limit involves the indeterminate product $x \cdot \ln(x)$. From the previous example we know that $\displaystyle\lim_{x \to 0^+} x \cdot \ln(x) = 0$ so we can conclude that:

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \cdot \ln(x)} = e^{\lim_{x \to 0^+} x \cdot \ln(x)} = e^0 = 1$$

because the function $f(u) = e^u$ is continuous everywhere. ◀

To use l'Hôpital's Rule on an expression involving exponents, f^g with the indeterminate form "0^0," "1^∞" or "∞^0," first convert it to an expression involving an indeterminate product by recognizing that $f^g = e^{g \cdot \ln(f)}$ and then determining the limit of $g \cdot \ln(f)$. The final result is $e^{\text{limit of } g \cdot \ln(f)}$.

Example 7. Evaluate $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$.

Solution. This expression has the form 1^∞ so we first use logarithms to convert the problem into a limit involving a product:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\infty} e^{x\cdot\ln\left(1+\frac{a}{x}\right)}$$

so now we need to compute $\lim_{x\to\infty} x\cdot\ln\left(1+\frac{a}{x}\right)$. This limit has the form "$\infty\cdot 0$" so we now convert the product to a quotient:

$$\lim_{x\to\infty} x\cdot\ln\left(1+\frac{a}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}$$

This last limit has the form "$\frac{0}{0}$" so we can finally apply l'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{\frac{-a}{x^2}}{1+\frac{a}{x}}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{a}{1+\frac{a}{x}} = \frac{a}{1} = a$$

and conclude that:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\infty} e^{x\cdot\ln\left(1+\frac{a}{x}\right)} = e^{\lim_{x\to\infty} x\cdot\ln\left(1+\frac{a}{x}\right)} = e^a$$

where we have again used the continuity of the function $f(u) = e^u$. ◀

3.7 Problems

In Problems 1–15, evaluate each limit. Be sure to justify any use of l'Hôpital's Rule.

1. $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$
2. $\lim_{x\to 2}\frac{x^4-16}{x^5-32}$
3. $\lim_{x\to 0}\frac{\ln(1+3x)}{5x}$
4. $\lim_{x\to\infty}\frac{e^x}{x^3}$
5. $\lim_{x\to 0}\frac{x\cdot e^x}{1-e^x}$
6. $\lim_{x\to 0}\frac{2^x-1}{x}$
7. $\lim_{x\to\infty}\frac{\ln(x)}{x}$
8. $\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}}$
9. $\lim_{x\to\infty}\frac{\ln(x)}{x^p}$ $(p>0)$
10. $\lim_{x\to 0}\frac{e^{3x}-e^{2x}}{4x}$
11. $\lim_{x\to 0}\frac{1-\cos(3x)}{x^2}$
12. $\lim_{x\to 0}\frac{1-\cos(2x)}{x}$
13. $\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}$
14. $\lim_{x\to 0}\frac{\cos(a+x)-\cos(a)}{x}$
15. $\lim_{x\to 0}\frac{1-\cos(x)}{x\cdot\cos(x)}$
16. Find a value for p so that $\lim_{x\to\infty}\frac{3x}{px+7} = 2$.
17. Find a value for p so that $\lim_{x\to 0}\frac{e^{px}-1}{3x} = 5$.
18. The limit $\lim_{x\to\infty}\frac{\sqrt{3x+5}}{\sqrt{2x-1}}$ has the indeterminate form "$\frac{\infty}{\infty}$." Why doesn't l'Hôpital's Rule work with this limit? (Hint: Apply l'Hôpital's Rule twice and see what happens.) Evaluate the limit without using l'Hôpital's Rule.
19. (a) Evaluate $\lim_{x\to\infty}\frac{e^x}{x}$, $\lim_{x\to\infty}\frac{e^x}{x^2}$ and $\lim_{x\to\infty}\frac{e^x}{x^5}$.
    (b) An algorithm is "exponential" if it requires $a\cdot e^{bn}$ steps (a, and b are positive constants). An algorithm is "polynomial" if it requires $c\cdot n^d$ steps. Show that polynomial algorithms require fewer steps than exponential ones for large values of n.

20. The problem $\lim_{x\to 0} \frac{x^2}{3x^2+x}$ appeared on a test. One student determined the limit was an indeterminate "$\frac{0}{0}$" form and applied l'Hôpital's Rule to get:

$$\lim_{x\to 0} \frac{x^2}{3x^2+x} = \lim_{x\to 0} \frac{2x}{6x+1} = \lim_{x\to 0} \frac{2}{6} = \frac{1}{3}$$

Another student also determined the limit was an indeterminate "$\frac{0}{0}$" form and wrote:

$$\lim_{x\to 0} \frac{x^2}{3x^2+x} = \lim_{x\to 0} \frac{2x}{6x+1} = \frac{0}{0+1} = 0$$

Which student is correct? Why?

In Problems 21–30, evaluate each limit. Be sure to justify any use of l'Hôpital's Rule.

21. $\lim_{x\to 0^+} \sin(x)\cdot\ln(x)$

22. $\lim_{x\to\infty} x^3 e^{-x}$

23. $\lim_{x\to 0^+} \sqrt{x}\cdot\ln(x)$

24. $\lim_{x\to 0^+} x^{\sin(x)}$

25. $\lim_{x\to\infty} \left(1-\frac{3}{x^2}\right)^x$

26. $\lim_{x\to 0} (1-\cos(3x))^x$

27. $\lim_{x\to 0} \left(\frac{1}{x}-\frac{1}{\sin(x)}\right)$

28. $\lim_{x\to\infty} [x-\ln(x)]$

29. $\lim_{x\to\infty} \left(\frac{x+5}{x}\right)^{\frac{1}{x}}$

30. $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{\frac{2}{x}}$

3.7 Practice Answers

1. Both numerator and denominator in the first limit are differentiable and both equal 0 when $x=0$, so we apply l'Hôpital's Rule:

$$\lim_{x\to 0} \frac{1-\cos(5x)}{3x} = \lim_{x\to 0} \frac{5\sin(5x)}{3} = \frac{0}{3} = 0$$

Both numerator and denominator in the second limit are differentiable and both equal 0 when $x=0$, so we apply l'Hôpital's Rule:

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2+2x-8} = \lim_{x\to 2} \frac{2x+1}{2x+2} = \frac{5}{6}$$

2. Both numerator and denominator are differentiable and both become arbitrarily large as $x\to\infty$, so we apply l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{x^2+e^x}{x^3+8x} = \lim_{x\to\infty} \frac{2x+e^x}{3x^2+8} = \lim_{x\to\infty} \frac{2+e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty$$

Note that we needed to apply l'Hôpital's Rule three times and that each stage involved an "$\frac{\infty}{\infty}$" indeterminate form.

3. Comparing A with e^n operations to B with $100\cdot\ln(n)$ operations we can apply l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{e^n}{100\ln(n)} = \lim_{n\to\infty} \frac{e^n}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n\cdot e^n}{100} = \infty$$

to show that B requires fewer operations than A.

Comparing B with $100\ln(n)$ operations to C with n^5 operations, we again apply l'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{100\ln(n)}{n^5} = \lim_{n\to\infty} \frac{\frac{100}{n}}{5n^4} = \lim_{n\to\infty} \frac{20}{n^5} = 0$$

to show that B requires fewer operations than C. So B requires the fewest operations of the three algorithms.

Comparing A to C we must apply l'Hôpital's Rule repeatedly:

$$\lim_{n\to\infty} \frac{e^n}{n^5} = \lim_{n\to\infty} \frac{e^n}{5n^4} = \lim_{n\to\infty} \frac{e^n}{20n^3} = \lim_{n\to\infty} \frac{e^n}{60n^2}$$
$$= \lim_{n\to\infty} \frac{e^n}{120n} = \lim_{n\to\infty} \frac{e^n}{120} = \infty$$

So A requires more operations than C and thus A requires the most operations of the three algorithms.