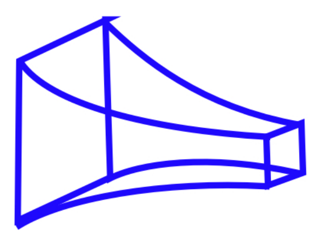
5
Applications of Definite Integrals

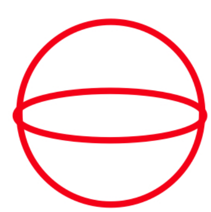
The previous chapter introduced the concepts of a definite integral as an "area" and as a limit of Riemann sums, demonstrated some of the properties of integrals, introduced some methods to compute values of definite integrals, and began to examine a few of their uses. This chapter focuses on several common applications of definite integrals.

An obvious goal of the chapter is to enable you to use integration when you encounter these particular applications later in mathematics or in other fields. A deeper goal is to illustrate the process of going from a problem to an integral, a process much broader than these particular applications. If you understand the process, then you can understand the use of integrals in many other fields and can even develop the integrals needed to solve problems in new areas. Another goal is to give you additional practice evaluating definite integrals.

Each section in this chapter follows the same basic format. First we describe a problem and present some background information. Then we approximate the solution to the basic problem using a Riemann sum. An exact answer comes from taking a limit of the Riemann sum, and we get a definite integral. After looking at several examples of the same basic application, we will examine some variations.



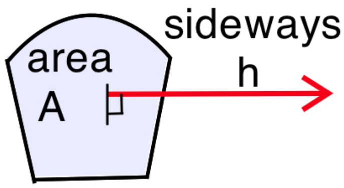
5.1 Volumes by Slicing



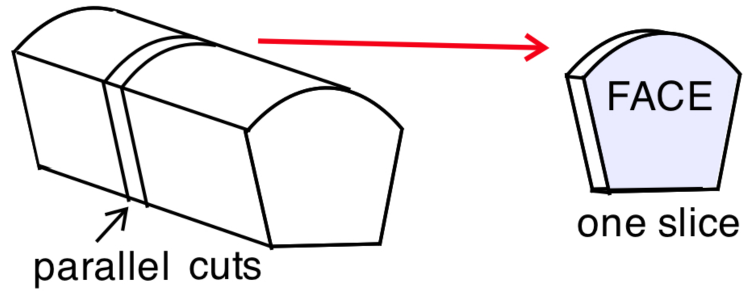
The previous chapter emphasized a geometric interpretation of definite integrals as "areas" in two dimensions. This section emphasizes another geometrical use of integration, computing volumes of solid three-dimensional objects such as those shown in the margin.

Our basic approach will involve cutting the whole solid into thin "slices" whose volumes we can approximate, adding the volumes of these "slices" together (to get a Riemann sum), and finally obtaining an exact answer by taking a limit of those sums to get a definite integral.

The Building Blocks: Right Solids



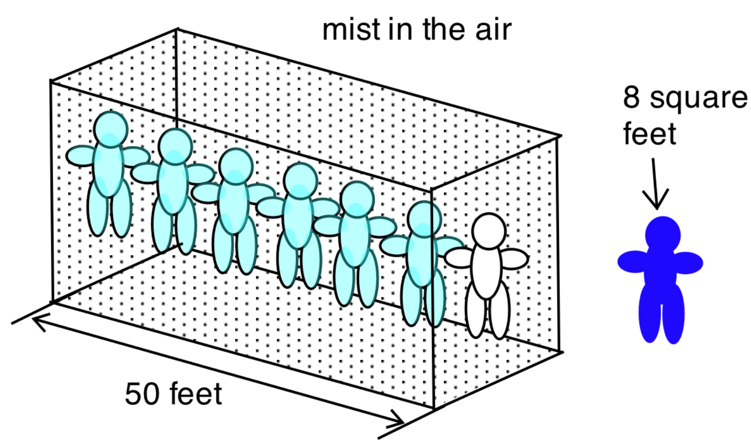
A **right solid** is a three-dimensional shape swept out by moving a planar region A some distance h along a line perpendicular to the plane of A (see margin). We call the region A a **face** of the solid and use the word "right" to indicate that the movement occurs along a line perpendicular—at a right angle—to the plane of A. Two parallel cuts produce one slice with two faces:



A slice has volume, and a face has area.

Example 1. Suppose a fine, uniform mist is suspended in the air and that every cubic foot of mist contains 0.02 ounces of water droplets. If you run 50 feet in a straight line through this mist, how wet do you get? Assume that the front (or a cross section) of your body has an area of 8 square feet.

Solution. As you run, the front of your body sweeps out a "tunnel" through the mist:

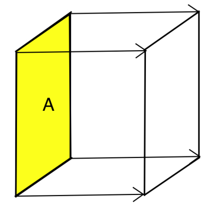


The volume of the "tunnel" is the area of the front of your body multiplied by the length of the tunnel:

$$\text{volume} = \left(8 \text{ ft}^2\right)(50 \text{ ft}) = 400 \text{ ft}^3$$

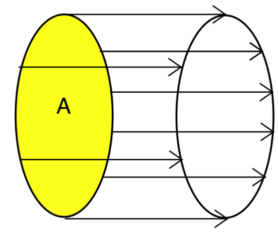
Because each cubic foot of mist held 0.02 ounces of water (which is now on you), you swept out a total of $\left(400 \text{ ft}^3\right)\left(0.02 \dfrac{\text{oz}}{\text{ft}^3}\right) = 8$ ounces of water. If the water were truly suspended and not falling, would it matter how fast you ran? ◀

If A is a rectangle, then the "right solid" formed by moving A along a line (see margin) is a 3-dimensional solid box B. The volume of B is:



$$(\text{area of } A)\,(\text{distance along the line}) = (\text{base})\,(\text{height})\,(\text{width})$$

If A is a circle with radius r meters (see margin), then the "right solid" formed by moving A along a line a distance of h meters is a right circular cylinder with volume equal to:

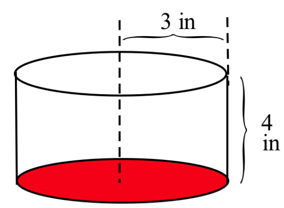


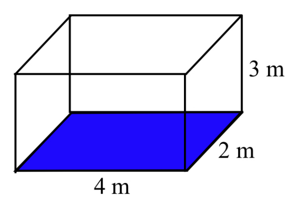
$$(\text{area of } A)\,(\text{distance along the line}) = \left[\pi\,(r\text{ ft})^2\right]\cdot[h\text{ ft}] = \pi r^2 h\text{ ft}^3$$

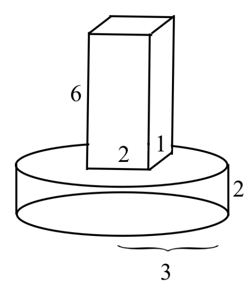
If we cut a right solid perpendicular to its axis (like slicing a block of cheese), then each face (cross-section) has the same two-dimensional shape and area. In general, if a 3-dimensional right solid B is formed by moving a 2-dimensional shape A along a line perpendicular to A, then the **volume** of B is defined to be:

$$(\text{area of } A)\cdot(\text{distance moved along the line perpendicular to } A)$$

Example 2. Calculate the volumes of the right solids in the margin.





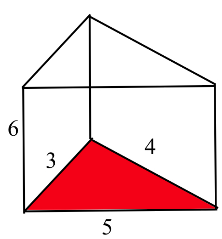


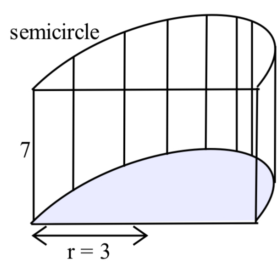
Solution. The cylinder is formed by moving the circular base with cross-sectional area $\pi r^2 = 9\pi\text{ in}^2$ a distance of 4 inches along a line perpendicular to the base, so the volume is $\left(9\pi\text{ in}^2\right)\cdot(4\text{ in}) = 36\pi\text{ in}^3$.

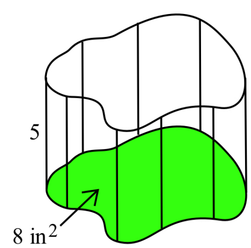
The volume of the box is $(\text{base area})\cdot(\text{distance base is moved}) = (8\text{ m}^2)\cdot(3\text{ m}) = 24\text{ m}^3$. We can also simply multiply "length times width times height" to get the same answer.

The last shape consists of two "easy" right solids with volumes $V_1 = \left(\pi\cdot 3^2\right)\cdot(2) = 18\pi\text{ cm}^3$ and $V_2 = (6)(1)(2) = 12\text{ cm}^3$, so the total volume is $(18\pi+12)\text{ cm}^3 \approx 68.5\text{ cm}^3$. ◀

Practice 1. Calculate the volumes of the right solids shown below.

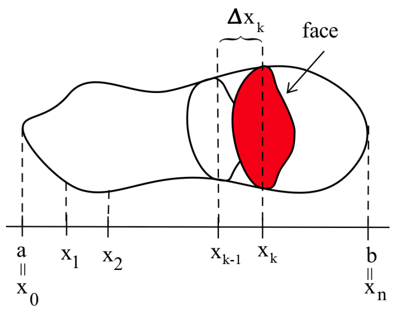






Volumes of General Solids

We can cut a general solid into "slices," each of which is "almost" a right solid if the cuts are close together. The volume of each slice will then be approximately equal to the volume of a right solid, so we can approximate the total volume of the entire solid by adding up the approximate volumes of the right-solid "slices."



First we position an x-axis below the solid shape (see margin) and let $A(t)$ be the area of the face formed when we cut the solid perpendicular to the x-axis where $x = t$. If $\mathcal{P} = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ is a partition of $[a,b]$ and we cut the solid at each x_k, then each slice of the solid is "almost" a right solid and the volume of each slice is approximately

$$(\text{area of a face of the slice})\,(\text{thickness of the slice}) \approx A\,(x_k) \cdot \Delta x_k$$

The total volume V of the solid is approximately equal to the sum of the volumes of the slices:

$$V = \sum (\text{volume of each slice}) \approx \sum A\,(x_k) \cdot \Delta x_k$$

which is a Riemann sum.

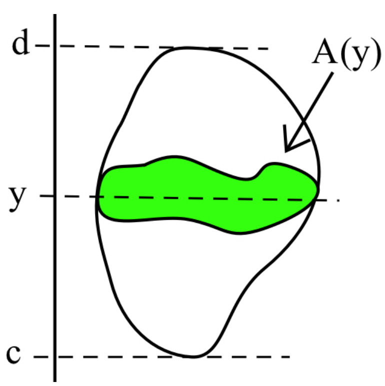
The limit, as the mesh of the partitions approaches 0 (taking thinner and thinner slices), of the Riemann sum is the definite integral of $A(x)$:

$$V \approx \sum A\,(x_k) \cdot \Delta x_k \longrightarrow \int_a^b A(x)\,dx$$

Volume by Slices Formula

If	S is a solid and $A(x)$ is the area of the face formed by a cut at x made perpendicular to the x-axis
then	the volume V of the part of S sitting above $[a,b]$ is:

$$V = \int_a^b A(x)\,dx$$

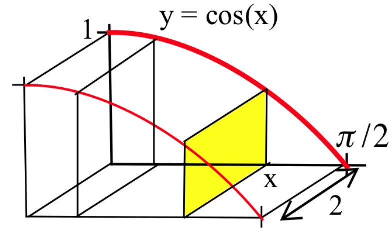


If S is a solid (see margin), and $A(y)$ is the area of a face formed by a cut at y perpendicular to the y-axis, then the volume of a slice with thickness Δy_k is approximately $A\,(y_k) \cdot \Delta y_k$. The volume of the part of S between cuts at $y = c$ and $y = d$ on the y-axis is therefore:

$$V = \int_c^d A(y)\,dy$$

Whether you slice a region with cuts perpendicular to the x-axis or cuts perpendicular to the y-axis depends on which slicing method results in slices with cross-sectional areas that are easiest to compute. Furthermore, slicing one way may result in a definite integral that is difficult to compute, while slicing the other way may result in a much easier definite integral (although you often can't tell which method will result in an easier integration process until you actually set up the integrals).

Example 3. For the solid shown in the margin, the cross-section formed by a cut at x is a rectangle with a base of 2 inches. (a) Find a formula for the approximate volume of the slice between x_{k-1} and x_k. (b) Compute the volume of the solid for x between 0 and $\frac{\pi}{2}$.



Solution. (a) The volume of a "slice" is approximately:

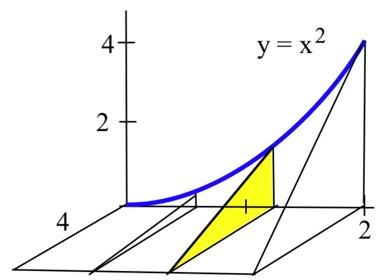
$$\begin{aligned}(\text{area of the face}) \cdot (\text{thickness}) &= (\text{base}) \cdot (\text{height}) \cdot (\text{thickness}) \\ &= (2 \text{ in})\,(\cos(x_k) \text{ in}) \cdot (\Delta x_k \text{ in}) \\ &= 2\cos(x_k)\Delta x_k \text{ in}^3\end{aligned}$$

(b) If we cut the solid into n slices of equal thickness Δx and add up the approximate volumes of the slices, we get a Riemann sum

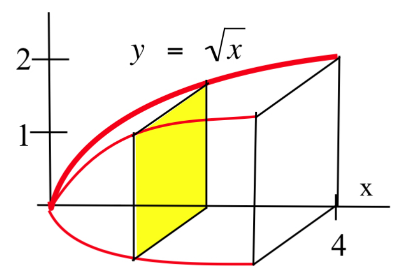
$$\sum_{k=1}^{n} 2\cos(x_k)\Delta x \longrightarrow \int_0^{\frac{\pi}{2}} 2\cos(x)\,dx = 2\sin(x)\Big|_0^{\frac{\pi}{2}} = 2$$

so the volume of the solid is 2 in^3. ◀

Practice 2. For the solid shown in the margin, the face formed by a cut at x is a triangle with a base of 4 inches. (a) Find a formula for the approximate volume of the slice between x_{k-1} and x_k. (b) Use a definite integral to compute the volume of the solid for x between 1 and 2.



Example 4. For the solid shown in margin, each face formed by a cut at x is a square. Compute the volume of the solid.



Solution. The volume of a "slice" is approximately:

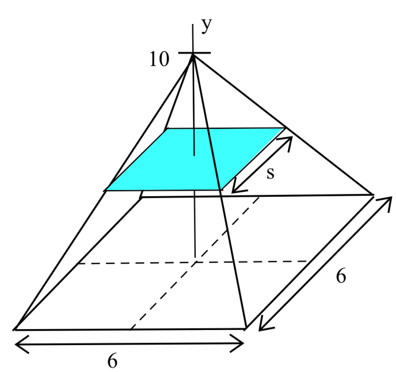
$$\begin{aligned}(\text{area of the face}) \cdot (\text{thickness}) &= (\text{base})^2 \cdot (\text{thickness}) \\ &= (\sqrt{x_k})^2 \cdot \Delta x_k = x_k \cdot \Delta x_k\end{aligned}$$

Adding up the approximate volumes of n slices, we get a Riemann sum that approximates the volume of the entire solid:

$$\sum_{k=1}^{n} x_k \cdot \Delta x_k \longrightarrow \int_0^4 x\,dx = \frac{1}{2}x^2\Big|_0^4 = 8$$

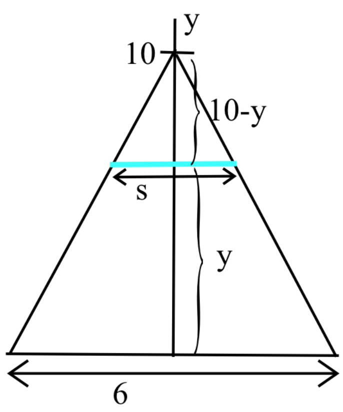
so the volume of the solid is 8. ◀

Example 5. Find the volume of the square-based pyramid shown in the margin.



Solution. Each cut perpendicular to the y-axis yields a square face, but in order to find the area of each square we need a formula for the length of one side s of the square as a function of y, the location of the cut. Using similar triangles (see margin), we know that:

$$\frac{s}{10-y} = \frac{6}{10} \quad \Rightarrow \quad s = \frac{6}{10}(10-y) = 6 - \frac{3}{5}y$$



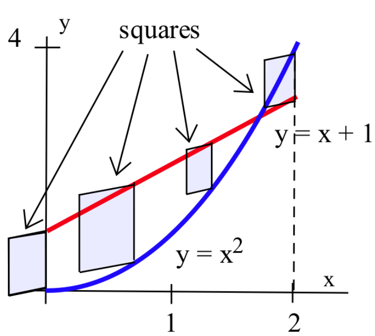
The rest of the solution is straightforward:

$$A(y) = (\text{side})^2 = \left[\frac{3}{5}(10-y)\right]^2 = \frac{9}{25}\left(100 - 20y + y^2\right)$$

so the volume of the solid is:

$$\begin{aligned} V &= \int_0^{10} A(y)\,dy = \int_0^{10} \frac{9}{25}\left(100-20y+y^2\right)\,dy \\ &= \frac{9}{25}\left[100y - 10y^2 + \frac{1}{3}y^3\right]_0^{10} \\ &= \frac{9}{25}\left[\left(1000 - 1000 + \frac{1000}{3}\right) - (0-0+0)\right] = 120 \end{aligned}$$

You may recall from geometry that the formula for the volume of a pyramid is $\frac{1}{3}Bh$ where B is the area of the base, which yields the same result as the definite integral: $\frac{1}{3}\left(6^2\right)(10) = 120$. ◀



Example 6. Form a solid with a base that is the region between the graphs of $f(x) = x+1$ and $g(x) = x^2$ for $0 \le x \le 2$ by building squares with heights (sides) equal to the vertical distance between the graphs of f and g (see margin). Find the volume of this solid.

Solution. The area of a square face is $A(x) = (\text{side})^2$ and the length of a side is either $f(x) - g(x)$ or $g(x) - f(x)$, depending on whether $f(x) \ge g(x)$ or $g(x) \ge f(x)$. We can express this side length as $|f(x) - g(x)|$ but the side length is squared in the area formula, so $A(x) = |f(x)-g(x)|^2 = (f(x)-g(x))^2$. Then:

$$\begin{aligned} V &= \int_a^b A(x)\,dx = \int_0^2 (f(x)-g(x))^2\,dx = \int_0^2 \left[(x+1) - x^2\right]^2\,dx \\ &= \int_0^2 \left[1 + 2x - x^2 - 2x^3 + x^4\right]\,dx \\ &= \left[x + x^2 - \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^2 \end{aligned}$$

which results in a volume of $\frac{26}{15}$. ◀

Wrap-Up

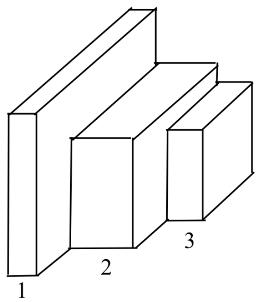
At first, all of these volumes may seem overwhelming—there are so many possible solids and formulas and different cases. If you concentrate on the differences, things can indeed seem very complicated.

Instead, focus on the pattern of cutting, finding areas of faces, volumes of slices, and adding those volumes. Then reason your way to a definite integral. Try to make cuts so the resulting faces have regular shapes (rectangles, triangles, circles) whose areas you can calculate easily. Try not to let the complexity of the whole solid confuse you. Sketch the shape of one face and label its dimensions. If you can find the area of one face in the middle of the solid, you can usually find the pattern for all of the faces — and then you can easily set up the integral.

5.1 Problems

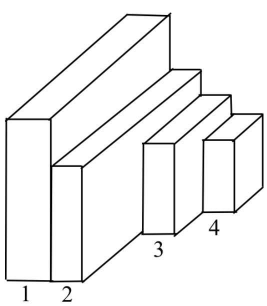
In Problems 1–5, compute the volume of the solid using the values provided in the table.

1.



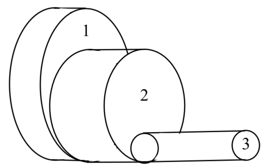
box	base	height	width
1	8	6	1
2	6	4	2
3	3	3	1

2.



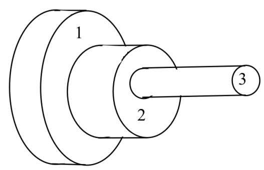
box	base	height	width
1	8	6	1
2	8	4	2
3	4	3	2
4	2	2	1

3.



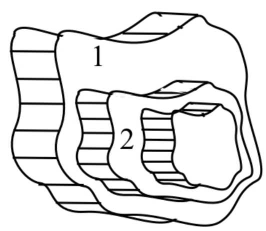
disk	radius	width
1	4	0.5
2	3	1.0
3	1	2.0

4.



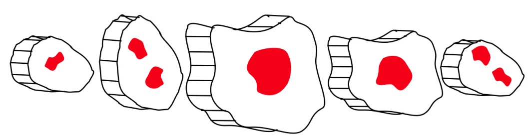
disk	diameter	width
1	8	0.5
2	6	1.0
3	2	2.0

5.



slice	face area	width
1	9	0.2
2	6	0.2
3	2	0.2

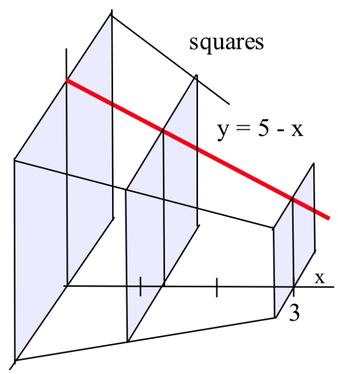
6. Five rock slices are embedded with mineral deposits. Use the information in the table to estimate the total rock volume.



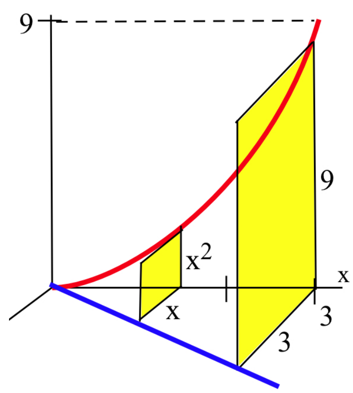
slice	face area	min. area	width
1	4	1	0.6
2	12	2	0.6
3	20	4	0.6
4	10	3	0.6
5	8	2	0.6

In Problems 7–12, represent the volume of each solid as a definite integral, then evaluate the integral.

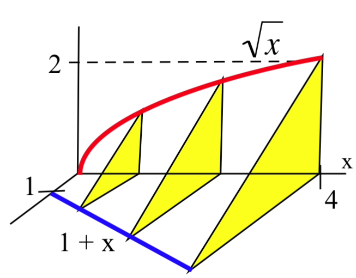
7. For $0 \le x \le 3$, each face is a square with height $5 - x$ inches.



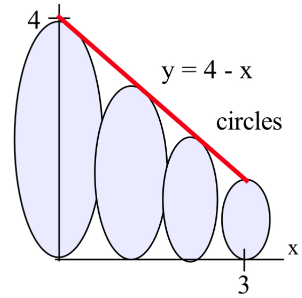
8. For $0 \le x \le 3$, each face is a rectangle with base x inches and height x^2 inches.



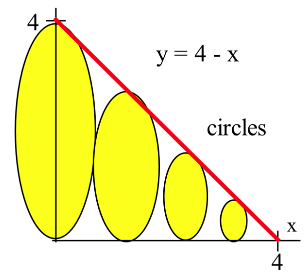
9. For $0 \le x \le 4$, each face is a triangle with base $x + 1$ m and height $\sqrt{x}$ m.



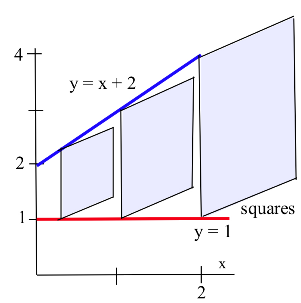
10. For $0 \le x \le 3$, each face is a circle with height (diameter) $4 - x$ m.



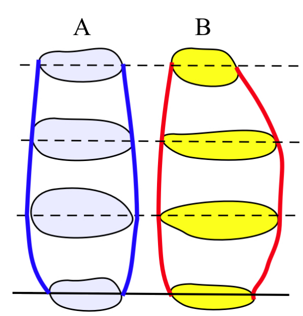
11. For $0 \le x \le 4$, each face is a circle with height (diameter) $4 - x$ m.



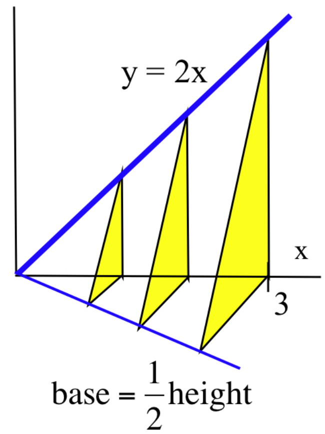
12. For $0 \le x \le 2$, each face is a square with a side extending from $y = 1$ to $y = x + 2$.

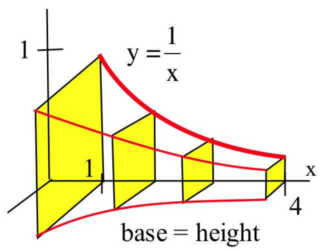


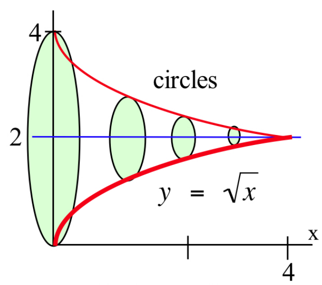
13. Suppose A and B are solids (see below) so that every horizontal cut produces faces of A and B that have equal areas. What can we conclude about the volumes of A and B? Justify your answer.

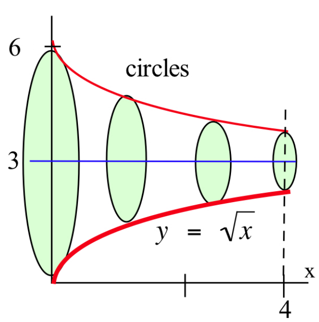


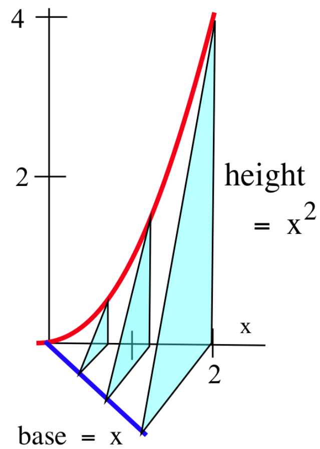
In 14–18, represent the volume of each solid as a definite integral, then evaluate the integral.

14. 

15. 

16. 

17. 

18. 

In 19–28, represent the volume of each solid as a definite integral, then evaluate the integral.

19. The base of a solid is the region between one arch of the curve $y = \sin(x)$ and the x-axis, and cross-sections ("slices") of the solid perpendicular to the base (and to the x-axis) are squares.

20. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis and the curve $y = \cos(x)$, and cross-sections ("slices") of the solid perpendicular to the base (and to the x-axis) are squares.

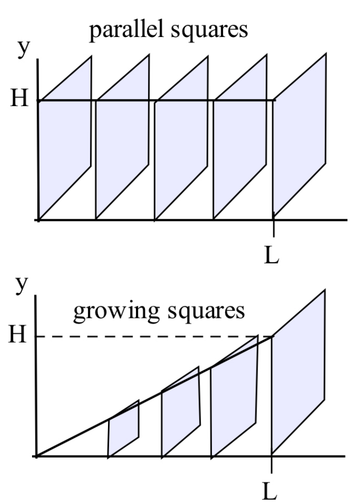
21. The base of a solid is the region in the first quadrant bounded by the x-axis, the y-axis and the curve $y = \cos(x)$, and slices perpendicular to the base (and to the x-axis) are semicircles.

22. The base of a solid is the region between one arch of the curve $y = \sin(x)$ and the x-axis, and slices perpendicular to the base (and to the x-axis) are equilateral triangles.

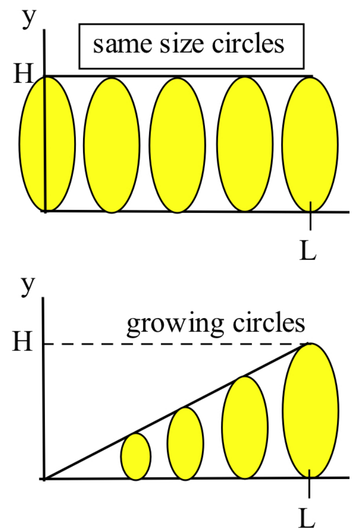
23. The base of a solid is the region bounded by the parabolas $y = x^2$ and $y = 3 + x - x^2$, and slices perpendicular to the base (and to the x-axis) are:

 (a) squares.
 (b) semicircles.
 (c) rectangles twice as tall as they are wide.
 (d) isosceles right triangles with a hypotenuse in the base of the solid.

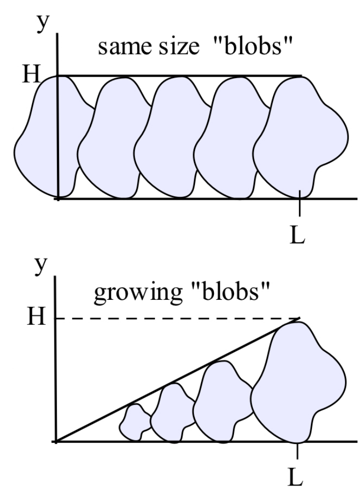
24. The base of a solid is the first-quadrant region bounded by the y-axis, the curve $y = \sin(x)$ and the curve $y = \cos(x)$, and slices perpendicular to the base (and to the x-axis) are:
    (a) squares.
    (b) semicircles.
    (c) rectangles twice as tall as they are wide.
    (d) isosceles right triangles with a hypotenuse in the base of the solid.
25. The base of a solid is the region bounded by the x-axis, the y-axis and the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are squares.
26. The base of a solid is the region bounded by the x-axis, the line $y = 3$ and the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are squares.
27. The base of a solid is the region bounded below by the line $y = 1$, on the left by the line $x = 2$ and above by the parabola $y = 8 - x^2$, and slices perpendicular to the base (and to the y-axis) are semicircles.
28. The base of a solid is the region bounded below by $y = 1$, on the left by $x = 2$ and above by $y = 8 - x^2$, and slices perpendicular to the base (and to the x-axis) are semicircles.
29. Calculate (a) the volume of the right solid in the top figure (b) the volume of the "right cone" in the bottom figure and (c) the ratio of the "right cone" volume to the right solid volume.



30. Calculate (a) the volume of the right solid in the top figure (b) the volume of the "right cone" in the bottom figure and (c) the ratio of the "right cone" volume to the right solid volume.



31. Calculate (a) the volume of the right solid in the top figure if each "blob" has area B (b) the volume of the "right cone" in the bottom figure, using "similar blobs" to conclude that the cross-section x units from the y-axis has area $A(x) = \frac{B}{L^2}x^2$ and (c) the ratio of the "right cone" volume to the right solid volume.



32. "Personal calculus": Describe a practical way to determine the volume of your hand and arm up to the elbow.

5.1 Practice Answers

1. triangular base: $V = (\text{base area}) \cdot (\text{height}) = \left(\frac{1}{2} \cdot 3 \cdot 4\right)(6) = 36$

   semicircular base: $V = (\text{base area}) \cdot (\text{height}) = \left(\frac{1}{2}\pi \cdot 3^2\right)(7) \approx 98.96$

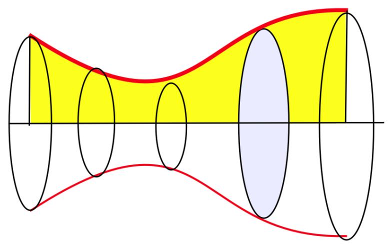
   "blob"-shaped base: $V = (\text{base area}) \cdot (\text{height}) = (8)(5) = 40 \text{ in}^3$

2. (a) The base of each triangular slice is 4 and the height is approximately ${x_k}^2$ so $A(x_k) \approx \frac{1}{2}(4)\left({x_k}^2\right) = 2{x_k}^2$ and the volume of the k-th slice is this approximately $2{x_k}^2 \cdot \Delta x_k$.

   (b) Adding up the approximate volumes of all n slices yields $\sum_{n=1}^{\infty} 2{x_k}^2 \cdot \Delta x_k$, which is a Riemann sum with limit:

$$\int_0^2 2x^2\,dx = \frac{2}{3}x^3\Big|_1^2 = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

5.2 *Volumes: Disks and Washers*

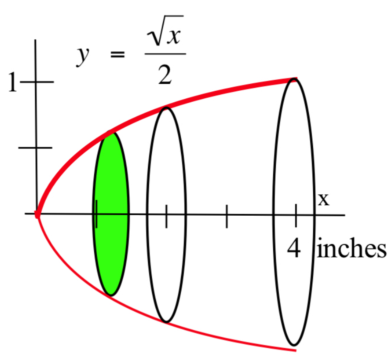


In the previous section, we computed volumes of solids for which we could determine the area of a cross-section or "slice." In this section, we restrict our attention to a special case in which the solid is generated by *rotating* a region in the xy-plane about a horizontal or vertical line. We call a solid formed in this way a **solid of revolution** and we call the line an **axis of rotation**.

If the axis of rotation coincides with a boundary of the region (as in the margin figure) then the cross-sections of the region perpendicular to the axis of rotation will be disks, making it relatively easy to find a formula for the area of a cross-section:

$$A(x) = \text{area of a disk} = \pi(\text{radius})^2$$

The radius is often a function of x, the location of the cross-section.



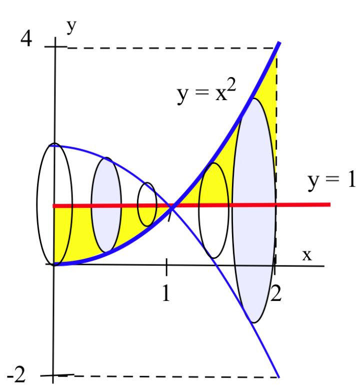
Example 1. Find the volume of the solid (shown in the margin) formed by rotating the region in the first quadrant bounded by the curve $y = \dfrac{\sqrt{x}}{2}$ and the line $x = 4$ about the x-axis.

Solution. Any slice perpendicular to the x-axis (and to the xy-plane) will yield a circular cross-section with radius equal to the distance between the curve $y = \dfrac{\sqrt{x}}{2}$ and the x-axis, so the volume of the region is given by:

$$V = \int_0^4 \pi \left[\frac{\sqrt{x}}{2}\right]^2 dx = \int_0^4 \pi \cdot \frac{x}{4}\, dx = \frac{\pi}{8}x^2\bigg|_0^4 = 2\pi$$

or about 6.28 cubic inches. ◀

Sometimes the boundary curve intersects the axis of rotation.



Example 2. The region between the graph of $f(x) = x^2$ and the horizontal line $y = 1$ for $0 \le x \le 2$ is revolved about the horizontal line $y = 1$ to form a solid (see margin). Compute the volume of the solid.

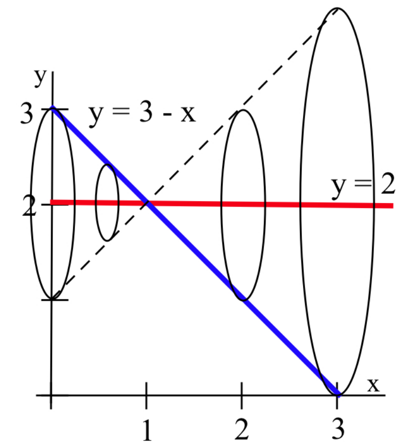
Solution. The margin figure shows cross-sections for several values of x, all of them disks. If $0 \le x \le 1$, then the radius of the disk is $r(x) = 1 - x^2$; if $1 \le x \le 2$, then $r(x) = x^2 - 1$. We could split up the volume computation into two separate integrals, using $A(x) = \pi\,[r(x)]^2 = \pi\left[1 - x^2\right]^2$ for $0 \le x \le 1$ and $A(x) = \pi\,[r(x)]^2 = \pi\left[x^2 - 1\right]^2$ for $1 \le x \le 2$, but:

$$\pi\left[x^2 - 1\right]^2 = \pi\left[-(1 - x^2)\right]^2 = \pi\left[1 - x^2\right]^2$$

for all x so we can instead compute the volume with a single integral:

$$V = \int_0^2 \pi\left[x^2 - 1\right]^2 dx = \pi \int_0^2 \left[x^4 - 2x^2 + 1\right] dx$$
$$= \pi\left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_0^2 = \pi\left[\frac{32}{5} - \frac{16}{3} + 2\right] = \frac{46\pi}{15}$$

or about 9.63. ◀



Practice 1. Find the volume of the solid formed by revolving the region between $f(x) = 3 - x$ and the horizontal line $y = 2$ about the line $y = 2$ for $0 \le x \le 3$ (see margin).

Volumes of Revolved Regions ("Disk Method")

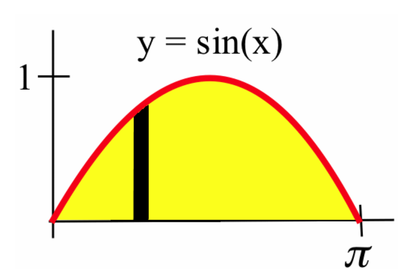
If	the region constrained by the graph of $y = f(x)$, the horizontal line $y = L$ and the interval $[a, b]$ is revolved about the horizontal line $y = L$
then	then the volume of the resulting solid is:

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\,(\text{radius})^2\,dx = \int_a^b \pi\,[f(x) - L]^2\,dx$$

We often refer to this technique as the "disk" method because revolving a thin rectangular slice of the region (that we might use in a Riemann sum to approximate the area of the region) results in a disk. If the region between the graph of f and the x-axis ($L = 0$) is revolved about the x-axis, then the previous formula reduces to:

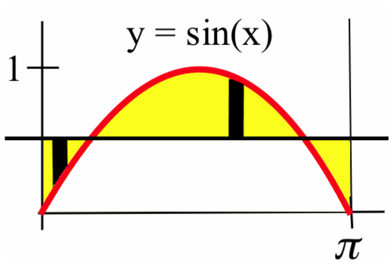
$$V = \int_a^b \pi\,[f(x)]^2\,dx$$

Example 3. Find the volume generated when the region between one arch of the sine curve (for $0 \le x \le \pi$) and (a) the x-axis is revolved about the x-axis and (b) the line $y = \frac{1}{2}$ is revolved about the line $y = \frac{1}{2}$.



Solution. (a) The radius of each circular slice (see margin) is just the height of the function $y = \sin(x)$:

$$V = \int_0^\pi \pi\,[\sin(x)]^2\,dx = \pi\int_0^\pi \sin^2(x)\,dx = \pi\int_0^\pi \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right] dx$$
$$= \pi\left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]_0^\pi = \pi\left[\frac{\pi}{2} - 0\right] - \pi\,[0 - 0] = \frac{\pi^2}{2} \approx 4.93$$

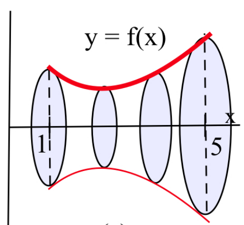


(b) Here we use the more general disk formula with $L = \frac{1}{2}$:

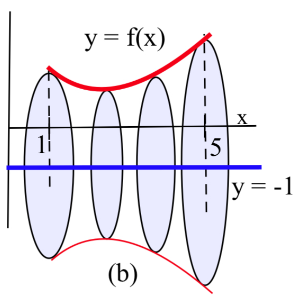
$$V = \int_0^{\pi} \pi \left[\sin(x) - \frac{1}{2}\right]^2 dx = \pi \int_0^{\pi} \left[\sin^2(x) - \sin(x) + \frac{1}{4}\right] dx$$
$$= \pi \int_0^{\pi} \left[\frac{1}{2} - \frac{1}{2}\cos(2x) - \sin(x) + \frac{1}{4}\right] dx$$
$$= \pi \left[\frac{3}{4}x - \frac{1}{4}\sin(2x) + \cos(x)\right]_0^{\pi}$$
$$= \pi \left[\frac{3\pi}{4} - 0 - 1\right] - \pi\,[0 - 0 + 1] = \frac{3\pi^2}{4} - 2\pi$$

or approximately 1.12. ◀

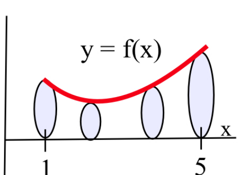
Practice 2. Find the volume generated when (a) the region between the parabola $y = x^2$ (for $0 \le x \le 2$) and the x-axis is revolved about the x-axis and (b) the region between the parabola $y = x^2$ (for $0 \le x \le 2$) and the line $y = 2$ is revolved about the line $y = 2$.



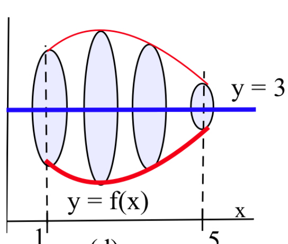
(a)



(b)



(c)



(d)

Example 4. Given that $\int_1^5 f(x)\,dx = 4$ and $\int_1^5 [f(x)]^2\,dx = 7$, represent the volume of each solid shown in the margin as a definite integral, and evaluate those integrals.

Solution. (a) Here the axis of rotation is $y = 0$ so:

$$V = \int_1^5 \pi\,(\text{radius})^2\,dx = \int_1^5 \pi\,[f(x)]^2\,dx = \pi \int_1^5 [f(x)]^2\,dx = 7\pi$$

(b) Here the axis of rotation is $y = -1$ so:

$$V = \int_1^5 \pi\,(\text{radius})^2\,dx = \int_1^5 \pi\,[f(x) - (-1)]^2\,dx$$
$$= \pi \int_1^5 [f(x) + 1]^2\,dx = \pi \int_1^5 \left[(f(x))^2 + 2f(x) + 1\right] dx$$
$$= \pi \left[\int_1^5 (f(x))^2\,dx + 2\int_1^5 f(x)\,dx + \int_1^5 1\,dx\right]$$
$$= \pi\,[7 + 2 \cdot 4 + (5 - 1)] = 19\pi$$

(c) This is not a solid of revolution, even though the cross-sections are disks. Each disk has diameter equal to the function height, so the radius of each disk is half that height, and the volume is:

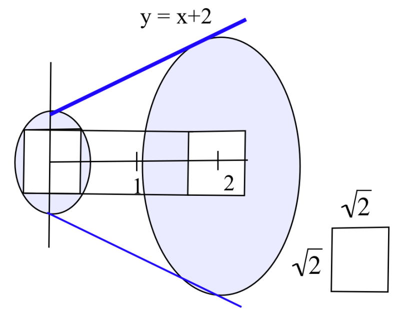
$$V = \int_1^5 \pi \left[\frac{f(x)}{2}\right]^2 dx = \frac{\pi}{4}\int_1^5 [f(x)]^2\,dx = \frac{\pi}{4} \cdot 7 = \frac{7\pi}{4}$$

The last one is left for you. ◀

Practice 3. Set up and evaluate an integral to compute the volume of the last solid shown in the margin.

Solids with Holes

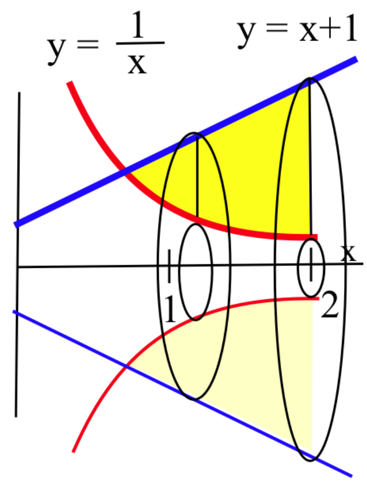
Some solids have "holes": for example, we might drill a cylindrical hole through a spherical solid (such as a ball bearing) to create a part for an engine. One approach involves using an integral (or using geometry) to compute the volume of the "outer" solid, then use another integral (or geometry) to compute the volume of the "hole" cut out of the original solid, and finally subtracting the second result from the first. You should be able to use this approach in the next problem.



Practice 4. Compute the volume of the solid shown in the margin.

A special case of a solid with a hole results from rotating a region bounded by two curves around an axis that does not intersect the region.

Example 5. Compute the volume of the solid shown in the margin.



Solution. The face for a slice made at x has area:

$$\begin{aligned}A(x) &= [\text{area of BIG circle}] - [\text{area of small circle}] \\ &= \pi\,[\text{BIG radius}]^2 - \pi\,[\text{small radius}]^2\end{aligned}$$

Here the BIG radius is the distance from the line $y = x + 1$ to the x-axis, or $R(x) = (x+1) - 0 = x + 1$; similarly, the small radius is the distance from the curve $y = \frac{1}{x}$ to the x-axis, or $r(x) = \frac{1}{x} - 0 = \frac{1}{x}$, hence the cross-sectional area is:

$$A(x) = \pi\,[x+1]^2 - \pi\left[\frac{1}{x}\right]^2 = \pi\left[x^2 + 2x + 1 - \frac{1}{x^2}\right]$$

The curves intersect where:

$$\begin{aligned}x + 1 = \frac{1}{x} \;&\Rightarrow\; x^2 + x = 1 \;\Rightarrow\; x^2 + x - 1 = 0 \\ &\Rightarrow\; x = \frac{-1 \pm \sqrt{1 - 4(-1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}\end{aligned}$$

Clearly we need $x > 0$ for this region, so the left endpoint of integration must be $x = \frac{-1+\sqrt{5}}{2}$ while the right endpoint is $x = 2$, so the volume of the solid is:

$$V = \int_{\frac{-1+\sqrt{5}}{2}}^{2} \pi\left[x^2 + 2x + 1 - \frac{1}{x^2}\right]\,dx = \pi\left[\frac{1}{3}x^3 + x^2 + x + \frac{1}{x}\right]_{\frac{-1+\sqrt{5}}{2}}^{2}$$

$$= \pi\left[\frac{2^3}{3} + 2^2 + 2 + \frac{1}{2}\right] - \pi\left[\frac{1}{3}\left(\frac{-1+\sqrt{5}}{2}\right)^3 + \left(\frac{-1+\sqrt{5}}{2}\right)^2 + \frac{-1+\sqrt{5}}{2} + \frac{2}{-1+\sqrt{5}}\right]$$

which simplifies to $\frac{\pi}{6}\left[50 - 5\sqrt{5}\right] \approx 20.33$. ◀

The previous Example extends the "disk" method to a more general technique often called the "washer" method because a big disk with a smaller disk cut out of the middle resembles a washer (a small flat ring used with nuts and bolts).

Volumes of Revolved Regions ("Washer Method")

If the region constrained by the graphs of $y = f(x)$ and $y = g(x)$ and the interval $[a, b]$ is revolved about a horizontal line

then the volume of the resulting solid is:

$$V = \int_a^b \left[\pi \left(R(x)\right)^2 - \pi \left(r(x)\right)^2 \right] \, dx$$

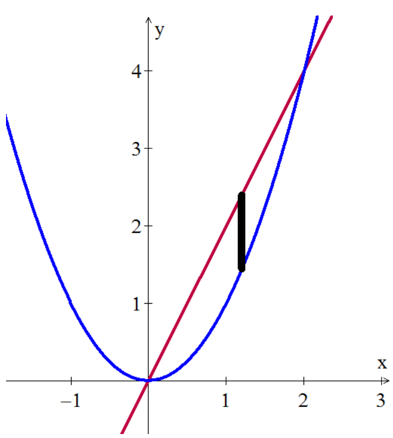
where $R(x)$ represents the distance from the axis of rotation to the farthest curve from that axis, and $r(x)$ represents the distance from the axis to the closest curve.

If $r(x) = 0$, the "washer" method becomes the "disk" method.

When applying the washer method, you should:

- graph the region
- draw a representative rectangular "slice" of that region
- check that revolving the slice about the axis of rotation results in a "washer"
- locate the limits of integration
- set up an integral
- evaluate the integral

If you are unable to find an antiderivative for the integrand of your integral, you can consult an integral table or use numerical methods to approximate the volume of the solid. You might also need to use numerical methods to locate where the boundary curves of the region intersect.



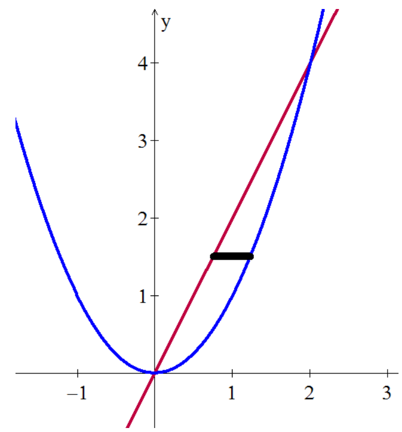
Example 6. Find the volume of the solid generated by rotating the region between the curves $y = 2x$ and $y = x^2$ about the (a) x-axis (b) y-axis (c) the line $x = -1$ (d) the line $y = 5$.

Solution. (a) The curves intersect where $x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0$, so the limits of integration should involve $x = 0$ and $x = 2$. Revolving a vertical slice of the region with width Δx about the x-axis yields a "washer" with big radius $R(x) = 2x - 0 = 2x$ (the line $y = 2x$ is farthest from the x-axis) and small radius $r(x) =$

$x^2 - 0 = x^2$ (the parabola is closest to the x-axis when $0 \le x \le 2$). So the volume of the solid is:

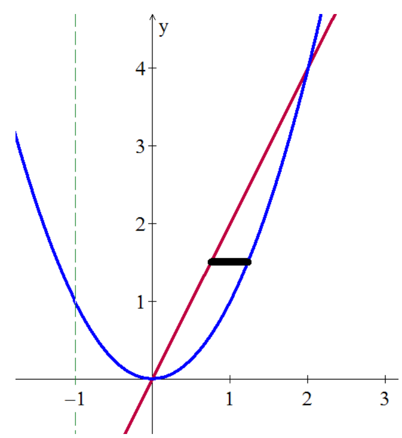
$$V = \int_0^2 \left[\pi(2x)^2 - \pi(x^2)^2\right] dx = \pi \int_0^2 \left[4x^2 - x^4\right] dx$$
$$= \pi \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \pi \left[\frac{32}{3} - \frac{32}{5}\right] - \pi\,[0-0] = \frac{64\pi}{15}$$

(b) A vertical slice revolved around the y-axis does **not** result in a "washer" so instead we try slicing horizontally. A horizontal slice of thickness Δy revolved around the y-axis does result in a washer. The big radius is the x-distance from the parabola (where $x = \sqrt{y}$) to the y-axis (where $x = 0$) so $R(y) = \sqrt{y}$. Similarly, the small radius is the distance from the line (where $x = \frac{y}{2}$) to the y-axis (where $x = 0$), so $r(y) = \frac{y}{2}$. Because the variable of integration is now y, we need y-values for the limits of integration. At the lower intersection point of the two curves, $x = 0 \Rightarrow y = 0$; at the upper intersection point, $x = 2 \Rightarrow y = x^2 = 2^2 = 4$. So the volume of the solid is:



$$V = \int_{y=0}^{y=4} \left[\pi\left(\sqrt{y}\right)^2 - \pi\left(\frac{y}{2}\right)^2\right] dy = \pi \int_0^4 \left[y - \frac{1}{4}y^2\right] dy$$
$$= \pi \left[\frac{1}{2}y^2 - \frac{1}{12}y^3\right]_0^4 = \pi\left[8 - \frac{16}{3}\right] = \frac{8\pi}{3}$$

(c) This solid resembles the one from part (b), except now the radii are both bigger because the region (and the curves that form the boundary of the region) are farther away from the axis of rotation: $R(x) = \sqrt{y} - (-1) = \sqrt{y} + 1$ and $r(x) = \frac{y}{2} - (-1) = \frac{y}{2} + 1$:



$$V = \int_{y=0}^{y=4} \left[\pi\left(\sqrt{y}+1\right)^2 - \pi\left(\frac{y}{2}+1\right)^2\right] dy$$
$$= \pi \int_0^4 \left[\left(y + 2\sqrt{y} + 1\right) - \left(\frac{1}{4}y^2 + y + 1\right)\right] dy$$
$$= \pi \int_0^4 \left[2y^{\frac{1}{2}} - \frac{1}{4}y^2\right] dy = \pi\left[\frac{4}{3}y^{\frac{3}{2}} - \frac{1}{12}y^3\right]_0^4 = \pi\left[\frac{32}{3} - \frac{16}{3}\right] = \frac{16\pi}{3}$$

(d) For this solid, slicing the region vertically as in part (a) results in washers, but here the "near" and "far" roles of the curves are reversed: the parabola is farthest away from $y = 5$ while the line is closest. The radii are $R(x) = 5 - x^2$ and $r(x) = 5 - 2x$:

$$V = \int_{x=0}^{x=2} \pi\left[(5-x^2)^2 - (5-2x)^2\right] dx = \frac{136\pi}{15}$$

The details of evaluating this definite integral are left to you. ◀

Practice 5. Find the volume of the solid generated by rotating the region between the curves $y = 2x$ and $y = x^2$ about the (a) the line $x = 5$ (b) the line $y = -5$.

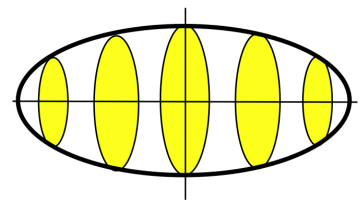
5.2 Problems

In Problems 1–12, find the volume of the solid generated when the region in the first quadrant bounded by the given curves is rotated about the x-axis.

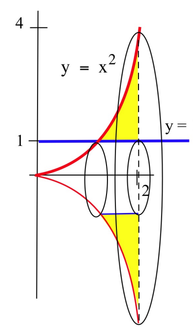
1. $y = x$, $x = 5$
2. $y = \sin(x)$, $x = \pi$
3. $y = \cos(x)$, $x = \frac{\pi}{3}$
4. $y = 3 - x$
5. $y = \sqrt{7 - x}$
6. $y = \sqrt[4]{9 - x}$
7. $y = 5 - x^2$
8. $x = 9 - y^2$
9. $x = 121 - y^2$
10. $x^2 + y^2 = 4$
11. $9x^2 + 25y^2 = 225$
12. $3x^2 + 5y^2 = 15$

In Problems 13–30, compute the volume of the solid formed when the region between the given curves is rotated about the specified axis.

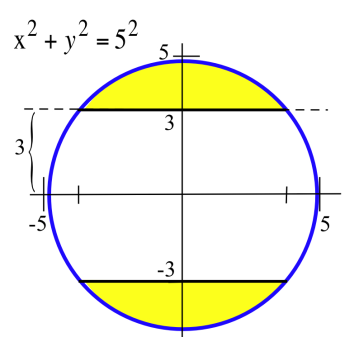
13. $y = x$, $y = x^4$ about the x-axis
14. $y = x$, $y = x^4$ about the y-axis
15. $y = x^2$, $y = x^4$ about the y-axis
16. $y = x^2$, $y = x^4$ about the x-axis
17. $y = x^2$, $y = x^3$ about the x-axis
18. $y = \sec(x)$, $y = 2\cos(x)$, $x = \frac{\pi}{3}$ about the x-axis
19. $y = \sec(x)$, $y = \cos(x)$, $x = \frac{\pi}{3}$ about the x-axis
20. $y = x$, $y = x^4$ about $y = 3$
21. $y = x$, $y = x^4$ about $y = -4$
22. $y = x$, $y = x^4$ about $x = -4$
23. $y = x$, $y = x^4$ about $x = 3$
24. $y = x$, $y = x^4$ about $x = 1$
25. $y = \sin(x)$, $y = x$, $x = 1$ about $y = 3$
26. $y = \sin(x)$, $y = x$, $x = \frac{\pi}{2}$ about $y = -2$
27. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $x = -2$
28. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $x = 4$
29. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $y = 2$
30. $y = \sqrt{x}$, $y = \sqrt[3]{x}$, about $y = -\sqrt{3}$
31. Use calculus to compute the volume of a sphere of radius 2. (A sphere is formed when the region bounded by the x-axis and the top half of the circle $x^2 + y^2 = 2^2$ is revolved about the x-axis.)
32. Use calculus to determine the volume of a sphere of radius r. (Revolve the region bounded by the x-axis and the top half of the circle $x^2 + y^2 = r^2$ about the x-axis.)
33. Compute the volume swept out when the top half of the elliptical region bounded by $\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$ is revolved around the x-axis (see figure below).



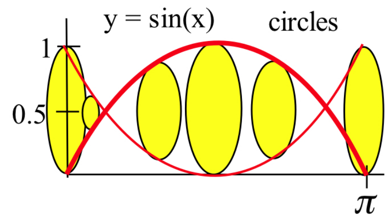
34. Compute the volume swept out when the top half of the elliptical region bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved around the x-axis.
35. Compute the volume of the region shown below.



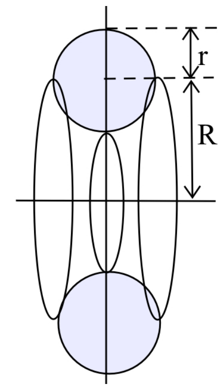
36. Compute the volume of a sphere of radius 5 with a hole of radius 3 drilled through its center.



37. Compute the volume of the region shown in the margin.



38. Determine the volume of the "doughnut" (called a "torus," see lower margin figure) generated by rotating a disk of radius r with center R units away from the x-axis about the x-axis.



39. (a) Find the **area** between $f(x) = \dfrac{1}{x}$ and the x-axis for $1 \le x \le 10$, $1 \le x \le 100$ and $1 \le x \le M$. What is the limit of the area for $1 \le x \le M$ when $M \to \infty$?

(b) Find the **volume** swept out when the region in part (a) is revolved about the x-axis for $1 \le x \le 10$, $1 \le x \le 100$ and $1 \le x \le M$. What is the limit of the volume for $1 \le x \le M$ when $M \to \infty$?

5.2 Practice Answers

1. $\displaystyle\int_0^3 \pi\Big[\,|(3-x)-2)|\,\Big]^2\,dx = \pi\int_0^3 (1-x)^2\,dx = \pi\int_0^3\left[1-2x+x^2\right]\,dx = \pi\left[x - x^2 + \frac{1}{3}x^3\right]_0^3 = 3\pi$

2. (a) Slicing the region vertically and rotating the slice about the x-axis results in disks, so the volume of the solid is:

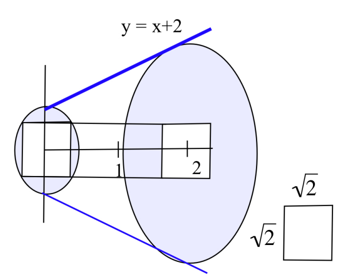
$$\int_0^2 \pi\left[x^2\right]^2\,dx = \pi\int_0^2 x^4\,dx = \pi\left[\frac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}$$

(b) Here the slices extend from $y = x^2$ to $y = 2$ so the radius of each disk is $2 - x^2$ and the volume is:

$$\int_0^2 \pi\left[2-x^2\right]^2\,dx = \pi\int_0^2\left[4-4x^2+x^4\right]\,dx = \pi\left[4x - \frac{4}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \frac{56\pi}{15}$$

3. $$\begin{aligned}\int_1^5 \pi\,[3-f(x)]^2\,dx &= \int_1^5 \pi\left[9-6f(x)+(f(x))^2\right]\,dx\\ &= \pi\left[\int_1^5 9\,dx - 6\int_1^5 f(x)\,dx + \int_1^5 [f(x)]^2\,dx\right] = \pi\,[36-6\cdot 4+7] = 19\pi\end{aligned}$$

4. The volume we want can be obtained by subtracting the volume of the "box" from the volume of the truncated cone generated by the rotated line segment. The volume of the truncated cone is:



$$\int_0^2 \pi\,[x+2]^2\,dx = \pi\int_0^2\left[x^2+4x+4\right]\,dx = \pi\left[\frac{1}{3}x^3 + 2x^2 + 4x\right]_0^2 = \frac{56\pi}{3}$$

while the volume of the box is $\left[\sqrt{2}\right]^2(2) = 4$ so the volume of the solid shown in the graph is $\dfrac{56\pi}{3} - 4 \approx 54.64$.

5. (a) Slicing the region vertically and rotating the slice about the line $x = 5$ results in something other than a washer, so we instead slice the region horizontally. The slice extends from $x = \frac{y}{2}$ (farthest from the axis of rotation) to $x = \sqrt{y}$ (closest), so the volume of the solid is:

$$\int_0^4 \left[\pi\left(5 - \frac{y}{2}\right)^2 - \pi\left(5 - \sqrt{y}\right)^2\right] dy = \frac{32\pi}{3}$$

(b) Slicing the region vertically and rotating the slice about the line $y = -5$ results in washers, so the volume is:

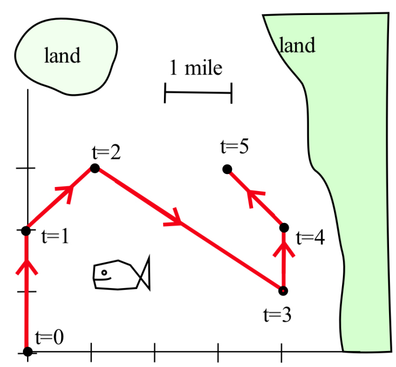
$$\int_0^2 \left[\pi\left(2x + 5\right)^2 - \pi\left(x^2 + 5\right)^2\right] dx = \frac{88\pi}{5}$$

5.3 *Arclength and Surface Area*

This section introduces two additional geometric applications of integration: finding the length of a curve and finding the area of a surface generated when you revolve a curve about a line. The general strategy remains the same: partition the problem into small pieces, approximate the solution on each small piece, add the small solutions together to form a Riemann sum and, finally, take the limit of the Riemann sum to get a definite integral.

Arclength: How Long Is a Curve?

In order to better understand an animal, biologists need to know how it moves through its environment and how far it travels. We need to know the length of the path it moves along. If we know the object's location at successive times, then we can easily calculate the distances between those locations and add them together to get a total (approximate) distance.



Example 1. In order to study the movement of whales, marine biologists implant a small transmitter on selected whales and track the location of a whale via satellite. Position data at one-hour time intervals over a five-hour period appears in the margin figure. How far did the whale swim during the first three hours?

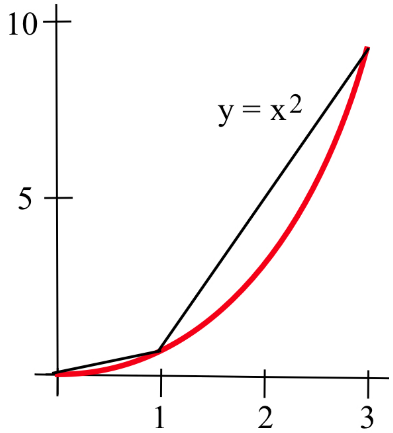
Solution. In moving from the point $(0,0)$ to the point $(0,2)$, the whale traveled *at least* 2 miles. Similarly, the whale traveled at least $\sqrt{(1-0)^2+(3-2)^2} = \sqrt{2} \approx 1.4$ miles during the second hour and at least $\sqrt{(4-1)^2+(1-3)^2} = \sqrt{13} \approx 3.6$ miles during the third hour. The scientist concluded that the whale swam at least $2+1.4+3.6=7$ miles during the three-hour period. ◀

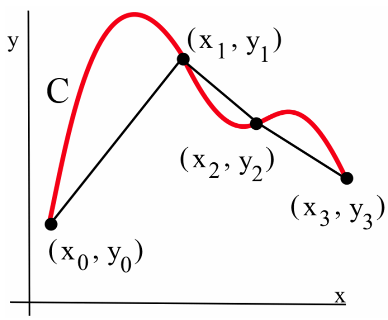
Practice 1. How far did the whale swim during the entire five-hour time period?

It is unlikely that the whale swam in a straight line from location to location, so its actual swimming distance was undoubtedly more than seven miles during the first three hours. Scientists might get better distance estimates by recording the whale's position over shorter, five-minute time intervals.

Our strategy for finding the length of a curve will resemble the one the scientist used, and if the locations are given by a formula, then we can calculate the successive locations over very short intervals and get very good approximations of the total path length.

Example 2. Use the points $(0,0)$, $(1,1)$ and $(3,9)$ to approximate the length of $y = x^2$ for $0 \le x \le 3$.





Solution. The lengths of the two line segments (see margin) are:

$$\sqrt{(1-0)^2+(1-0)^2} = \sqrt{1+1} = \sqrt{2} \approx 1.41$$

and:

$$\sqrt{(3-1)^2+(9-1)^2} = \sqrt{4+64} = \sqrt{68} \approx 8.25$$

so the length of the curve is approximately $1.41 + 8.25 = 9.66$. ◀

Practice 2. Get a better approximation of the length of $y = x^2$ for $0 \le x \le 3$ by using the points $(0,0)$, $(1,1)$, $(2,4)$ and $(3,9)$. Is your approximation longer or shorter than the actual length?

For a curve $\mathcal{C}$ (see margin), pick some points (x_k, y_k) along $\mathcal{C}$ and connect those points with line segments. Then the sum of the lengths of the line segments will approximate the length of $\mathcal{C}$. We can think of this as pinning a string to the curve at the selected points, and then measuring the length of the string as an approximation of the length of the curve. Of course, if we only pick a few points (as in the margin), then the total length approximation will probably be rather poor, so eventually we want lots of points (x_k, y_k) close together all along $\mathcal{C}$.

Label these points so that (x_0, y_0) is one endpoint of $\mathcal{C}$ and (x_n, y_n) is the other endpoint, and so that the subscripts increase as we move along $\mathcal{C}$. Then the distance between the successive points (x_{k-1}, y_{k-1}) and (x_k, y_k) is:

$$\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

and the total length of these line segments is simply the sum of the successive lengths:

$$\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

This summation does not have the form $\sum g(c_k) \cdot \Delta x_k$ so it is not a Riemann sum. It is, however, algebraically equivalent to an expression very much like a Riemann sum that will lead us to a definite integral representation for the length of $\mathcal{C}$.

If $\mathcal{C}$ is given by $y = f(x)$ for $a \le x \le b$, so that y is a function of x, we can factor $(\Delta x_k)^2$ from inside the radical and simplify:

$$\text{length of } C \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 \left[1 + \frac{(\Delta y_k)^2}{(\Delta x_k)^2}\right]}$$

$$= \sum_{k=1}^{n} (\Delta x_k) \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} = \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k$$

to get an expression that looks more like a Riemann sum. The $\frac{\Delta y_k}{\Delta x_k}$ inside the radical should remind you of two things: the slope of a line segment (it is, in fact, the slope of the k-th line segment in our approximation of the curve $\mathcal{C}$) and a derivative, $\frac{dy}{dx}$. If $f(x)$ is both continuous and differentiable, then the Mean Value Theorem guarantees that there is some number c_k between x_{k-1} and x_k so that:

Review Section 3.2 if you need to refresh your memory about the hypotheses and conclusions of the Mean Value Theorem.

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta y_k}{\Delta x_k}$$

in which case we can write:

$$\sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta x_k$$

This last expression *is* a Riemann sum, so it converges to a definite integral:

$$\sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta x_k \longrightarrow \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

In order to be sure that the sum converges to the integral, we need the resulting integrand to be an integrable function. If we require $f'(x)$ to be continuous on $[a, b]$ then the integrand will be a composition of continuous functions, hence continuous, and we know that a function that is continuous on a closed interval is integrable.

This definite integral provides us with a formula for the length of a curve $\mathcal{C}$ given by $y = f(x)$ for $a \le x \le b$.

Arclength Formula: $y = f(x)$ version

If	$\mathcal{C}$ is a curve given by $y = f(x)$ for $a \le x \le b$
and	$f'(x)$ exists and is continuous on $[a, b]$
then	the length L of $\mathcal{C}$ is given by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example 3. Compute the length of $y = x^2$ for $0 \le x \le 3$.

Solution. Here $f(x) = x^2 \Rightarrow f'(x) = 2x$ so the length of this curve is:

$$\int_0^3 \sqrt{1 + [2x]^2}\, dx = \int_0^3 \sqrt{1 + 4x^2}\, dx$$

Unfortunately we do not (yet) have a technique to find an antiderivative of this integrand, but we can use numerical methods (such as Simpson's Rule, or a calculator or computer) to determine that the value of the integral is approximately 9.7471 (compare this with the answers from Example 2 and Practice 2). ◀

You will eventually be able to find an exact value for this definite integral using techniques developed in Section 8.4.

Practice 3. Compute the length of $y = x^2$ between $(1, 1)$ and $(4, 16)$.

Practice 4. Represent the length of one period of $y = \sin(x)$ as a definite integral, then find the length of this curve (using technology to approximate the value of the definite integral, if necessary).

More Arclength Formulas

Not all interesting curves are graphs of functions of the form $y = f(x)$. For a curve given by $x = g(y)$ we can mimic the previous argument (or simply swap x and y) to arrive at another arclength formula:

Arclength Formula: $x = g(y)$ version

If	$\mathcal{C}$ is a curve given by $x = g(y)$ for $c \le y \le d$
and	$g'(y)$ exists and is continuous on $[c, d]$
then	the length L of $\mathcal{C}$ is given by:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Practice 5. Compute the length of $x = \sqrt{y}$ between $(1, 1)$ and $(4, 16)$.

Review Section 2.5 to refresh your memory about parametric equations.

A curve $\mathcal{C}$ can also be described using parametric equations, where functions $x(t)$ and $y(t)$ give the coordinates of a point on the curve specified by a parameter t. We often think of t as "time," so that $(x(t), y(t))$ represents the position of a particle in the xy-plane t seconds (or minutes or hours) after time $t = 0$. In Section 2.5, we discovered that the speed of such a particle at time t is given by:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The distance traveled by the particle and the length of the curve will be equal as long as the particle does not traverse any part of the curve more than once on the interval $\alpha \le t < \beta$.

To find the distance the particle travels between times $t = \alpha$ and $t = \beta$, we could then integrate this speed function, which would also tell us the length of the curve.

Arclength Formula (Parametric Version)

If	$\mathcal{C}$ is a curve given by $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$
and	$x'(t)$ and $y'(t)$ exist and are continuous on $[\alpha, \beta]$
then	the length L of $\mathcal{C}$ is given by:

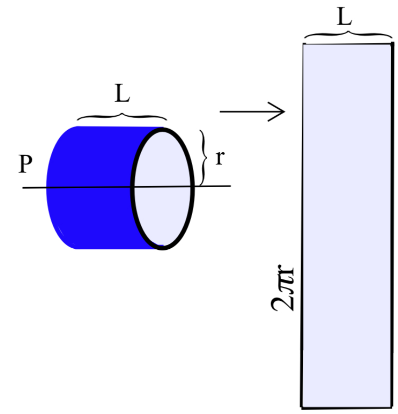
$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Practice 6. Compute the length of the parametric curve given by the functions $x(t) = \cos(t)$ and $y(t) = \sin(t)$ for $0 \le t \le 2\pi$.

Practice 7. Compute the length of the parametric path given by the functions $x(t) = 1 + 3t$ and $y(t) = 4t$ for $1 \le t \le 3$.

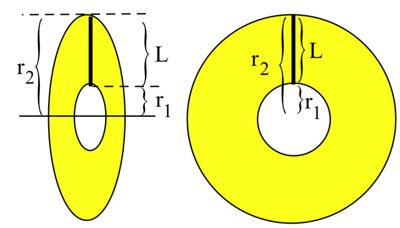
Areas of Surfaces of Revolution

In the previous section, we revolved a *region* in the xy-plane about a horizontal or vertical axis to create a *solid*, then used an integral to compute the volume of that solid. If we instead rotate a *curve* about an axis, we get a *surface*, whose **surface area** we can also compute using an integral. Just as the integral formulas for arclength came from the simple distance formula, the integral formulas for the area of a surface of revolution come from the formula for revolving a single line segment.



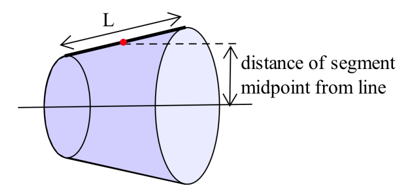
If we rotate a line segment of length L parallel to a line P (see margin) about the line P, then the resulting surface (a cylinder) can be "unrolled" and laid flat. This flattened surface is a rectangle with area $A = 2\pi \cdot r \cdot L$.

If we rotate a line segment of length L perpendicular to a line P and not intersecting P (see second margin figure) about the line P, then the resulting surface is the region between two concentric circles (an "annulus") and its area is:



$$
\begin{aligned}
A &= (\text{area of large circle}) - (\text{area of small circle}) \\
&= \pi (r_2)^2 - \pi (r_1)^2 = \pi \left[(r_2)^2 - (r_1)^2\right] = \pi (r_2 + r_1)(r_2 - r_1) \\
&= 2\pi \left(\frac{r_2 + r_1}{2}\right) L
\end{aligned}
$$

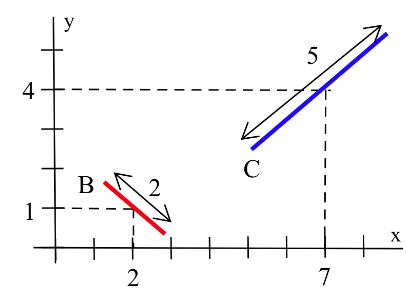
The expression $\frac{r_2+r_1}{2}$ represents the distance of the *midpoint* of the line segment L from the axis of rotation P and $2\pi\left(\frac{r_2+r_1}{2}\right)$ is the *distance* this midpoint travels when we revolve the line segment about the axis. It turns out that this pattern holds when we revolve *any* line segment of length L that does not intersect a line P about the line P (see margin):



See Problem 52 for a proof.

$$
\begin{aligned}
A &= (\text{distance traveled by segment midpoint}) \cdot (\text{length of line segment}) \\
&= 2\pi\,(\text{distance of segment midpoint from line } P) \cdot L
\end{aligned}
$$

Example 4. Compute the area of the surface generated when each line segment in the margin figure is rotated about the x-axis and the y-axis.



Solution. Line segment B has length $L = 2$ and its midpoint is at $(2, 1)$, which is 1 unit from the x-axis and 2 units from the y-axis. When B is rotated about the x-axis, the surface area is therefore:

$$2\pi \cdot (\text{distance of midpoint from } x\text{-axis}) \cdot 2 = 2\pi(1)2 = 4\pi$$

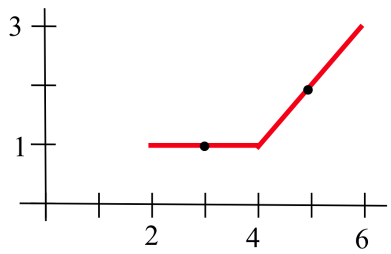
and when B is rotated about the y-axis, the surface area is:

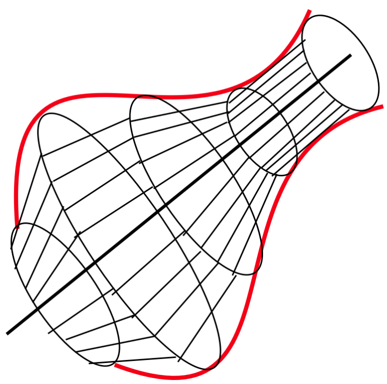
$$2\pi \cdot (\text{distance of midpoint from } y\text{-axis}) \cdot 2 = 2\pi(2)2 = 8\pi$$

Line segment $\mathcal{C}$ has length 5 and its midpoint is at $(7, 4)$. When $\mathcal{C}$ is rotated about the x-axis, the resulting surface area is:

$$2\pi \cdot (\text{distance of midpoint from } x\text{-axis}) \cdot 5 = 2\pi(4)5 = 40\pi$$

When $\mathcal{C}$ is rotated about the y-axis, the distance of the midpoint from the axis is 7, so the surface area is $2\pi(7)5 = 70\pi$. ◀





Practice 8. Find the area of the surface generated when the graph in the margin is rotated about each coordinate axis.

When we rotate a *curve* $\mathcal{C}$ (that does not intersect a line P, as in the second margin figure) about the line P, we also get a surface. To approximate the area of that surface, we can use the same strategy we used to approximate the length of a curve: select some points (x_k, y_k) along the curve, connect the points with line segments, calculate the surface area of each rotated line segment, and add together the surface areas of the rotated line segments.

The rotated line segment with endpoints (x_{k-1}, y_{k-1}) and (x_k, y_k) has midpoint:

$$(\overline{x}_k, \overline{y}_k) = \left(\frac{x_{k-1} + x_k}{2}, \frac{y_{k-1} + y_k}{2}\right)$$

and length:

$$L = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

If we rotate $\mathcal{C}$ about the x-axis, the distance from the midpoint of the k-th line segment to the x-axis is $\overline{y}_k$ so the surface area of the k-th rotated line segment will be:

$$\begin{aligned} 2\pi\left(\overline{y}_k\right)L &= 2\pi\left(\frac{y_{k-1} + y_k}{2}\right)\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= 2\pi\left(\frac{y_{k-1} + y_k}{2}\right)\sqrt{1 + \left[\frac{\Delta y_k}{\Delta x_k}\right]^2}\,\Delta x_k \end{aligned}$$

If $\mathcal{C}$ is given by $y = f(x)$ for $a \le x \le b$, and $f'(x)$ is continuous on $[a, b]$, we can appeal to the Mean Value Theorem to find a c_k with $x_{k-1} < c_k < x_k$ and $f'(c_k) = \frac{\Delta y_k}{\Delta x_k}$ so that our last expression becomes:

$$2\pi\left(\frac{f(x_{k-1}) + f(x_k)}{2}\right)\sqrt{1 + [f'(c_k)]^2}\,\Delta x_k$$

Adding up these approximations, we get:

$$\sum_{k=1}^{n} 2\pi\left(\frac{f(x_{k-1}) + f(x_k)}{2}\right)\sqrt{1 + [f'(c_k)]^2}\,\Delta x_k$$

This is not actually a Riemann sum, because the values x_{k-1}, x_k and c_k are not all the same. Proving that this sum converges to a definite integral requires some more advanced techniques.

which converges to a definite integral:

$$\int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

that gives us a formula for the surface area of the revolved curve.

Example 5. Compute the area of the surface generated when the curve $y = 2 + x^2$ for $0 \le x \le 3$ is rotated about the x-axis.

Solution. Here $f(x) = 2 + x^2 \;\Rightarrow\; f'(x) = 2x$ so, using the integral formula we just obtained:

$$\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx = \int_0^3 2\pi\left(2+x^2\right)\sqrt{1+4x^2}\,dx$$

We do not (yet) know how to find an antiderivative for this integrand, but numerical approximation yields a result of 383.8. ◀

You will eventually be able to find an exact value for this definite integral using techniques developed in Section 8.4.

More Sufrace Area Formulas

If a curve $\mathcal{C}$ given by $y = f(x)$ for $a \le x \le b$ is instead rotated about the y-axis, then the distance from the midpoint of the k-th line segment to the axis of rotation is $\overline{x}_k$. Replacing $\overline{y}_k$ with $\overline{x}_k$ in our work on the previous page yields the formula:

$$\int_a^b 2\pi x\sqrt{1+[f'(x)]^2}\,dx$$

for the area of the surface generated by revolving $\mathcal{C}$ about the y-axis (assuming, as before, that $f'(x)$ is continuous for $a \le x \le b$).

Example 6. Compute the area of the surface generated when the curve $y = 2 + x^2$ for $0 \le x \le 3$ is rotated about the y-axis.

Solution. Here again $f(x) = 2 + x^2 \;\Rightarrow\; f'(x) = 2x$ so, using our newest integral formula:

$$\int_a^b 2\pi x\sqrt{1+[f'(x)]^2}\,dx = \int_0^3 2\pi x\sqrt{1+4x^2}\,dx$$

We can find an antiderivative of this integrand using substitution:

$$u = 1+4x^2 \;\Rightarrow\; du = 8x\,dx \;\Rightarrow\; \frac{1}{8}\,du = x\,dx$$

The integral limits become $u = 1 + 4(0)^2 = 1$ and $u = 1 + 4(3)^2 = 37$, so the surface area is:

$$\int_{u=1}^{u=37} 2\pi\cdot\frac{1}{8}\sqrt{u}\,du = \frac{\pi}{4}\int_1^{37} u^{\frac{1}{2}}\,du = \frac{\pi}{4}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{37} = \frac{\pi}{6}\left[37\sqrt{37}-1\right]$$

or approximately 117.3. ◀

Wrap-Up

Developing formulas for the area of a surface generated by rotating a curve $x = g(y)$ for $c \le y \le d$ (or by parametric equations) present little additional difficulty. In future chapters, however, we will develop much more general—yet simpler—formulas for arclength and surface area. While the integral formulas developed in this section can be useful, more importantly their development served to illustrate yet again how relatively simple approximation formulas can lead us—via Riemann sums—to integral formulas. We will see this process again and again.

See Problems 48–51.

5.3 Problems

1. The locations (in feet, relative to an oak tree) at various times (in minutes) for a squirrel spotted in a back yard appear in the table below:

time	north	east
0	10	7
5	25	27
10	1	45
15	13	33
20	24	40
25	10	23
30	0	14

At least how far did the squirrel travel during the first 15 minutes?

2. The squirrel in the previous problem traveled at least how far during the first 30 minutes?
3. Use the partition $\{0,1,2\}$ to estimate the length of $y = 2^x$ between the points $(0,1)$ and $(2,4)$.
4. Use the partition $\{1,2,3,4\}$ to estimate the length of $y = \dfrac{1}{x}$ between the points $(1,1)$ and $\left(4, \dfrac{1}{4}\right)$.

The graphs of the functions in Problems 5–8 are line segments. Calculate each length (a) using the distance formula between two points and (b) by setting up and evaluating an appropriate arclength integral.

5. $y = 1 + 2x$ for $0 \le x \le 2$.
6. $y = 5 - x$ for $1 \le x \le 4$,
7. $x = 2 + t$, $y = 1 - 2t$ for $0 \le t \le 3$.
8. $x = -1 - 4t$, $y = 2 + t$ for $1 \le t \le 4$.
9. Calculate the length of $y = \dfrac{2}{3}x^{\frac{3}{2}}$ for $0 \le x \le 4$.
10. Calculate the length of $y = 4x^{\frac{3}{2}}$ for $1 \le x \le 9$.

Very few functions of the form $y = f(x)$ lead to integrands of the form $\sqrt{1 + [f'(x)]^2}$ that have elementary antiderivatives. In 11—14, $1 + \left[f'(x)\right]^2$ ends up being a perfect square, so you can evaluate the resulting arclength integral using antiderivatives.

11. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ for $1 \le x \le 5$.
12. $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ for $1 \le x \le 9$.
13. $y = \dfrac{x^5}{5} + \dfrac{1}{12x^3}$ for $1 \le x \le 5$.
14. $y = \dfrac{x^6}{6} + \dfrac{1}{16x^4}$ for $4 \le x \le 25$.

In Problems 15–23, represent each length as a definite integral, then evaluate the integral (using technology, if necessary).

15. The length of $y = x^2$ from $(0,0)$ to $(1,1)$.
16. The length of $y = x^3$ from $(0,0)$ to $(1,1)$.
17. The length of $y = \sqrt{x}$ from $(1,1)$ to $(9,3)$.
18. The length of $y = \ln(x)$ from $(1,0)$ to $(e,1)$.
19. The length of $y = \sin(x)$ from $(0,0)$ to $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$ and from $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$ to $\left(\dfrac{\pi}{2}, 1\right)$.
20. The length of the ellipse $x(t) = 3\cos(t)$, $y(t) = 4\sin(t)$ for $0 \le t \le 2\pi$.
21. The length of the ellipse $x(t) = 5\cos(t)$, $y(t) = 2\sin(t)$ for $0 \le t \le 2\pi$.
22. A robot programmed to be at location $x(t) = t\cos(t)$, $y(t) = t\sin(t)$ at time t will travel how far between $t = 0$ and $t = 2\pi$?
23. How far will the robot in the previous problem travel between $t = 10$ and $t = 20$?
24. As a tire of radius R rolls, a pebble stuck in the tread will travel a "cycloid" path, given by $x(t) = R \cdot (t - \sin(t))$, $y(t) = R \cdot (1 - \cos(t))$. As t increases from 0 to 2π, the tire makes one complete revolution and travels forward $2\pi R$ units. How far does the pebble travel?
25. Referring to the previous problem, as a tire with a 1-foot radius rolls forward 1 mile, how far does a pebble stuck in the tire tread travel?
26. Graph $y = x^n$ for $n = 1$, 3, 10 and 20. As the value of n becomes large, what happens to the graph of $y = x^n$? Estimate the value of:

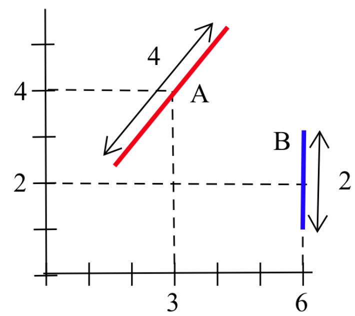
$$\lim_{n\to\infty} \int_{x=0}^{x=1} \sqrt{1 + [n \cdot x^{n-1}]^2}\, dx$$

27. Find the point on the curve $f(x) = x^2$ for $0 \le x \le 4$ that will divide the curve into two equally long pieces. Find the points that will divide the segment into three equally long pieces.

28. Find the pattern for the functions in Problems 11–14. If $y = \frac{x^n}{n} + \frac{1}{Ax^p}$, how must A and p be related to n?

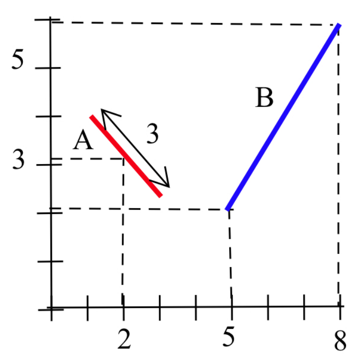
29. Use the formulas for A and p from the previous problem with $n = \frac{3}{2}$ and find a new function $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{Ax^p}$ so that $1 + \left[\frac{dy}{dx}\right]^2$ is a perfect square.

30. Find the surface area when each line segment in the figure below is rotated about the (a) x-axis and (b) y-axis.



31. Find the surface area when each line segment in the figure above is rotated about the line (a) $y = 1$ and (b) $x = -2$.

32. Find the surface area when each line segment in the figure below is rotated about the line (a) $y = 1$ and (b) $x = -2$.



33. Find the surface area when each line segment in the figure above is rotated about the (a) x-axis and (b) y-axis.

34. A line segment of length 2 with midpoint $(2, 5)$ makes an angle of θ with the horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the y-axis? Explain your reasoning.

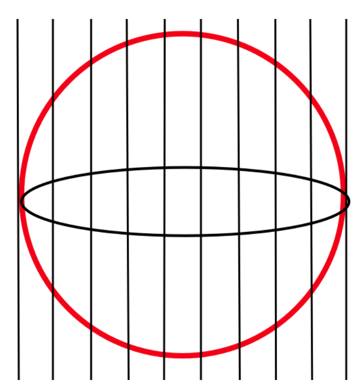
35. A line segment of length 2 with one end at $(2, 5)$ makes an angle of θ with the horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the x-axis? Explain your reasoning.

In Problems 36–43, when the given curve is rotated about the given axis, represent the area of the resulting surface as a definite integral, then evaluate that integral using technology.

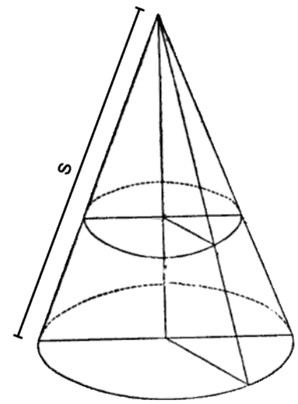
36. $y = x^3$ for $0 \le x \le 2$ about the y-axis
37. $y = 2x^3$ for $0 \le x \le 1$ about the y-axis
38. $y = x^2$ for $0 \le x \le 2$ about the x-axis
39. $y = 2x^2$ for $0 \le x \le 1$ about the x-axis
40. $y = \sin(x)$ for $0 \le x \le \pi$ about the x-axis
41. $y = x^3$ for $0 \le x \le 2$ about the x-axis
42. $y = \sin(x)$ for $0 \le x \le \frac{\pi}{2}$ about the y-axis
43. $y = x^2$ for $0 \le x \le 2$ about the y-axis

44. Find the area of the surface formed when the graph of $y = \sqrt{4 - x^2}$ is rotated about the x-axis:
    (a) for $0 \le x \le 1$.
    (b) for $1 \le x \le 2$.
    (c) for $2 \le x \le 3$.

45. Show that if a thin hollow sphere is sliced into pieces by equally spaced parallel cuts (see below), then each piece has the same weight. (Hint: Does each piece have the same surface area?)



46. Interpret the result of the previous problem for an orange sliced by equally spaced parallel cuts.
47. A hemispherical cake with a uniformly thick layer of frosting is sliced with equally spaced parallel cuts. Does everyone get the same amount of cake? The same amount of frosting?
48. Devise a formula for the area of the surface generated by revolving the curve $x = g(y)$ for $c \le y \le d$ about the (a) x-axis and (b) y-axis.
49. Use the answer to the previous problem to find the area of the surface generated by revolving $x = e^y$ for $0 \le y \le 1$ about (a) the x-axis and (b) the y-axis.
50. Devise a formula for the area of the surface generated by revolving the curve given by parametric equations $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$ about the (a) x-axis and (b) y-axis.
51. Use the answer to the previous problem to find the area of the surface generated by revolving the curve given by $x = \cos(t)$ and $y = \sin(t)$ for $0 \le t \le \frac{\pi}{2}$ about (a) the x-axis and (b) the y-axis.
52. The surface generated by revolving a line segment of length L about a line P (that does not intersect the line segment) is the **frustrum** of a cone: the surface that results from taking a larger cone of radius r_2 and removing a smaller cone of radius r_1 ("chopping off the top"). We know from geometry that the surface area of a cone is $\pi r s$ where r is the radius of the cone and s is the **slant height**:



(a) If s_1 is the slant height of the smaller cone that is removed from the bigger cone, show that:

$$s_1 + L = \frac{r_2 L}{r_2 - r_1}$$

(Hint: Use similar triangles.)

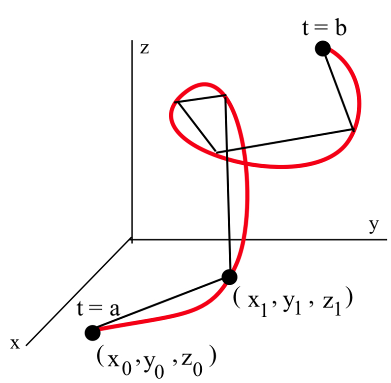
(b) Show that the surface area of the frustrum is:

$$\pi r_2 (s_1 + L) - \pi r_1 s_1$$

(c) Show that this quantity equals:

$$\pi (r_1 + r_2) L$$

(d) Show that this last quantity is the product of the distance traveled by the midpoint and the length of the line segment.



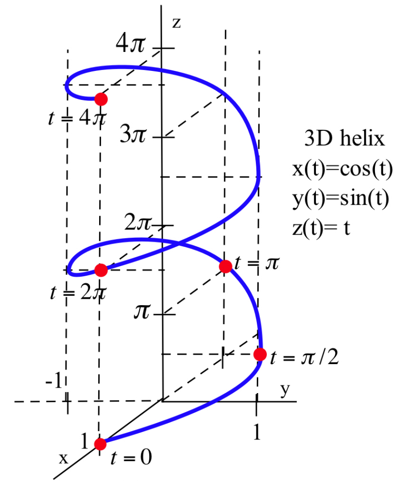
3-D Arclength

If a 3-dimensional curve $\mathcal{C}$ (see margin) is given parametrically by $x = x(t)$, $y = y(t)$ and $z = z(t)$ for $\alpha \le t \le \beta$, then we can easily extend the arclength formula to three dimensions:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The remaining problems in this section use this formula to provide you with a preview of calculus in higher dimensions.

53. Find the length of the helix (see figure) given by $x = \cos(t)$, $y = \sin(t)$, $z = t$ for $0 \le t \le 4\pi$.
54. Find the length of the line segment given by $x = t$, $y = t$, $z = t$ for $0 \le t \le 1$.
55. Find the length of the curve given by $x = t$, $y = t^2$, $z = t^3$ for $0 \le t \le 1$.
56. Find the length of the "stretched helix" given by $x = \cos(t)$, $y = \sin(t)$, $z = t^2$ for $0 \le t \le 2\pi$.
57. Find the length of the curve given by $x = 3\cos(t)$, $y = 2\sin(t)$, $z = \sin(7t)$ for $0 \le t \le 2\pi$.



5.3 Practice Answers

1. At least $2 + \sqrt{2} + \sqrt{13} + 1 + \sqrt{2} \approx 9.43$ miles.
2. $L \approx \sqrt{2} + \sqrt{10} + \sqrt{26} <$ actual length
3. $\int_1^4 \sqrt{1 + [2x]^2}\,dx = \int_1^4 \sqrt{1 + 4x^2}\,dx \approx 15.34$
4. $\int_0^{2\pi} \sqrt{1 + [\cos(x)]^2}\,dx \approx 7.64$
5. Here $g(y) = \sqrt{y} = y^{\frac{1}{2}} \Rightarrow g'(y) = \frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$ so the arclength is:

$$\int_1^{16} \sqrt{1 + \left[\frac{1}{2\sqrt{y}}\right]^2}\,dy = \int_1^{16} \sqrt{1 + \frac{1}{4y}}\,dy \approx 15.34$$

This answer is the same as the answer to Practice 3. Should that surprise you?

6. Here $x'(t) = -\sin(t)$ and $y'(t) = \cos(t)$ so the arclength is:

$$\int_0^{2\pi} \sqrt{[-\sin(t)]^2 + [\cos(t)]^2}\,dt = \int_0^{2\pi} \sqrt{1}\,dt = 2\pi$$

The curve in question is a circle of radius 1. Does the answer from the integral formula agree with the answer you can obtain using simple geometry?

7. Here $x'(t) = 3$ and $y'(t) = 4$ so the arclength is:

$$\int_1^3 \sqrt{[3]^2 + [4]^2}\,dt = \int_1^3 5\,dt = 10$$

The "curve" is a line segment from $(4, 4)$ to $(10, 12)$. Does the answer from the integral formula agree with the answer you can obtain using simple geometry?

8. The surface area of the horizontal segment revolved about x-axis is $2\pi(1)(2) = 4\pi \approx 12.57$ while the surface area of other segment revolved about the x-axis is $2\pi(2)(\sqrt{8}) \approx 35.54$, so the total surface area is approximately $12.57 + 35.54 = 48.11$ square units.

   The surface area of the horizontal segment revolved about y-axis is $2\pi(3)(2) = 12\pi \approx 37.70$ while the surface area of the other segment revolved about the y-axis is $2\pi(5)(\sqrt{8}) \approx 88.86$, so the total surface area is approximately $37.70 + 88.86 = 126.56$ square units.

5.4 More Work

In Section 4.7 we investigated the problem of calculating the work done in lifting an object using a cable. This section continues that investigation and extends the process to handle situations in which the applied force or the distance—or both—may vary. The method we used before turns up again here. The first step is to divide the problem into small "slices" so that the force and distance vary only slightly on each slice. Then we calculate the work done for each slice, approximate the total work by adding together the work for each slice (to get a Riemann sum) and, finally, take a limit of that Riemann sum to get a definite integral representing the total work.

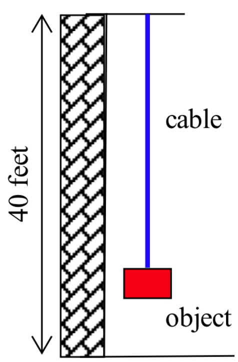
There are so many possible variations in work problems that it is vital you understand the process.

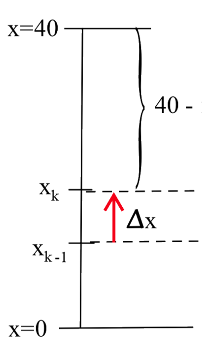
Recall that the work done on an object by a constant force is defined to be the magnitude of the force applied to the object multiplied by the distance over which the force is applied:

$$\text{work} = (\text{force}) \cdot (\text{distance})$$

A similar example appeared in Section 4.7, but it provides a good illustration of the process of dividing a problem into pieces and analyzing each piece.

Example 1. A 10-pound object is lifted 40 feet from the ground to the top of a building using a cable that weighs $\frac{1}{2}$ pound per foot (see margin figure). How much work is done?





Solution. The work done on the object is simply:

$$W = F \cdot d = (10 \text{ lbs}) \cdot (40 \text{ ft}) = 400 \text{ ft-lbs}$$

For the rope, we can partition it (see second margin figure) into n small pieces, each with length Δx. Each small piece of rope weighs:

$$\left(\frac{1}{2}\frac{\text{lb}}{\text{ft}}\right)(\Delta x \text{ ft}) = \frac{1}{2}\Delta x \text{ lb}$$

and the k-th slice of rope is lifted a distance of (approximately) $40 - x_k$ feet, so the work done on the k-th slice of rope is (approximately):

$$W_k = F_k \cdot d_k = \left(\frac{1}{2}\Delta x \text{ lb}\right) \cdot \Big((40 - x_k)\ \text{ft}\Big) = \frac{1}{2}(40 - x_k)\,\Delta x \text{ ft-lbs}$$

and the total work done to lift the rope is therefore:

$$\sum_{k=1}^{n} \frac{1}{2}(40 - x_k)\,\Delta x \longrightarrow \int_0^{40} \frac{1}{2}(40 - x)\, dx$$

Evaluating this integral yields:

$$\frac{1}{2}\left[40x - \frac{1}{2}x^2\right]_0^{40} = \frac{1}{2}[1600 - 800] = 400 \text{ ft-lbs}$$

so the total work done to lift the object is $400 + 400 = 800$ ft-lbs. ◀

Practice 1. How much work is done lifting a 130-pound injured person to the top of a 30-foot cliff using a stretcher that weighs 10 pounds and a cable weighing 2 pounds per foot?

Work in the Metric System

All of the work problems we have considered so far measured force in pounds and distance in feet, so that work was measured in "foot-pounds." In the metric system, we often measure distance in meters (m) and force in **newton**s (N). According to Newton's second law of motion:

$$\text{force} = (\text{mass}) \cdot (\text{acceleration})$$

or, more succinctly, $F = ma$. The force in many work problems is the weight of an object, so the acceleration in question is the acceleration due to gravity, denoted by g. Near sea level on Earth, $g \approx 9.80665 \,\frac{\text{m}}{\text{sec}^2}$, although the value 9.81 is commonly used in computations. An object with a mass of 10 kg would thus have a weight of:

$$mg = (10 \text{ kg}) \cdot \left(9.81 \,\frac{\text{m}}{\text{sec}^2}\right) = 98.1 \,\frac{\text{kg} \cdot \text{m}}{\text{sec}^2} = 98.1\,\text{N}$$

Virtually all countries other than the United States — along with U.S. scientists and engineers — use the metric system, so you need to know how to solve work problems using metric units.

Sir Isaac Newton (1643–1727) not only invented calculus, he formulated the laws of motion and universal gravitation in physics (among many other accomplishments).

Example 2. An object with a mass of 10 kg is lifted 40 m from the ground to the top of a building using a 40-meter cable with a mass of 20 kg. How much work is done?

Solution. The work done on the object is:

$$W = F \cdot d = mg \cdot d = (10 \text{ kg}) \left(9.81 \,\frac{\text{m}}{\text{sec}^2}\right) \cdot (40 \text{ m}) = 3924 \text{ N-m}$$

or 3,924 joules (a **joule**, abbreviated "J," is 1 N-m). The cable has total mass 20 kg and is 40 m long, so it has a linear density of:

$$\frac{20 \text{ kg}}{40 \text{ m}} = \frac{1}{2}\,\frac{\text{kg}}{\text{m}}$$

This unit for work is named after another English physicist, James Prescott Joule (1818–1889).

We can partition the cable into n small pieces, each with length Δx, so each small piece of cable has a mass of:

$$\left(\frac{1}{2}\,\frac{\text{kg}}{\text{m}}\right)(\Delta x \text{ m}) = \frac{1}{2}\Delta x \text{ kg}$$

and thus has a weight of:

$$F = mg = \left(\frac{1}{2}\Delta x \text{ kg}\right)\left(9.81 \,\frac{\text{m}}{\text{sec}^2}\right) = 4.905\Delta x \,\frac{\text{kg} \cdot \text{m}}{\text{sec}^2} = 4.905\Delta x \text{ N}$$

The k-th slice of cable is lifted a distance of approximately $40 - x_k$ m, so the work done on the k-th slice of cable is:

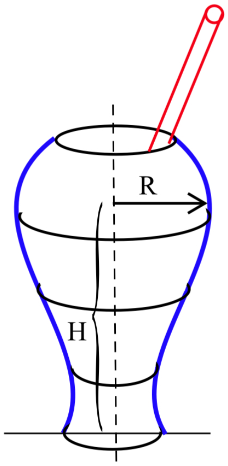
$$W_k = F_k \cdot d_k = (4.905\Delta x \text{ N}) \cdot \Big((40 - x_k) \text{ m}\Big) = 4.905\,(40 - x_k)\,\Delta x \text{ N-m}$$

and the total work done lifting the cable is therefore:

$$\sum_{k=1}^{n} 4.905\,(40 - x_k)\,\Delta x \;\longrightarrow\; \int_0^{40} 4.905\,(40 - x)\; dx = 3924 \text{ J}$$

so the total work done to lift the object is $3924 + 3924 = 7848$ J. ◀

Much of this process should look familiar. Compare the solution of Example 2 to that of Example 1 on the previous page.



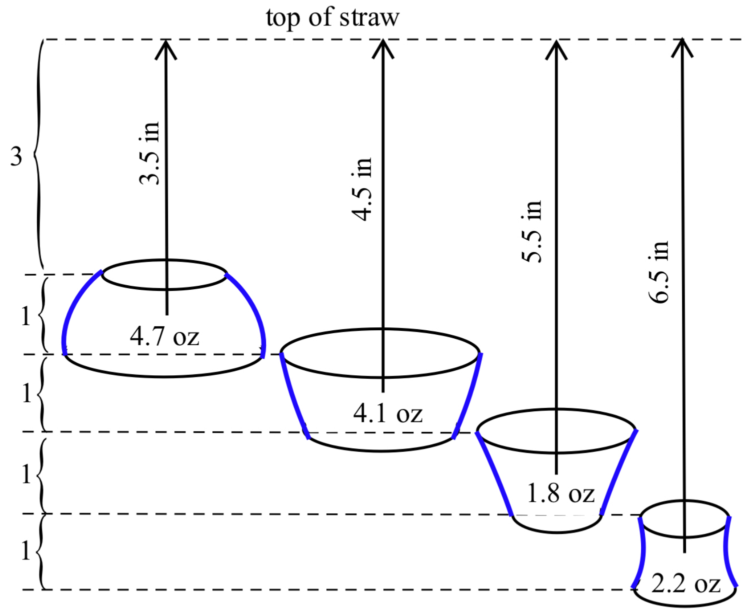
height	radius
4	1.4
3	1.6
2	1.5
1	1.0
0	1.1

Practice 2. How much work is done lifting an injured person of mass 50 kg to the top of a 30-meter cliff using a stretcher of mass 5 kg and a 30-meter cable of mass 10 kg?

Lifting Liquids

Example 3. A cola glass (see margin figure) has dimensions given in the margin table. Approximately how much work do you do when you drink a cola glass full of water by sucking it through a straw to a point 3 inches above the top edge of the glass?

Solution. The table partitions the water into 1-inch "slices":



The work needed to move each slice is approximately the weight of the slice times the distance the slice is moved. We can use the radius at the bottom of each slice to approximate the volume — and then the weight — of the slice, and a point halfway up each slice to calculate the distance the slice is moved. For the top slice:

You might wonder why the displacement is not computed by taking the distance from the bottom of the straw up to the top of the straw, but when computing work we need to use the *net* displacement.

Water's density is $62.5\frac{\text{lb}}{\text{ft}^3} = 0.5787\frac{\text{oz}}{\text{in}^3}$.

$$\text{weight} = (\text{volume})\,(\text{density}) \approx \pi\,(1.6\,\text{in})^2\,(1\,\text{in})\left(0.5787\,\frac{\text{oz}}{\text{in}^3}\right) \approx 4.7\,\text{oz}$$

and the distance this slice travels is roughly 3.5 inches, so:

$$W = F \cdot d \approx (4.7\text{ oz})\,(3.5\text{ in}) \approx 16.4\text{ oz-in}$$

For the next slice:

$$\text{weight} = (\text{volume})\,(\text{density}) \approx \pi\,(1.5\,\text{in})^2\,(1\,\text{in})\left(0.5787\,\frac{\text{oz}}{\text{in}^3}\right) \approx 4.1\,\text{oz}$$

and the distance this slice travels is roughly 4.5 inches, so:

$$W = F \cdot d \approx (4.1\text{ oz})\,(4.5\text{ in}) \approx 18.4\text{ oz-in}$$

The work for the last two slices is $(1.8\,\text{oz})\,(5.5\,\text{in}) = 9.9$ oz-in and $(2.2\,\text{oz})\,(6.5\,\text{in}) = 14.3$ oz-in. The total work is then sum of the work needed to raise each slice of water:

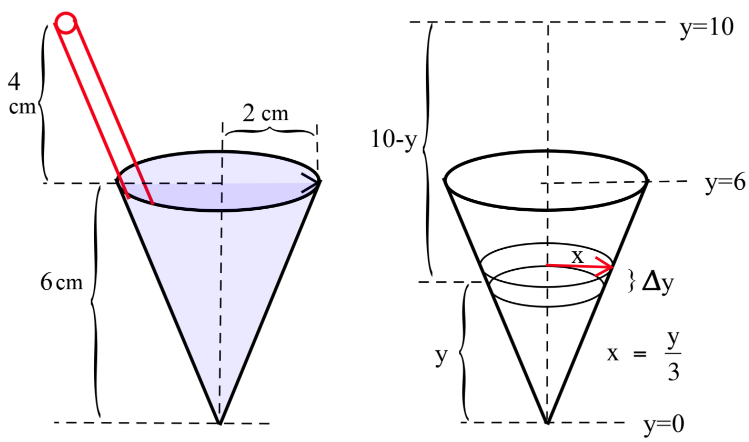
$$(16.4\,\text{oz-in}) + (18.4\,\text{oz-in}) + (9.9\,\text{oz-in}) + (14.3\,\text{oz-in}) = 59.0\,\text{oz-in}$$

or about 0.31 ft-lbs. ◀

Practice 3. Approximate the total work needed to raise the water in Example 3 by using the top radius of each slice to approximate its weight and the midpoint of each slice to approximate the distance the slice is raised.

If we knew the radius of the cola glass at *every* height, then we could improve our approximation by taking thinner and thinner slices. In fact, we could have formed a Riemann sum, taken the limit of the Riemann sum as the thickness of the slices approached 0, and obtained a definite integral. In the next Example we *do* know the radius of the container at every height.

Example 4. Find the work needed to raise the water in the cone shown below to the top of the straw.



In this example, both the force and the distance vary, and each depends on the height of the "slice" above the bottom of the cone.

Solution. We can partition the cone to get n "slices" of water. The work done raising the k-th slice is the product of the distance the slice is raised and the force needed to move the slice (the weight of the slice). For any c_k in the subinterval $[y_{k-1}, y_k]$, the slice is raised a distance of approximately $(10 - c_k)$ cm. Each slice is approximately a right circular cylinder, so its volume is:

If you want, you can choose $c_k = y_k$ like you did in Practice 3.

$$\pi\,(\text{radius})^2\,\Delta y$$

At a height y above the bottom of the cone, the radius of the cylinder is $x = \frac{y}{3}$ so at a height c_k the radius is $\frac{1}{3}c_k$; the mass of each slice is

To see this, use similar triangles in the right-hand figure above:

$$\frac{x}{y} = \frac{2}{6} \Rightarrow x = \frac{y}{3}$$

therefore:

$$(\text{volume})(\text{density}) \approx \pi\,(\text{radius})^2\,(\Delta y)\left(1\,\frac{\text{g}}{\text{cm}^3}\right) = \pi\left(\frac{1}{3}c_k\text{ cm}\right)^2(\Delta y\text{ cm})\left(1\,\frac{\text{g}}{\text{cm}^3}\right) = \frac{\pi}{9}\,(c_k)^2\,\Delta y\text{ g}$$

In the metric system, a **gram** (abbreviated "g") is defined as the mass of one cubic centimeter of water, so the density of water is:

$$1\,\frac{\text{g}}{\text{cm}^3} = 1{,}000\,\frac{\text{kg}}{\text{m}^3}$$

so the force required to raise the k-th slice is:

$$F_k = m_k \cdot g \approx \left[\frac{\pi}{9}\,(c_k)^2\,\Delta y\text{ g}\right]\cdot\left[981\,\frac{\text{cm}}{\text{sec}^2}\right] = 109\pi\,(c_k)^2\,\Delta y\,\frac{\text{g-cm}}{\text{sec}^2}$$

In the g–cm–sec version of the metric system, the standard unit of force is a **dyne** (abbreviated "dyn"), which is $1\,\frac{\text{g-cm}}{\text{sec}^2}$:

$$1\text{ N} = 100{,}000\text{ dyn}$$

and the work required to lift the k-th slice is:

$$W_k = F_k \cdot d_k \approx \left[109\pi\,(c_k)^2\,\Delta y\text{ dyn}\right]\cdot[(10-c_k)\text{ cm}] = 109\pi\,(c_k)^2\,(10-c_k)\,\Delta y\text{ dyn-cm}$$

In the g–cm–sec version of the metric system, the standard unit of work is called an **erg**, which is 1 dyn-cm:

$$1\text{ J} = 10{,}000{,}000\text{ erg}$$

We can then add the work done on all n slices to get a Riemann sum:

$$W \approx \sum_{k=1}^{n} 109\pi\,(c_k)^2\,(10-c_k)\,\Delta y \longrightarrow \int_{y=0}^{y=6} 109\pi y^2(10-y)\,dy$$

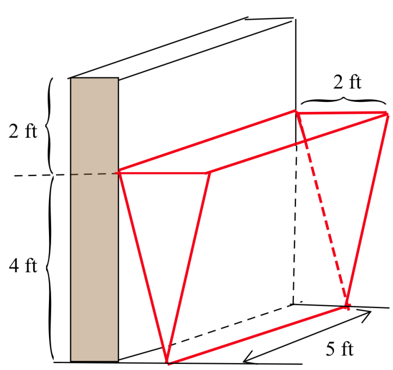
Evaluating this integral is relatively straightforward:

$$W = 109\pi\int_0^6\left(10y^2-y^3\right)dy = 109\pi\left[\frac{10}{3}y^3-\frac{1}{4}y^4\right]_0^6 = 109\pi\,[720-324] = 43164\pi\text{ erg}$$

We integrate from $y=0$ to $y=6$ because the bottom slice of water is at a height of 0 cm and the top slice of water is at a height of 6 cm.

or about $135{,}604\text{ erg} = 0.0135604$ J. ◀

Practice 4. How much work is done drinking just the top 3 cm of the water in Example 4?

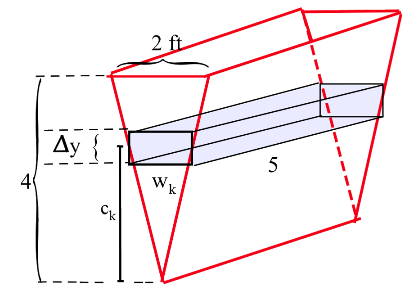


Example 5. The trough shown in the margin is filled with a liquid weighing 70 pounds per cubic foot. How much work is done pumping the liquid over the wall next to the trough?

Solution. As before, we can partition the height of the trough to get n "slices" of liquid (see margin figure at top of next page). To form a Riemann sum for the total work, we need the weight of a typical slice and the distance that slice is raised. The weight of the k-th slice is:

$$(\text{volume})\cdot(\text{density}) \approx (\text{length})\,(\text{width})\,(\text{height})\cdot\left(70\,\frac{\text{lb}}{\text{ft}^3}\right)$$

The length of each slice is 5 feet, and the height of each slice is Δy feet, but the width of each slice (w_k) varies and depends on how far the slice



is above the bottom of the trough (c_k). Using similar triangles on the edge of the trough, we can observe that:

$$\frac{w_k}{c_k} = \frac{2}{4} \Rightarrow w_k = \frac{c_k}{2}$$

so the weight of the k-th slice is therefore:

$$(5 \text{ ft}) \left(\frac{c_k}{2} \text{ ft}\right) (\Delta y \text{ ft}) \cdot \left(70 \frac{\text{lb}}{\text{ft}^3}\right) = 175 c_k \Delta y \text{ lb}$$

The k-th slice is raised from a height of c_k feet to a height of 6 feet, through a distance of $6 - c_k$ feet, so the work done on the k-th slice is:

$$W_k = F_k \cdot d_k \approx \left[175 c_k \Delta y \text{ lb}\right] \cdot \left[(6 - c_k) \text{ ft}\right] = 175 c_k (6 - c_k) \Delta y \text{ lb-ft}$$

Adding up the work done on all n slices yields a Riemann sum that converges to a definite integral:

$$\sum_{k=1}^{n} 175 c_k (6 - c_k) \Delta y \longrightarrow \int_0^4 175 y (6 - y)\, dy$$

We integrate from $y = 0$ to $y = 4$ because the bottom slice of liquid is at a height of 0 feet and the top slice of liquid is at a height of 4 feet.

Evaluating the integral is straightforward:

$$175 \int_0^4 \left(6y - y^2\right) dy = 175 \left[3y^2 - \frac{1}{3} y^3\right]_0^4 = \frac{14000}{3}$$

or about 4,667 ft-lbs. ◀

You can generally handle "raise the liquid" problems by partitioning the height of the container and then focusing on a typical slice.

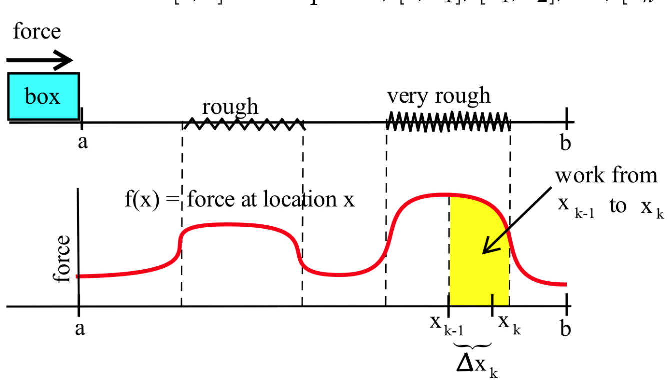
If you can calculate the weight of a typical slice and the distance it is raised, the rest of the steps are straightforward: form a Riemann sum, let it converge to a definite integral, and evaluate the integral to get the total work.

Practice 5. How much work is done pumping half of the liquid over the wall in Example 5?

Work Moving an Object Along a Straight Path

If you push a box along a flat surface (as in the figure below) that is smooth in some places and rough in others, at some places you only need to push the box lightly and in other places you have to push hard. If $f(x)$ is the amount of force needed at location x, and you want to push the box along a straight path from $x = a$ to $x = b$, then we can partition the interval $[a, b]$ into n pieces, $[a, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, b]$:

The force $f(x)$ discussed here is the minimum force required to counteract the **kinetic friction** between the box and the surface at any point. You will learn more about friction in physics and engineering classes.

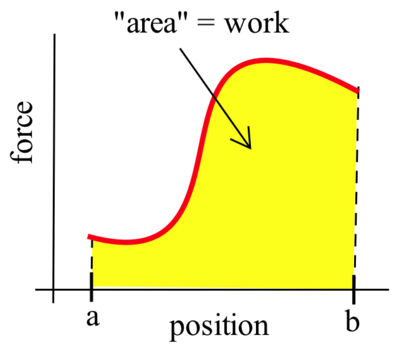


The work required to move the box through the k-th subinterval, from x_{k-1} to x_k, is approximately:

$$(\text{force}) \cdot (\text{distance}) \approx f(c_k) \cdot (x_k - x_{k-1}) = f(c_k) \cdot \Delta x_k$$

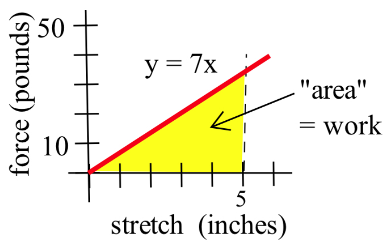
for any c_k in the subinterval $[x_{k-1}, x_k]$. The total work is the sum of the work along these n pieces, which is a Riemann sum that converges to a definite integral:

$$\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \longrightarrow \int_a^b f(x)\,dx$$



This has a simple geometric interpretation. If $f(x)$ is the force applied at position x, then the work done to move the object from position $x = a$ to position $x = b$ is the area under the graph of f between $x = a$ and $x = b$ (see margin). This formula applies in more general situations, as demonstrated in the next Example.

Example 6. If a force of $7x$ pounds is required to keep a spring stretched x inches past its natural length, how much work will be done stretching the spring from its natural length ($x = 0$) to five inches beyond its natural length ($x = 5$)?

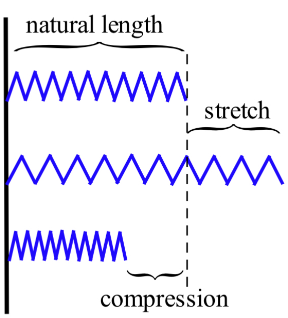


Solution. According to the formula we just developed:

$$\text{work} = \int_a^b f(x)\,dx = \int_0^5 7x\,dx = \left[\frac{7}{2}x^2\right]_0^5 = \frac{175}{2} = 87.5\,\text{in-lbs}$$

or about 7.29 ft-lbs. (See margin for a graphical interpretation.) ◀

Practice 6. How much work is done to stretch the spring in Example 6 from 5 inches past its natural length to 10 inches past its natural length?



The preceding spring example is an application of a physical principle discovered by English physicist Robert Hooke (1635–1703), a contemporary of Newton.

> **Hooke's Law**: The force $f(x)$ needed to keep a spring stretched (or compressed) x units beyond its natural length is proportional to the distance x: $f(x) = kx$ for some constant k.

In fact, Hooke's Law holds for most solid objects, at least for limited forces: "Nor is it observable in these bodies only, but in all other springy bodies whatsoever, whether metal, wood, stones, baked earth, hair, horns, silk, bones, sinews, glass and the like." — Robert Hooke, *De Potentia Restitutiva, or Of Spring*

We call the "k" in Hooke's Law a "spring constant." It varies from spring to spring (depending on the materials and dimensions of the spring — and even on the temperature of the spring), but remains constant for each spring as long as the spring is not overextended or overcompressed. Most bathroom scales use compressed springs — and Hooke's Law — to measure a person's weight.

Example 7. A spring has a natural length of 43 cm when hung from a ceiling. A mass of 40 grams stretches it to a length of 75 cm. How much work is done stretching the spring from a length of 63 cm to a length of 93 cm?

Solution. First we need to use the given information to find the value of k, the spring constant. A mass of 40 g produces a stretch of $75 - 43 = 32$ cm. Substituting $x = 32$ cm and $f(x) = 40\,\text{g} \cdot 981\,\frac{\text{cm}}{\text{sec}^2}$ into Hooke's Law $f(x) = kx$, we have:

$$40(981) = k(32) \;\Rightarrow\; k = \frac{4905}{4}$$

The length of 63 cm represents a stretch of 20 cm beyond the spring's natural length, while the length of 93 cm represents a 50-cm stretch. The work done is therefore:

$$\int_{20}^{50} \frac{4905}{4} x\,dx = \left[\frac{4905}{8}x^2\right]_{20}^{50} = 613.125\left[50^2 - 20^2\right] = 1287562.5 \text{ ergs}$$

or about 0.129 joules. ◀

Practice 7. A spring has a natural length of 3 inches when hung from a ceiling, and a force of 2 pounds stretches it to a length of 8 inches. How much work is done stretching the spring from a length of 5 inches to a length of 10 inches?

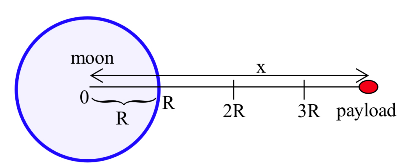
Lifting a Payload

Calculating the work required to lift a payload from the surface of a moon (or any body with no atmosphere) can be accomplished using a similar computation. Newton's Law of Universal Gravitation says that the gravitational force between two bodies of mass M and m is:

$$F = \frac{GMm}{x^2}$$

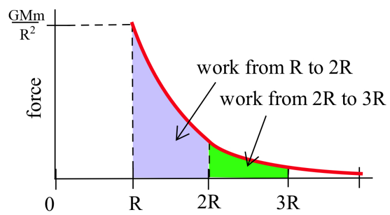
where $G \approx 6.67310^{-11}\,\text{N}\left(\frac{\text{m}}{\text{kg}}\right)^2$ is the **gravitational constant** and x is the distance between (the centers of) the two bodies.

If the moon has a radius of R m and mass M, the payload has mass m and x measures the distance (in meters) from payload to the center of the moon (so $x \geq R$), then the total amount of work done lifting the payload from the surface of the moon (an altitude of 0, where $x = R$) to an altitude of R (where $x = R + R = 2R$) is:



$$\int_R^{2R} \frac{GMm}{x^2}\,dx = GMm\left[\frac{-1}{x}\right]_R^{2R} = GMm\left[\frac{-1}{2R} + \frac{1}{R}\right] = \frac{GMm}{2R}$$

Practice 8. How much work is required to lift the payload from an altitude of R m above the surface ($x = 2R$) to an altitude of $2R$ m?



The appropriate areas under the force graph (see margin) illustrate why the work to lift the payload from $x = R$ to $x = 2R$ is much larger than the work to lift it from $x = 2R$ to $x = 3R$. In fact, the work to lift the payload from $x = 2R$ to x = $100R$ is $0.49GMmR^{-1}$, which is less than the $0.5GMmR^{-1}$ needed to lift it from $x = R$ to $x = 2R$.

The real-world problem of lifting a payload turns out to be much more challenging, because the rocket doing the lifting must also lift itself (more work) and the mass of the rocket keeps changing as it burns up fuel. Lifting a payload from a moon (or planet) with an atmosphere is even more difficult: the atmosphere produces friction, and the frictional force depends on the density of the atmosphere (which varies with height), the speed of the rocket and the shape of the rocket. Life can get complicated.

Power

Scottish engineer James Watt (1736–1819) devised horsepower to compare the output of steam engines with the power of draft horses.

In physics, **power** is defined as the rate of work done per unit of time. One traditional measurement of power, **horsepower** (abbreviated "hp"), originated with James Watt's determination in 1782 that a horse could turn a mill wheel of radius 12 feet 144 times in an hour while exerting a force of 180 pounds. Such a horse would travel:

$$144\ \frac{\text{rev}}{\text{hr}} \cdot 2\pi(12)\ \frac{\text{ft}}{\text{rev}} \cdot \frac{1}{60}\ \frac{\text{hr}}{\text{min}} = \frac{288\pi}{5}\ \frac{\text{ft}}{\text{min}}$$

and so it would produce work at a rate of:

$$(180\text{ lb})\left(\frac{288\pi}{5}\ \frac{\text{ft}}{\text{min}}\right) = 10368\pi\ \frac{\text{ft-lb}}{\text{min}} \approx 32572\ \frac{\text{ft-lb}}{\text{min}}$$

which Watt subsequently rounded to:

$$33000\ \frac{\text{ft-lb}}{\text{min}} = 550\ \frac{\text{ft-lb}}{\text{sec}} = 1\text{ horsepower}$$

$1\text{ hp} \approx 746\text{ W}$

The metric unit of power, called a **watt** (abbreviated "W") in Watt's honor, is equivalent to 1 joule per second.

Example 8. How long will it take for a 1-horsepower electric pump to pump all of the liquid in the trough from Example 5 over the wall?

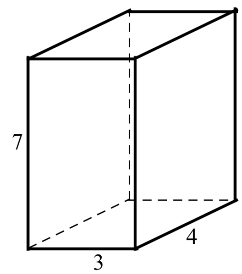
Solution. Power (P) is the rate at which work (W) is done, so:

$$P = \frac{W}{t} \Rightarrow t = \frac{W}{P} = \frac{\frac{14000}{3}\text{ ft-lbs}}{1\text{ hp}} = \frac{\frac{14000}{3}\text{ ft-lbs}}{550\ \frac{\text{ft-lbs}}{\text{sec}}} = \frac{280}{33}\text{ sec}$$

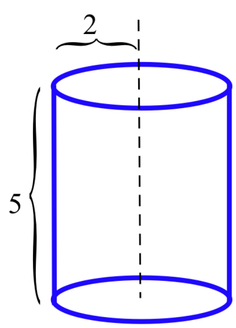
or about 8.5 seconds. ◀

5.4 Problems

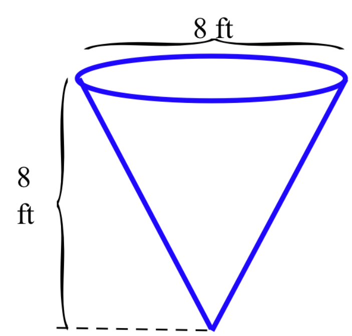
1. A tank 4 feet long, 3 feet wide and 7 feet tall (see below) is filled with water. How much work is required to pump the water out over the top edge of the tank?



2. A tank 4 feet long, 3 feet wide and 6 feet tall is filled with a oil with a density of 60 pounds per cubic foot.
   (a) How much work is needed to pump all of the oil over the top edge of the tank?
   (b) How much work is needed to pump the top 3 feet of oil over the top edge of the tank?
3. A tank 5 m long, 2 m wide and 4 m tall is filled with an oil of density 900 kg/m^3.
   (a) How much work is needed to pump all of the oil over the top edge of the tank?
   (b) How much work is needed to pump the top 10 m^3 of oil over the top edge of the tank?
   (c) How long does it take for a 200-watt pump to empty the tank?
4. A cylindrical aquarium with radius 2 feet and height 5 feet (see below) is filled with salt water (which has a density of 64 pounds per cubic foot).

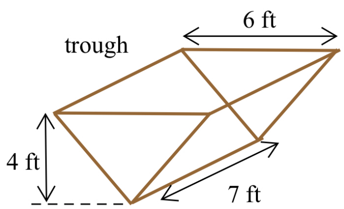


   (a) How much work is done pumping all of the water over the top edge of the aquarium?
   (b) How long does it take for a $\frac{1}{2}$-horsepower pump to empty the tank? A $\frac{1}{4}$-horsepower pump? Which pump does more work?
   (c) If the aquarium is only filled to a height of 4 feet with sea water, how much work is required to empty it?
5. A cylindrical barrel with a radius of 1 m and a height of 6 m is filled with water.
   (a) How much work is done pumping all of the water over the top edge of the barrel?
   (b) How much work is done pumping the top 1 m of water to a point 2 m above the top edge of the barrel?
   (c) How long will it take a $\frac{1}{2}$-horsepower pump to remove half of the water from the barrel?
6. The conical container shown below is filled with oats that weigh 25 pounds per ft^3.
   (a) How much work is done lifting all of the grain over the top edge of the cone?
   (b) How much work is required to lift the top 2 feet of grain over the top edge of the cone?



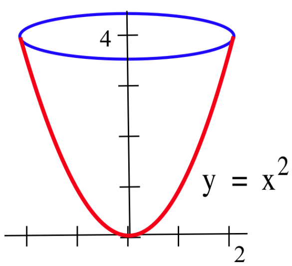
7. If you and a friend share the work equally in emptying the conical container in the previous problem, what depth of grain should the first person leave for the second person to empty?

8. A trough (see below) is filled with pig slop weighing 80 pounds per ft^3. How much work is done lifting all the slop over the top of the trough?



9. In the preceding problem, how much work is done lifting the top 14 ft^3 of slop over the top edge of the trough?

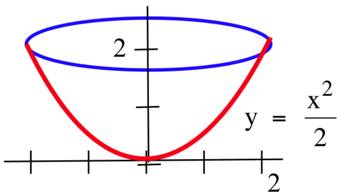
10. The parabolic container shown below (with a height of 4 m) is filled with water.



(a) How much work is done pumping all of the water over the top edge of the tank?

(b) How much work is done pumping all of the water to a point 3 m above the top of the tank?

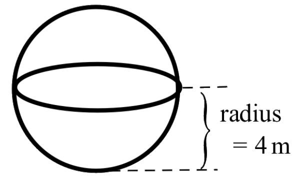
11. The parabolic container shown below (with a height of 2 m) is filled with water.



(a) How much work is done pumping all of the water over the top edge of the tank?

(b) How much work is done pumping all of the water to a point 3 m above the top of the tank?

12. A spherical tank with radius 4 m is full of water. How much work is done lifting all of the water to the top of the tank?



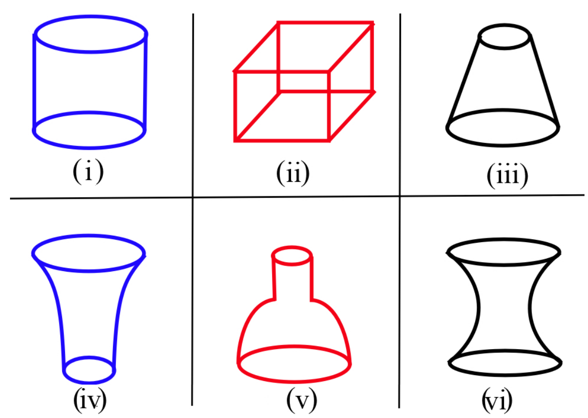
13. The spherical tank shown above is filled with water to a depth of 2 m. How much work is done lifting all of that water to the top of the tank?

14. A student said, "I've got a shortcut for these tank problems, but it doesn't always work. I figure the weight of the liquid and multiply that by the distance I have to move the 'middle point' in the water. It worked for the first five problems and then it didn't."

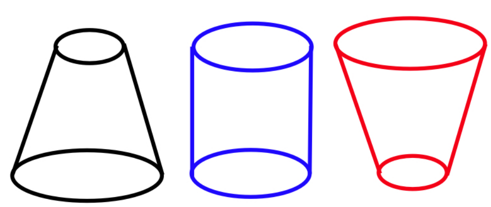
(a) Does this "shortcut" really give the right answer for the first five problems?

(b) How do the containers in the first five problems differ from the others?

(c) For which of the containers shown below will the "shortcut" work?



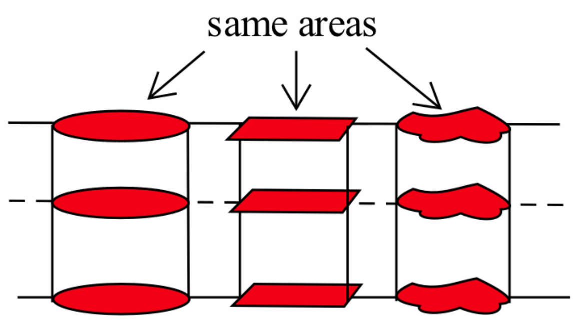
15. All of the containers shown below have the same height and hold the same volume of water.



(a) Which requires the most work to empty? Justify your response with a detailed explanation.

(b) Which requires the least work to empty?

16. All of the containers shown below have the same total height and at each height x above the ground they all have the same cross-sectional area.



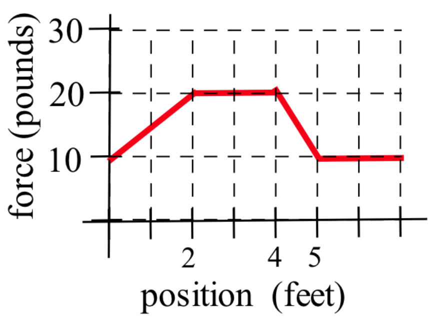
(a) Which requires the most work to empty? Justify your response with a detailed explanation.

(b) Which requires the least work to empty?

17. The figure below shows the force required to move a box along a rough surface. How much work is done pushing the box:

(a) from $x = 0$ to $x = 5$ feet?

(b) from $x = 3$ to $x = 5$ feet?



18. How much work is done pushing the box in the figure above:

(a) from $x = 3$ to $x = 7$ feet?

(b) from $x = 0$ to $x = 7$ feet?

19. A spring requires a force of $6x$ ounces to keep it stretched x inches past its natural length. How much work is done stretching the spring:

(a) from its natural length ($x = 0$) to 3 inches beyond its natural length?

(b) from its natural length to 6 inches beyond its natural length?

20. A spring requires a force of $5x$ dyn to keep it compressed x cm from its natural length. How much work is done compressing the spring:

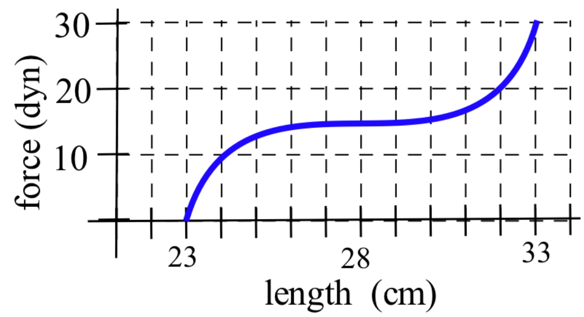
(a) 7 cm from its natural length?

(b) 10 cm from its natural length?

21. The figure below shows the force required to keep a spring that does not obey Hooke's Law stretched beyond its natural length of 23 cm. About how much work is done stretching it:

(a) from a length of 23 cm to a length 33 cm?

(b) from a length of 28 cm to a length 33 cm?



22. Approximately how much work is done stretching the defective spring in the previous problem:

(a) from a length of 23 cm to a length 26 cm?

(b) from a length of 30 cm to a length 35 cm?

23. A 3-kg object attached to a spring hung from the ceiling stretches the spring 15 cm. How much work is done stretching the spring 4 more cm?

24. A 2-lb fish stretches a spring 3 in. How much work is done stretching the spring 3 more inches?

25. A payload of mass 100 kg sits on the surface of the asteroid Ceres, a dwarf planet that is the largest object in the asteroid belt between Mars and Jupiter. Ceres has diameter 950 km and mass 896×10^{18} kg. How much work is required to lift the payload from the asteroid's surface to an altitude of (a) 10 km? (b) 100 km? (c) 500 km?

26. Calculate the amount of work required to lift *you* from the surface of the Earth's moon to an altitude of 100 km above the moon's surface. (The moon's radius is approximately 1,737.5 km and its mass is about 7.35×10^{22} kg.)

27. Calculate the amount of work required to lift *you* from the surface of the Earth's moon (see previous problem) to an altitude of:
    (a) 200 km.
    (b) 400 km.
    (c) 10,000 km.

28. An object located at the origin repels you with a force inversely proportional to your distance from the object (so that $f(x) = \frac{k}{x}$ where x is your distance from the object, measured in feet). When you are 10 feet away from the origin, the repelling force is 0.1 pound. How much work must you do to move:
    (a) from $x = 20$ to $x = 10$?
    (b) from $x = 10$ to $x = 1$?
    (c) from $x = 1$ to $x = 0.1$?

29. An object located at the origin repels you with a force inversely proportional to the square of your distance from the object (so that $f(x) = \frac{k}{x^2}$ where x is your distance from the object, measured in meters). When you are 10 m away from the origin, the repelling force is 0.1 N. How much work must you do to move:
    (a) from $x = 20$ to $x = 10$?
    (b) from $x = 10$ to $x = 1$?
    (c) from $x = 1$ to $x = 0.1$?

30. A student said "I've got a 'work along a line' shortcut that always seems to work. I figure the average force and then multiply by the total distance. Will it always work?"
    (a) Will it? Justify your answer. (Hint: What is the formula for "average force"?)
    (b) Is this a shortcut?

Work Along a Curved Path

If the location of a moving object is defined parametrically as $x = x(t)$ and $y = y(t)$ for $a \le t \le b$ (where t often represents time), and the force required to overcome friction at time t is given as $f(t)$, we can represent the work done moving along the (possibly curved) path as a definite integral. Partitioning $[a, b]$ into n subintervals of the form $[t_{k-1}, t_k]$, we can choose any c_k in $[t_{k-1}, t_k]$ and approximate the force required on $[t_{k-1}, t_k]$ by $f(c_k)$ so that the work done between $t = t_{k-1}$ and $t = t_k$ is approximately:

$$f(c_k) \cdot \sqrt{[\Delta x_k]^2 + [\Delta y_k]^2} = f(c_k) \cdot \sqrt{\left[\frac{\Delta x_k}{\Delta t_k}\right]^2 + \left[\frac{\Delta y_k}{\Delta t_k}\right]^2} \cdot \Delta t_k$$

The total work done between times $t = a$ and $t = b$ is then:

$$\sum_{k=1}^{n} f(c_k) \cdot \sqrt{\left[\frac{\Delta x_k}{\Delta t_k}\right]^2 + \left[\frac{\Delta y_k}{\Delta t_k}\right]^2} \cdot \Delta t_k \longrightarrow \int_a^b f(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

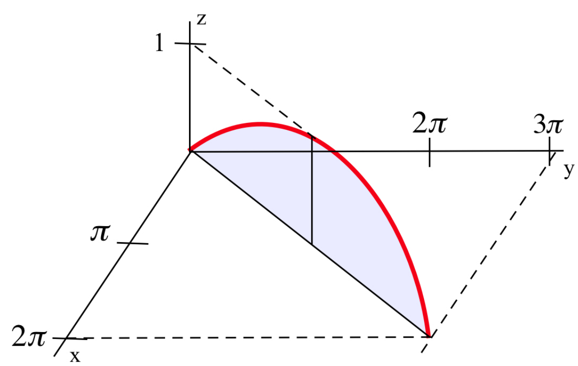
In 31–35, find the work done as an object is moved along the given parametric path (with distance measured in meters), where $f(t)$ (in newtons) is the force required at time t (in seconds). If necessary, approximate the value of the integral using technology.

31. $f(t) = t$, $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \le t \le 2\pi$

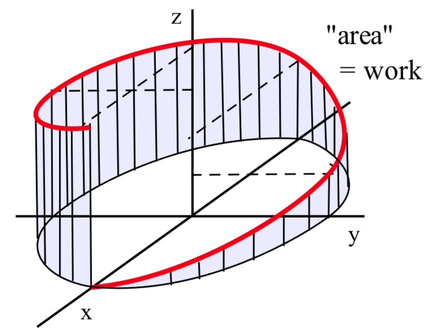
32. $f(t) = t$, $x(t) = t$, $y(t) = t^2$, $0 \le t \le 1$

33. $f(t) = t$, $x(t) = t^2$, $y(t) = t$, $0 \le t \le 1$

34. $f(t) = \sin(t)$, $x(t) = 2t$, $y(t) = 3t$, $0 \le t \le \pi$:



35. $f(t) = t$, $x(t) = \cos(t)$, $y(t) = \sin(t)$, $0 \le t \le 2\pi$:



(Can you find a geometric way to calculate the shaded area?)

5.4 Practice Answers

1. The work done lifting the person and the stretcher is:

$$(130 \text{ lb} + 10 \text{ lb}) \cdot (30 \text{ ft}) = 4200 \text{ ft-lbs}$$

The work done lifting a small piece of cable with length Δx ft at an initial height of x feet above the ground is:

$$\left(2\ \frac{\text{lb}}{\text{ft}}\right)(\Delta x \text{ ft})\Big((30 - x) \text{ ft}\Big) = (60 - 2x)\Delta x \text{ ft-lbs}$$

so the work done lifting the cable is:

$$\sum_{k=1}^{n} (60 - 2x)\Delta x \longrightarrow \int_0^{30} (60 - 2x)\, dx = 900 \text{ ft-lbs}$$

and the total work is $4200 + 900 = 5100$ ft-lbs.

2. The work done lifting the person and the stretcher is:

$$(50 \text{ kg} + 5 \text{ kg}) \cdot \left(9.81\ \frac{\text{m}}{\text{sec}^2}\right) \cdot (30 \text{ m}) = (55 \text{ N})\,(30 \text{ m}) = 16186.5 \text{ J}$$

The work done lifting a small piece of cable with length Δx m at an initial height of x m above the ground is:

$$\left(\frac{1}{3}\ \frac{\text{kg}}{\text{m}}\right)\left(9.81\ \frac{\text{m}}{\text{sec}^2}\right)(\Delta x \text{ m})\Big((30 - x) \text{ m}\Big) = 3.27(30 - x)\Delta x \text{ J}$$

so the work done lifting the cable is:

$$\sum_{k=1}^{n} 3.27(30 - x)\Delta x \longrightarrow 3.27\int_0^{30} (30 - x)\, dx = 1471.5 \text{ J}$$

and the total work is $16186.5 + 1471.5 = 17658$ J.

3. The total work done is approximately

$$\left[\pi(1.4)^2(3.5) + \pi(1.6)^2(4.5) + \pi(1.5)^2(5.5) + \pi(1.0)^2(6.5)\right](0.5787)$$

or 67.73 oz-in $\approx$ 0.35 ft-lbs.

4. We can use the same integral as in the solution to Example 4, but instead integrate from $y = 3$ to $y = 6$:

$$W = 109\pi \int_3^6 \left(10y^2 - y^3\right)\, dy = 109\pi \left[\frac{10}{3}y^3 - \frac{1}{4}y^4\right]_3^6$$
$$= 109\pi \left[(720 - 324) - \left(90 - \frac{81}{4}\right)\right] = 35561.25\pi \text{ erg}$$

or about 111,719 erg $=$ 0.0111719 J.

5. The total amount of liquid in the trough is $\frac{1}{2} \cdot 4 \cdot 2 \cdot 5 = 20$ ft^3, so we need to lift the top 10 ft^3 of liquid out of the trough. To find the height separating the bottom 10 ft^3 of liquid from the rest, we can recall that (from our similar-triangles computation), the width at height h is $w = \frac{h}{2}$, so the volume of liquid between height $y = 0$ and height $y = h$ is:

$$10 = \frac{1}{2} \cdot h \cdot \frac{h}{2} \cdot 5 \;\Rightarrow\; h^2 = 8 \;\Rightarrow\; h = 2\sqrt{2}$$

The work to lift the top 10 ft^3 of liquid is thus:

$$175 \int_{2\sqrt{2}}^4 \left(6y - y^2\right)\, dy = 175 \left[3y^2 - \frac{1}{3}y^3\right]_{2\sqrt{2}}^4$$
$$= 175 \left[\left(48 - \frac{64}{3}\right) - \left(24 - \frac{16\sqrt{2}}{3}\right)\right]$$

or about 1,786.6 ft-lbs.

6. We can use the same integral as in the solution to Example 6, but instead integrate from $x = 5$ to $x = 10$:

$$\int_5^{10} 7x\, dx = \left[\frac{7}{2}x^2\right]_5^{10} = 350 - \frac{175}{2} = 262.5 \text{ in-lbs} = 21.875 \text{ ft-lbs}$$

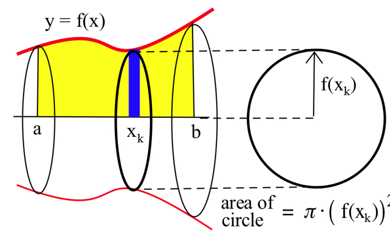
7. According to Hooke's Law, 2 lb $= k \cdot (8 \text{ in} - 3 \text{ in}) \;\Rightarrow\; k = \dfrac{2}{5}$, so stretching the spring from $5 - 3 = 2$ in to $10 - 3 = 7$ in beyond its natural length requires:

$$\int_2^7 \frac{2}{5}x\, dx = \left[\frac{1}{5}x^2\right]_2^7 = 9 \text{ in-lb} = \frac{3}{4} \text{ ft-lb}$$

8. The work required to lift the payload from $x = 2R$ to $x = 3R$ is:

$$\int_{2R}^{3R} \frac{GMm}{x^2}\, dx = GMm \left[\frac{-1}{x}\right]_{2R}^{3R} = GMm \left[\frac{-1}{3R} + \frac{1}{2R}\right] = \frac{GMm}{6R}$$

5.5 Volumes: Tubes



In Section 5.2, we devised the "disk" method to find the volume swept out when a region is revolved about a line. To find the volume swept out when revolving a region about the x-axis (see margin), we made cuts perpendicular to the x-axis so that each slice was (approximately) a "disk" with volume $\pi\,(\text{radius})^2 \cdot (\text{thickness})$. Adding the volumes of these slices together yielded a Riemann sum. Taking a limit as the thicknesses of the slices approached 0, we obtained a definite integral representation for the exact volume that had the form:

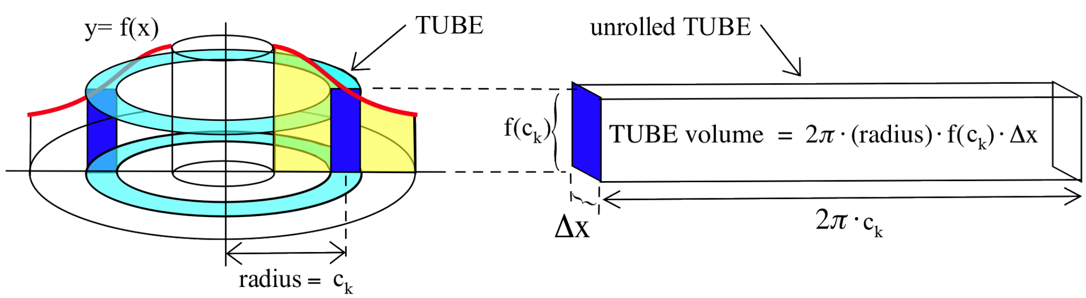
$$\int_a^b \pi\,[f(x)]^2\,dx$$

The disk method, while useful in many circumstances, can be cumbersome if we want to find the volume when a region defined by a curve of the form $y = f(x)$ is revolved about the y-axis or some other vertical line. To revolve the region about the y-axis, the disk method requires that we rewrite the original equation $y = f(x)$ as $x = g(y)$. Sometimes this is easy: if $y = 3x$ then $x = \frac{y}{3}$. But sometimes it is not easy at all: if $y = x + e^x$, then we cannot solve for x as an elementary function of y.

Refer to Examples 6(b) and 6(c) from Section 5.2 to refresh your memory.

The "Tube" Method

Partition the x-axis (as we did in the "disk" method) to cut the region into thin, almost-rectangular vertical "slices." When we revolve one of these slices about the y-axis (see below), we can approximate the volume of the resulting "tube" by cutting the "wall" of the tube and rolling it out flat:



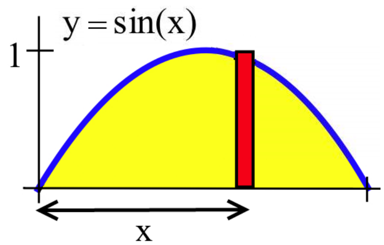
to get a thin, solid rectangular box. The volume of the tube is approximately the same as the volume of the solid box:

$$\begin{aligned} V_{\text{tube}} \approx V_{\text{box}} &= (\text{length}) \cdot (\text{height}) \cdot (\text{thickness}) \\ &= (2\pi \cdot [\text{radius}]) \cdot (\text{height}) \cdot (\Delta x_k) \\ &= (2\pi c_k)\Big(f(c_k)\Big) \cdot \Delta x_k \end{aligned}$$

where c_k is (as usual) any point chosen from the interval $[x_{k-1}, x_k]$.

The volume swept out when we revolve the whole region about the y-axis is (approximately) the sum of the volumes of these "tubes," which is a Riemann sum that converges to a definite integral:

$$\sum_{k=1}^{n} (2\pi c_k)\left(f(c_k)\right) \cdot \Delta x_k \longrightarrow \int_a^b 2\pi x \cdot f(x)\, dx$$



Example 1. Use a definite integral to represent the volume of the solid generated by rotating the region between the graph of $y = \sin(x)$ (for $0 \le x \le \pi$) and the x-axis around the y-axis.

Solution. Slicing this region vertically (see margin for a representative slice), yields slices with width Δx and height $\sin(x)$. Rotating a slice located x units away from the y-axis results in a "tube" with volume:

$$2\pi\,(\text{radius})\,(\text{height})\,(\text{thickness}) = 2\pi\,(x)\left(\sin(x)\right)\Delta x$$

where the radius of the tube (x) is the distance from the slice to the y-axis and the height of the tube is the height of the slice ($\sin(x)$). Adding the volumes of all such tubes yields a Riemann sum that converges to a definite integral:

$$\int_0^{\pi} 2\pi\,(\text{radius})\,(\text{height})\; dx = \int_0^{\pi} 2\pi x \sin(x)\, dx$$

You'll learn how to find an antiderivative for $x \sin(x)$ in Section 8.2 or, for now, you can look for this pattern in Appendix I.

We don't (yet) know how to find an antiderivative for $x \sin(x)$ but we can use technology (or a numerical method from Section 4.9) to compute the value of the integral, which turns out to be $2\pi^2 \approx 19.74$. ◀

Practice 1. Use a definite integral to compute the volume of the solid generated by rotating the region in the first quadrant bounded by $y = 4x - x^2$ about the y-axis.

If we had sliced the region in Example 1 horizontally instead of vertically, the rotated slices would have resulted in "washers"; applying the "washer" method from Section 5.2 yields the integral:

$$\int_0^1 \pi\left[(\pi - \arcsin(y))^2 - (\arcsin(y))^2\right] dy$$

Furthermore, the washer-method integral in this situation is more challenging to set up than the integral using the tube method, so the tube method is the most efficient choice on all counts.

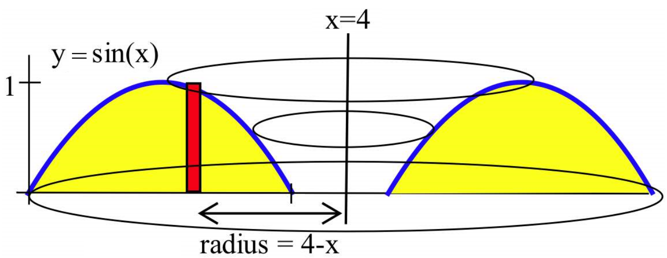
The value of this integral is also $2\pi^2$, but finding an antiderivative for this integrand will be much more challenging than finding an antiderivative for $x \sin(x)$.

Rotating About Other Axes

The "tube" method extends easily to solids generated by rotating a region about any vertical line (not just the y-axis).

Example 2. Use a definite integral to represent the volume of the solid generated by rotating the region between the graph of $y = \sin(x)$ (for $0 \le x \le \pi$) and the x-axis around the line $x = 4$.

Solution. The region is the same as the one in Example 1, but here we're rotating that region about a different vertical line:



Vertical slices again generate tubes when rotated about $x = 4$; the only difference here is that the radius for a slice located x units away from y-axis is now $4 - x$ (the distance from the axis of rotation to the slice). The volume integral becomes:

$$\int_0^\pi 2\pi\,(\text{radius})\,(\text{height})\;dx = \int_0^\pi 2\pi(4-x)\cdot\sin(x)\,dx$$

which turns out to be $2\pi(8-\pi) \approx 4.8584$. ◄

Use technology (or a table of integrals) to verify this numerical result.

Practice 2. Use a definite integral to compute the volume of the solid generated by rotating the region in the first quadrant bounded by $y = 4x - x^2$ about the line $x = -7$.

More General Regions

The "tube" method also extends easily to more general regions.

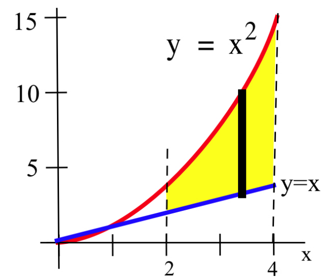
> **Volumes of Revolved Regions ("Tube Method")**
>
> If the region constrained by the graphs of $y = f(x)$ and $y = g(x)$ and the interval $[a, b]$ is revolved about a vertical line $x = c$ that does not intersect the region
>
> then the volume of the resulting solid is:
>
> $$V = \int_a^b 2\pi\cdot|x-c|\cdot|f(x)-g(x)|\;dx$$

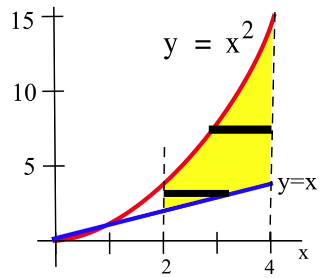
Many textbooks refer to this method as the "method of cylindrical shells" or the "shell method," but "cylindrical shells" is a mouthful (compared with "tube") and "shell method" is not precise, as shells are not necessarily cylindrical.

The absolute values appear in the general formula because the radius and the height are both distances, hence both must be positive.

You can ensure that these ingredients in your tube-method integral will be positive by always subtracting smaller values from larger values: think "right − left" for x-values and "top − bottom" for y-values.

Example 3. Compute the volume of the solid generated by rotating the region between the graphs of $y = x$ and $y = x^2$ for $2 \le x \le 4$ around the y-axis using (a) vertical slices and (b) horizontal slices.





Solution. (a) Vertical slices (see margin) result in tubes when rotated about the y-axis, and a slice x units away from the y-axis results in a tube of radius x and height $x^2 - x$, so the volume of the solid is:

$$\int_2^4 2\pi x \left[x^2 - x\right] dx = 2\pi \int_2^4 \left[x^3 - x^2\right] dx = 2\pi \left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_2^4$$
$$= 2\pi \left[\left(64 - \frac{64}{3}\right) - \left(4 - \frac{8}{3}\right)\right] = \frac{248\pi}{3}$$

or about 259.7. (b) Horizontal slices result in washers when rotated about the y-axis, but we have a new problem: the lower slices (where $2 \le y \le 4$) extend from the line $x = 2$ on the left to the line $y = x$ on the right, while the upper slices (where $4 \le y \le 16$) extend from the parabola $y = x^2$ on the left to the line $x = 4$ on the right. This requires us to use *two* integrals to compute the volume:

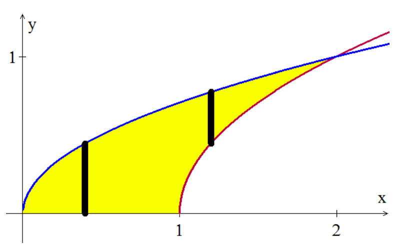
$$\int_{y=2}^{y=4} \pi \left[y^2 - 2^2\right] dy + \int_{y=4}^{y=16} \pi \left[4^2 - (\sqrt{y})^2\right] dy$$

Evaluating these integrals is straightforward, but setting them up was more time-consuming than using the tube method.

Evaluating these integrals also results in a volume of $\frac{248\pi}{3} \approx 259.7$. ◀

Practice 3. Find the volume of the solid formed by rotating the region between the graphs of $y = x$ and $y = x^2$ for $2 \le x \le 4$ around $x = 13$.

Practice 4. Compute the volume of the solid generated by rotating the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$, $y = x + 1$ and $x = 4$ around (a) the y-axis (b) the x-axis.



Example 4. Compute the volume of the solid swept out by rotating the region in the first quadrant between the graphs of $y = \sqrt{\frac{x}{2}}$ and $y = \sqrt{x - 1}$ about the x-axis.

Solution. Graphing the region (see margin), it is apparent that the curves intersect where:

$$\sqrt{\frac{x}{2}} = \sqrt{x - 1} \Rightarrow \frac{x}{2} = x - 1 \Rightarrow x = 2$$

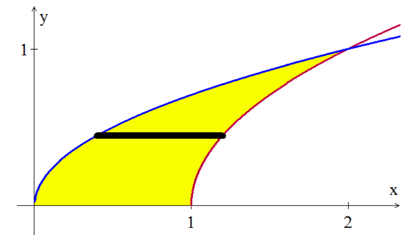
Slicing the region vertically results in two cases: when $0 \le x \le 1$, the slice extends from the x-axis to the curve $y = \sqrt{\frac{x}{2}}$; when $1 \le x \le 2$, the slice extends from $y = \sqrt{x - 1}$ to $y = \sqrt{\frac{x}{2}}$. Rotating the first type of slice about the x-axis results in a disk; rotating the second type of slice about the x-axis results in a washer. Using the disk method for the first interval and the washer method for the second interval, the volume of the solid is:

Both types of slices are perpendicular to the x-axis, so the width of each slice is of the form Δx and our integrals should involve dx.

$$\int_0^1 \pi \left[\sqrt{\frac{x}{2}}\right]^2 dx + \int_1^2 \pi \left[\left(\sqrt{\frac{x}{2}}\right)^2 - \left(\sqrt{x - 1}\right)^2\right] dx$$

Evaluating these integrals is straightforward:

$$\pi \int_0^1 \frac{x}{2}\, dx + \pi \int_1^2 \left[\frac{x}{2} - (x-1)\right] dx = \pi \left[\frac{x^2}{4}\right]_0^1 + \pi \left[x - \frac{x^2}{4}\right]_1^2 = \frac{\pi}{2}$$



If you had instead sliced the region horizontally, you would only need one type of slice (see margin). Rotating a horizontal slice around the x-axis results in a tube. Because this slice is perpendicular to the y-axis, the thickness of the slice is of the form Δy, so the tube-method integral will include a dy and we will need to formulate the radius and "height" of the tube in terms of y. The radius of the slice is merely y, the distance between the slice and the x-axis. The "height" of the slice is its length, which is the distance between the two curves. The left-hand curve is:

$$y = \sqrt{\frac{x}{2}} \;\Rightarrow\; y^2 = \frac{x}{2} \;\Rightarrow x = 2y^2$$

and the right-hand curve is:

$$y = \sqrt{x-1} \;\Rightarrow\; y^2 = x - 1 \;\Rightarrow\; x = y^2 + 1$$

so the distance between the two curves is:

$$\left(y^2+1\right) - \left(2y^2\right) = 1 - y^2$$

This application of the tube method rotates a horizontal slice around a horizontal axis; in previous tube-method applications we have only rotated a vertical slice about a vertical axis. Either option results in a tube, and the general formula on page 433 can be further extended to this new situation — as we have done here — by swapping the roles of x and y,

The curves intersect where: $y^2 + 1 = 2y^2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$; from the graph we can see that the bottom of the region corresponds to $y = 0$ and the top of the region is at $y = 1$. Applying the tube method, the volume of the solid is:

$$\int_{y=0}^{y=1} 2\pi y \cdot \left[1 - y^2\right] dy = 2\pi \int_0^1 \left[y - y^3\right] dy = 2\pi \left[\frac{y}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$

which agrees with the result above from the disk+washer method. ◀

Practice 5. Compute the volume of the solid swept out by rotating the region in the first quadrant between the graphs of $y = \sqrt{\frac{x}{2}}$ and $y = \sqrt{x-1}$ about (a) the line $x = 5$ (b) the line $y = 5$.

Which Method Is Best?

In theory, both the washer method and the tube method will work for any volume-of-revolution problem involving a horizontal or vertical axis. In practice, however, one of these methods is usually easier to use than the other — but which one is easier depends on the particular region and type of axis. As we have seen, challenges may include:

We will investigate a method for computing volumes of solids formed by rotating a region around "tilted" axes in Section 5.6.

- The necessity to split the region into two (or more) pieces, resulting in two (or more) integrals.

- The difficulty (or impossibility) of solving an equation of the form $y = f(x)$ for x or an equation of the form $x = g(y)$ for y.
- The difficulty (or impossibility) of finding an antiderivative for the resulting integrand.

With experience (and lots of practice) you will begin to develop an intuition for which method might be the best choice for a particular situation. Sketching the region along with representative horizontal and vertical slices is a vital first step.

The method that avoids the need to split the region up into more than one piece is often—but not always—the superior choice. Avoiding the need to find an inverse function for a boundary curve should also be a priority. Finally, if you need an exact value and one method results in a challenging antiderivative search, start over and try the other method.

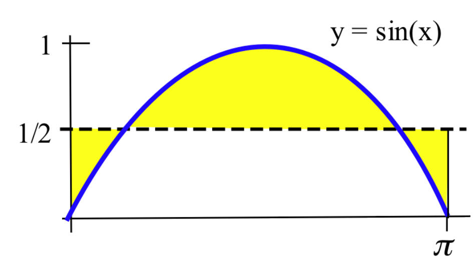
5.5 Problems

In Problems 1–6, sketch the region and calculate the volume swept out when the region is revolved about the specified vertical line.

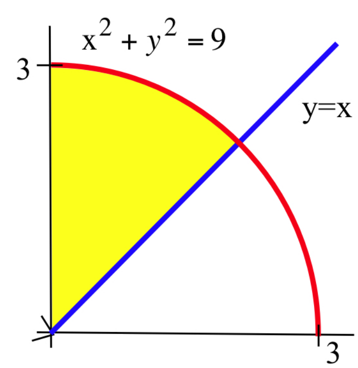
1. The region in the first quadrant between the curve $y = \sqrt{1-x^2}$ and the x-axis is rotated about the y-axis.
2. The region in the first quadrant between the curve $y = 2x - x^2$ and the x-axis is rotated about the y-axis.
3. The region in the first quadrant between between $y = 2x$ and $y = x^2$ for $0 \le x \le 3$ is rotated about the line $x = 4$.
4. The region in the first quadrant between the curve $y = \dfrac{1}{1+x^2}$, the x-axis and the line $x = 3$ is rotated about the y-axis.
5. The region between $y = \dfrac{1}{x}$, $y = \dfrac{1}{3}$ and $x = 1$ is rotated about the line $x = 5$.
6. The region between $y = x$, $y = 2x$, $x = 1$ and $x = 3$ is rotated about the line $x = 1$.

In Problems 7–11, use a definite integral to represent the volume swept out when the given region is revolved about the y-axis, then use technology to evaluate the integral.

7. The region in the first quadrant between the graphs of $y = \ln(x)$, $y = x$ and $x = 4$.
8. The region in the first quadrant between the graphs of $y = e^x$, $y = x$ and $x = 2$.
9. The region between $y = x^2$ and $y = 6 - x$ for $1 \le x \le 4$.
10. The shaded region in the figure below.



11. The shaded region in the figure below.



5.5 Problems

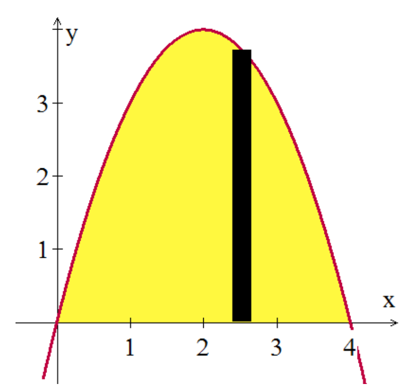
In Problems 12–30, set up an integral to calculate the volume swept out when the region between the given curves is rotated about the specified axis, using any appropriate method (disks, washers, tubes). If possible, work out an exact value of the integral; otherwise, use technology to find an approximate numerical value.

12. $y = x$, $y = x^4$, about the y-axis
13. $y = x^2$, $y = x^4$, about the y-axis
14. $y = x^2$, $y = x^4$, about the x-axis
15. $y = \sin(x^2)$, $y = 0$, $x = 0$, $x = \sqrt{\pi}$, about $x = 0$
16. $y = \cos(x^2)$, $y = 0$, $x = 0$, $x = \frac{\sqrt{\pi}}{2}$, about $x = 0$
17. $y = \dfrac{1}{\sqrt{1-x^2}}$, $y = 0$, $x = 0$, $x = \frac{1}{2}$, about $x = 0$
18. $y = \dfrac{1}{\sqrt{1-x^2}}$, $y = 0$, $x = 0$, $x = \frac{1}{2}$, about $y = 0$
19. $y = x$, $y = x^4$, about $x = 3$
20. $y = x$, $y = x^4$, about $y = 3$
21. $y = x$, $y = x^4$, about $y = -3$
22. $y = x$, $y = x^4$, about $x = -3$
23. $y = \dfrac{1}{1+x^2}$, $y = 0$, $x = 0$, $x = 1$ about $x = 2$
24. $y = \dfrac{1}{1+x^2}$, $y = 0$, $x = 1$, $x = \sqrt{3}$, about $x = 2$
25. $y = \dfrac{1}{1+x^2}$, $y = 1$, $x = 1$, about $x = -2$
26. $y = \dfrac{1}{1+x^2}$, $y = \frac{1}{2}$, about $x = 1$
27. $y = \sqrt{x-2}$, $y = \sqrt{x-1}$, $y = 0$, $x = 3$, about $x = 4$
28. $y = \sqrt{x-2}$, $y = \sqrt{x-1}$, $y = 0$, $x = 3$, about $x = -4$
29. $y = \sqrt{x-2}$, $y = \sqrt{x-1}$, $y = 0$, $x = 3$, about $y = 4$
30. $y = \sqrt{x-2}$, $y = \sqrt{x-1}$, $y = 0$, $x = 3$, about $y = -4$

5.5 Practice Answers

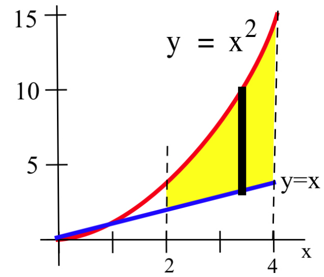
1. Graph the region (see margin) and note that the curve $y = 4x - x^2$ intersects the x-axis where $4x - x^2 = 0 \Rightarrow x(4-x) = 0 \Rightarrow x = 0$ or $x = 4$. Rotating a vertical slice around the y-axis results in a tube with radius x (the distance between the slice and the y-axis) and height $4x - x^2$ so the volume of the solid is:

$$\int_0^4 2\pi x \left(4x - x^2\right)\,dx = 2\pi \int_0^4 \left[4x^2 - x^3\right]\,dx$$
$$= 2\pi \left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4 = \frac{128\pi}{3} \approx 134$$



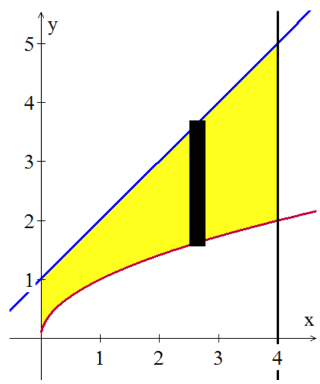
2. The region here is identical to the region in Practice 1, but we are now rotating a slice around the axis $x = -7$, so the radius of the resulting tube is $x - (-7) = x + 7$ (the distance from the slice at location x to the axis of rotation). The volume of the solid is therefore:

$$\int_0^4 2\pi(x+7)\left(4x - x^2\right)\,dx = 2\pi \int_0^4 \left[28x - 3x^2 - x^3\right]\,dx$$
$$= 2\pi \left[14x^2 - x^3 - \frac{1}{4}x^4\right]_0^4 = 192\pi \approx 603$$



3. Rotating a vertical slice (see margin figure) about the line $x = 13$ results in a tube with radius $13 - x$ and height $x^2 - x$, so the volume of the solid is:

$$\int_2^4 2\pi(13-x)(x^2-x)\,dx = 2\pi\int_2^4\left[-13x+14x^2-x^3\right]\,dx$$
$$= 2\pi\left[-\frac{13}{2}x^2+\frac{14}{3}x^3-\frac{1}{4}x^4\right]_2^4 = 2\pi\left[\frac{392}{3}-\frac{10}{3}\right] = \frac{764\pi}{3} \approx 800$$

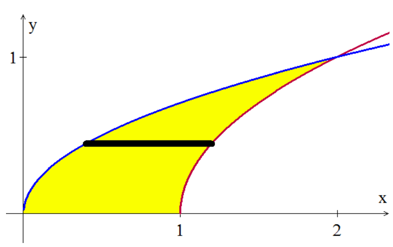


4. Graph the region (see margin) and draw a representative vertical slice. (Horizontal slices would require splitting the region into two pieces — why?) (a) Rotating the vertical slice about the y-axis results in a tube of radius x (the distance from the slice to the y-axis) and height $(x+1)-\sqrt{x}$, and the region sits between $x = 0$ and $x = 4$ so the volume of the solid is:

$$\int_0^4 2\pi x\left[x+1-x^{\frac{1}{2}}\right]\,dx = 2\pi\int_0^4\left[x^2+x-x^{\frac{3}{2}}\right]\,dx$$
$$= 2\pi\left[\frac{1}{3}x^3+\frac{1}{2}x^2-\frac{2}{5}x^{\frac{5}{2}}\right]_0^4 = \frac{496\pi}{15} \approx 104$$

(b) Rotating the vertical slice around the x-axis results in a washer with big radius $x+1$ (the distance from the x-axis to the curve farthest from the x-axis) and small radius $\sqrt{x}$ (the distance from the x-axis to the closer curve) so the volume of the solid is:

$$\int_0^4 \pi\left[(x+1)^2-\left(\sqrt{x}\right)^2\right]\,dx = \pi\int_0^4\left[x^2+x+1\right]\,dx = \frac{100\pi}{3} \approx 105$$



5. This region is the same as the one in Example 4, where it was apparent that slicing horizontally resulted in a single type of slice (compared with vertical slices, which required us to split the region into two pieces). (a) Rotating a horizontal slice around the vertical line $x = 5$ results in washers with thickness Δy (so our integral will involve dy), big radius $5 - 2y^2$ (the distance between the axis of rotation and the farthest curve) and small radius $5 - \left(y^2+1\right) = 4 - y^2$ (the distance between the axis of rotation and closest curve). Applying the washer method, the volume of the solid is:

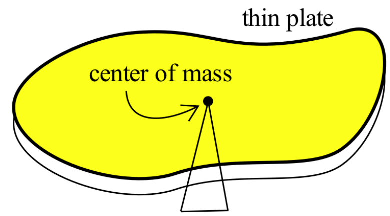
$$\int_0^1 \pi\left[\left(5-2y^2\right)^2-\left(4-y^2\right)^2\right]\,dy = \frac{14\pi}{5} \approx 8.8$$

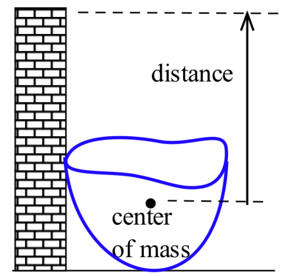
(b) Rotating a horizontal slice around the horizontal line $y = 5$ results in a tube of radius $5 - y$ (the distance between the slice and the axis of rotation) and "height" $1 - y^2$ (the length of the slice). Applying the tube method, the volume of the solid is:

$$\int_0^1 2\pi(5-y)\left(1-y^2\right)\,dy = \frac{37\pi}{6} \approx 19.4$$

5.6 *Moments and Centers of Mass*

This section develops a method for finding the center of mass of a thin, flat shape — the point at which the shape will balance without tilting (see margin). Centers of mass are important because in many applied situations an object behaves as though its entire mass is located at its center of mass. For example, the work required to pump the water in a tank to a higher point is the same as the work required to move a small object with the same mass located at the tank's center of mass to the higher point (see margin), a much easier problem (if we know the mass and the center of mass of the water). Volumes and surface areas of solids of revolution can also become easy to calculate if we know the center of mass of the region being revolved.



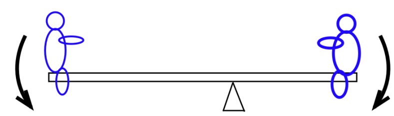


work = (force)(distance)

Point-Masses in One Dimension

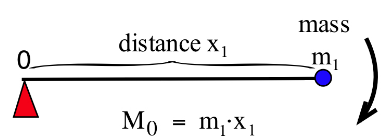
Before investigating the centers of mass of complicated regions, we consider **point-masses** (and systems of point-masses), first in one dimension and then in two dimensions.

Two people with different masses can position themselves on a seesaw so that the seesaw balances (see margin). The person on the right causes the seesaw to "want to turn" clockwise about the fulcrum, and the person on the left causes it to "want to turn" counterclockwise. If these two "tendencies" are equal, the seesaw will balance on the fulcrum. A measure of this tendency to turn about the fulcrum is called the **moment** about the fulcrum of the system, and its magnitude is the product of the mass and the distance from the mass to the fulcrum.



In this seesaw example, we need to imagine that the seesaw is constructed using a very lightweight — yet sturdy — substance, so that its mass is negligible compared with the masses of the two people.

In general, the **moment about the origin**, M_0, produced by a mass m_1 at a location x_1 is $m_1 \cdot x_1$, the product of the mass and the "signed distance" of the point-mass from the origin (see margin). For a **system** of n masses $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$, respectively, the total mass of the system is:



$$m = m_1 + m_2 + \cdots + m_n = \sum_{k=1}^{n} m_k$$

and the moment about the origin of the system is:

$$M_0 = m_1 \cdot x_1 + m_2 \cdot x_2 + \cdots + m_n \cdot x_n = \sum_{k=1}^{n} m_k \cdot x_k$$

If the moment about the origin is positive, then the system "tends to rotate" clockwise about the origin. If the moment about the origin is negative, then the system "tends to rotate" counterclockwise about the origin. If the moment about the origin is zero, then the system does not tend to rotate in either direction about the origin: it balances on a fulcrum located at the origin.

The **moment about a point $x = p$**, M_p, produced by a mass m_1 at location $x = x_1$ is the product of the mass and the signed distance of x_1 from the point p: $m_1 \cdot (x_1 - p)$. The moment about a point $x = p$ produced by masses $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$, respectively, is:

$$M_p = m_1(x_1 - p) + m_2(x_2 - p) + \cdots + m_n(x_n - p) = \sum_{k=1}^{n} m_k(x_k - p)$$

The point at which a system of point-masses balances is called the **center of mass** of the system, written $\overline{x}$ (pronounced "x-bar"). Because the system balances at $x = \overline{x}$, the moment about $\overline{x}$, $M_{\overline{x}}$, must be 0. Using this fact (and summation properties), we obtain a formula for $\overline{x}$:

You have seen this "bar" notation before, in conjunction with the average value of a function. Here we can think of $\overline{x}$ as a "weighted average".

$$\begin{aligned} 0 = M_{\overline{x}} &= \sum_{k=1}^{n} m_k \cdot (x_k - \overline{x}) = \left[\sum_{k=1}^{n} m_k \cdot x_k\right] - \left[\sum_{k=1}^{n} m_k \cdot \overline{x}\right] \\ &= \left[\sum_{k=1}^{n} m_k \cdot x_k\right] - \overline{x} \cdot \left[\sum_{k=1}^{n} m_k\right] = M_0 - \overline{x} \cdot m \end{aligned}$$

We can factor $\overline{x}$ out of the second sum because it is constant.

so $\overline{x} \cdot m = M_0$ and solving for $\overline{x}$ yields the following formula.

> The **center of mass of a system of point-masses** $m_1, m_2, \ldots, m_n$ at locations $x_1, x_2, \ldots, x_n$ is:
>
> $$\overline{x} = \frac{M_0}{m} = \frac{\sum_{k=1}^{n} m_k \cdot x_k}{\sum_{k=1}^{n} m_k}$$

A single point-mass with mass m (the total mass of the system) located at $\overline{x}$ (the center of mass of the system) produces the same moment about any point on the line as the whole system:

$$\begin{aligned} M_p &= \sum_{k=1}^{n} m_k(x_k - p) = \left[\sum_{k=1}^{n} m_k x_k\right] - p\left[\sum_{k=1}^{n} m_k\right] = M_0 - pm \\ &= m\left(\frac{M_0}{m} - p\right) = m(\overline{x} - p) \end{aligned}$$

For many purposes, we can think of the mass of the entire system as being "concentrated at $\overline{x}$."

k	m_k	x_k
1	2	-3
2	3	4
3	1	6
4	5	-2
5	3	4

Example 1. Find the center of mass of the system consisting of the first three point-masses listed in the margin table.

Solution. $m = 2 + 3 + 1 = 6$ and $M_0 = (2)(-3) + (3)(4) + (1)(6) = 12$ so:

$$\overline{x} = \frac{M_0}{m} = \frac{12}{6} = 2$$

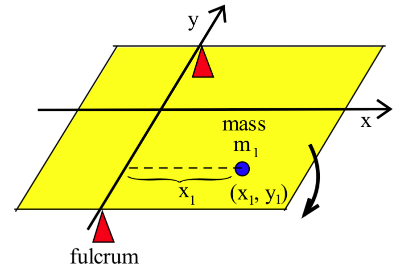
The system of three point-masses will balance on a fulcrum at $x = 2$. ◀

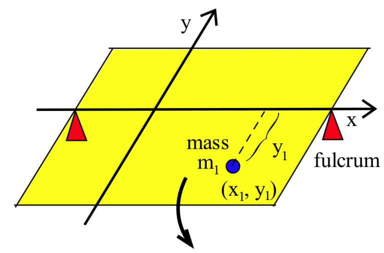
Practice 1. Find the center of mass of the system consisting of the last three point-masses listed in the margin table.

Point-Masses in Two Dimensions

The concepts of moments and centers of mass extend nicely from one dimension to a system of masses located at points in a plane. For a "knife edge" fulcrum located along the y-axis (see margin), the moment of a point-mass with mass m_1 located at the point (x_1, y_1) is the product of the mass and the signed distance of the point-mass from the y-axis: $m_1 \cdot x_1$. This "tendency to rotate about the y-axis" is called the **moment about the y-axis**, written M_y. Here, $M_y = m_1 \cdot x_1$. Similarly, a point-mass with mass m_1 located at the point (x_1, y_1) has a **moment about the x-axis** (see margin): $M_x = m_1 \cdot y_1$.

As in the seesaw example, we need to imagine that the point-masses are sitting on a thin—yet strong—plate of negligible mass compared with the point-masses.





For a **system of masses** m_k located at the points (x_k, y_k), the total mass of the system is (as before):

$$m = m_1 + m_2 + \cdots + m_n = \sum_{k=1}^{n} m_k$$

while the moment about the y-axis is:

$$M_y = m_1 \cdot x_1 + m_2 \cdot x_2 + \cdots + m_n \cdot x_n = \sum_{k=1}^{n} m_k \cdot x_k$$

and the moment about the x-axis is:

$$M_x = m_1 \cdot y_1 + m_2 \cdot y_2 + \cdots + y_n \cdot x_n = \sum_{k=1}^{n} m_k \cdot y_k$$

At first, it may seem confusing that the formula for M_y would involve x and the formula for M_x would involve y, but keep in mind that an equation for the y-axis is $x = 0$, so we could write the moment about the y-axis as $M_{x=0}$ and the moment about the x-axis as $M_{y=0}$.

The **center of mass** of this two-dimensional system is a point $(\overline{x}, \overline{y})$ such that any line that passes through this point is a "balancing fulcrum" for the system. So we need the moment about any such line—including $x = \overline{x}$ and $y = \overline{y}$—to be zero:

If we can find such a point, then the system will balance on a single "point-fulcrum" located at the center of mass.

$$0 = M_{x=\overline{x}} = \sum_{k=1}^{n} m_k (x_k - \overline{x}) = \left[\sum_{k=1}^{n} m_k \cdot x_k\right] - \overline{x}\left[\sum_{k=1}^{n} m_k\right] = M_y - \overline{x}m$$

so $\overline{x} = \dfrac{M_y}{m}$, and similar arithmetic shows that $\overline{y} = \dfrac{M_x}{m}$.

> The **center of mass of a system of point-masses** $m_1, m_2, \ldots, m_n$ at locations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is the point $(\overline{x}, \overline{y})$ where:
>
> $$\overline{x} = \frac{M_y}{m} = \frac{\sum_{k=1}^{n} m_k \cdot x_k}{\sum_{k=1}^{n} m_k} \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{\sum_{k=1}^{n} m_k \cdot y_k}{\sum_{k=1}^{n} m_k}$$

A single point-mass with mass m (the total mass of the system) located at $(\overline{x}, \overline{y})$ (the center of mass of the system) produces the same moment about any line as the whole system does about that line. For many purposes, we can think of the mass of the entire system being "concentrated at $(\overline{x}, \overline{y})$."

The arithmetic needed to prove this statement is similar to arithmetic we did to prove the corresponding assertion for a one-dimensional system.

k	m_k	x_k	y_k
1	2	-3	4
2	3	4	-7
3	1	6	-2
4	5	-2	1
5	3	4	-6

Example 2. Find the center of mass of the system consisting of the first three point-masses listed in the margin table.

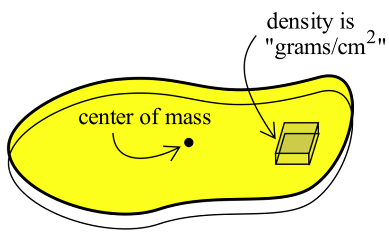
Solution. $m = 2+3+1 = 6$ and $M_y = (2)(-3)+(3)(4)+(1)(6) = 12$ while $M_x = (2)(4)+(3)(-7)+(1)(-2) = -15$ so:

$$\overline{x} = \frac{M_y}{m} = \frac{12}{6} = 2 \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{-15}{6} = -2.5$$

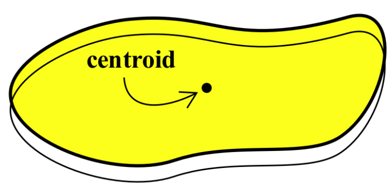
The system of three point-masses will balance on any fulcrum passing through the point $(2, -2.5)$. ◀

Practice 2. Find the center of mass of the system consisting of all five point-masses listed in the margin table.

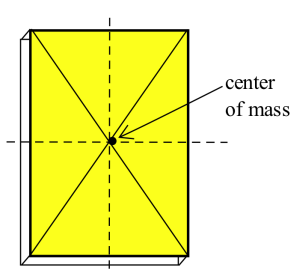
Centroid of a Region



When we move from discrete point-masses to continuous regions in a plane, we move from finite sums and arithmetic to limits of Riemann sums, definite integrals and calculus. The following discussion extends ideas and calculations from point-masses to uniformly thin, flat plates (called **lamina**) that have a uniform density throughout (given as mass per area, such as "grams per cm^2"). The center of mass of one of these plates is the point $(\overline{x}, \overline{y})$ at which the plate balances without tilting. It turns out that for plates with uniform density, the center of mass $(\overline{x}, \overline{y})$ depends only on the shape (and location) of the region of the plane covered by the plate and not on the (constant) density. In these uniform-density situations, we call the center of mass the **centroid** of the region. Throughout the following discussion, you should notice that each finite sum that appeared in the discussion of point-masses has an integral counterpart for these thin plates.

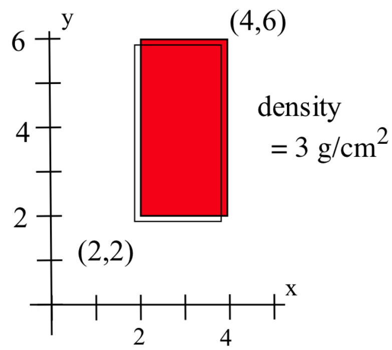


Rectangles



The components of a Riemann sum typically involve areas of rectangles, so it should come as no surprise that the basic shape used to extend point-mass concepts to regions is the rectangle. The total mass of a rectangular plate is the product of the area of the plate and its (constant) density: $m = \text{mass} = (\text{area}) \cdot (\text{density})$. We will assume that the center of mass of a thin, rectangular plate is located halfway up and halfway across the rectangle, at the point where the diagonals of the rectangle cross (see margin).

The moments of the rectangle about an axis can be found by treating the rectangle as a single point-mass with mass m located at the center of mass of the rectangle.



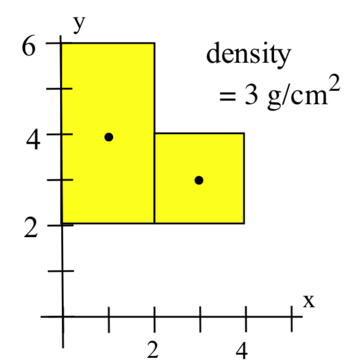
Example 3. Find the moments about the x-axis, y-axis and the line $x = 5$ of the thin, rectangular plate shown in the margin.

Solution. The density of the plate is 3 g/cm^2 and the area of the plate is $(2\,\text{cm})(4\,\text{cm}) = 8\,\text{cm}^2$ so the total mass is:

$$m = \left(8 \text{ cm}^2\right)\left(3 \frac{\text{g}}{\text{cm}^2}\right) = 24 \text{ g}$$

The center of mass of the rectangular plate is $(\overline{x}, \overline{y}) = (3, 4)$. The moment about the x-axis is the product of the mass and the signed distance of the mass from the x-axis: $M_x = (24\,\text{g})(4\,\text{cm}) = 96\,\text{g-cm}$. Similarly, $M_y = (24\,\text{g})(3\,\text{cm}) = 72\,\text{g-cm}$. The moment about the line $x = 5$ is $M_{x=5} = (24\,\text{g})([5-3] \text{ cm}) = 48\,\text{g-cm}$. ◀

To find the moments and center of mass of a plate made up of several rectangular regions, we can simply treat each of the rectangular pieces as a point-mass concentrated at its center of mass, then treat the plate as a system of discrete point-masses.

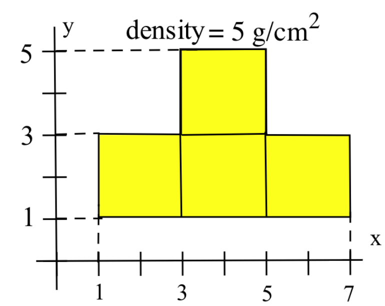


Example 4. Find the centroid of the region in the margin figure.

Solution. We can divide the plate into two rectangular plates, one with mass 24 g and center of mass $(1, 4)$, and the other with mass 12 g and center of mass $(3, 3)$. The total mass of the pair of point-masses is $m = 24 + 12 = 36$ g, and the moments about the axes are $M_x = (24 \text{ g})(4 \text{ cm}) + (12 \text{ g})(3 \text{ cm}) = 132$ g-cm and $M_y = (24 \text{ g})(1 \text{ cm}) + (12 \text{ g})(3 \text{ cm}) = 60$ g-cm. So:

$$\overline{x} = \frac{M_y}{m} = \frac{60 \text{ g-cm}}{36 \text{ g}} = \frac{5}{3} \text{ cm} \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{132 \text{ g-cm}}{36 \text{ g}} = \frac{11}{3} \text{ cm}$$

The centroid of the plate is located at $\left(\frac{5}{3}, \frac{11}{3}\right)$. ◀

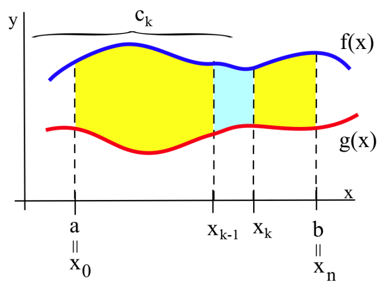


Practice 3. Find the centroid of the region in the margin figure.

To find the center of mass of a thin, non-rectangular plate, we will "slice" the plate into narrow, almost-rectangular plates and treat the collection of almost-rectangular plates as a system of point-masses located at the centers of mass of the almost-rectangles. The total mass and moments about the axes for the system of point-masses will be Riemann sums. By taking limits as the widths of the almost-rectangles approach 0, we will obtain exact values for the mass and moments as definite integrals

$\overline{x}$ *for a Region*

The Greek letter ρ (pronounced "row," as in "row your boat") is often used to represent the density of a region.



Suppose $f(x) \geq g(x)$ on the interval $[a, b]$ and $\mathcal{R}$ is a plate of uniform density ($= \rho$) sitting on the region between the graphs of $f(x)$ and $g(x)$ and the lines $x = a$ and $x = b$ (see margin figure). If we partition the interval $[a, b]$ into n subintervals of the form $[x_{k-1}, x_k]$ and choose the points c_k to be the midpoints of these subintervals, then the slice between vertical cuts at $x = x_{k-1}$ and $x = x_k$ is approximately rectangular and has mass approximately equal to:

$$\begin{aligned}(\text{area})\,(\text{density}) &= (\text{height})\,(\text{width})\,(\text{density}) \\ &\approx [f(c_k) - g(c_k)] \cdot (x_{k-1} - x_k) \cdot \rho \\ &= \rho\,[f(c_k) - g(c_k)]\,\Delta x_k\end{aligned}$$

So the mass of the whole plate is approximately

$$m = \sum_{k=1}^{n} \rho\,[f(c_k) - g(c_k)]\,\Delta x_k \longrightarrow \int_a^b \rho\,[f(x) - g(x)]\,dx = \rho \cdot A$$

where A is the area of the region $\mathcal{R}$.

The moment about the y-axis of each almost-rectangular slice is the product of the mass of the slice (m) and the distance from the centroid of the almost-rectangle to the y-axis. The x-coordinate of that centroid is located at $x = c_k$, so the distance from the centroid to the y-axis is $c_k - 0 = c_k$. The moment of the almost-rectangle about the y-axis is therefore:

$$m_k \cdot c_k = (\rho\,[f(c_k) - g(c_k)]\,\Delta x_k) \cdot c_k$$

so the moment of the entire plate about the y-axis is (approximately):

$$M_y = \sum_{k=1}^{n} \rho c_k \cdot [f(c_k) - g(c_k)]\,\Delta x_k \longrightarrow \int_a^b \rho x \cdot [f(x) - g(x)]\,dx$$

The x-coordinate of the centroid of the plate is therefore:

$$\overline{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x \cdot [f(x) - g(x)]\,dx}{\rho \int_a^b [f(x) - g(x)]\,dx} = \frac{\int_a^b x \cdot [f(x) - g(x)]\,dx}{\int_a^b [f(x) - g(x)]\,dx}$$

The density constant ρ is a factor of both M_y and m, so it cancels and has no effect on the value of $\overline{x}$. The value of $\overline{x}$ depends only on the shape and location of the region $\mathcal{R}$.

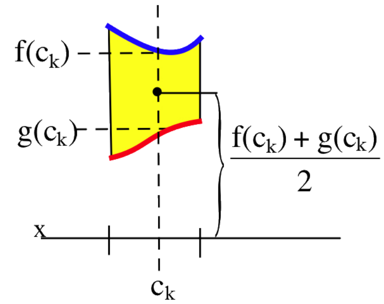
If the bottom boundary of $\mathcal{R}$ is the x-axis, then $g(x) = 0$ and the previous formulas simplify to:

$$m = \rho \int_a^b f(x)\,dx, \quad M_y = \rho \int_a^b x f(x)\,dx \quad \text{and} \quad \overline{x} = \frac{M_y}{m} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

Practice 4. Find the x-coordinate of the centroid of the region between $f(x) = x^2$, the x-axis and $x = 2$.

$\overline{y}$ *for a Region*

To find $\overline{y}$, the y-coordinate of the centroid of $\mathcal{R}$, we need to find M_x, the moment of $\mathcal{R}$ about the x-axis. For vertical partitions of $\mathcal{R}$ (see margin), the moment of each narrow strip about the x-axis, M_x, is the product of the strip's mass and the signed distance between the centroid of the strip and the x-axis. We've already computed the mass:



$$m_k = \rho\left[f\left(c_k\right) - g\left(c_k\right)\right]\Delta x_k$$

Because each strip is nearly rectangular, the centroid of the k-th strip is roughly halfway up the strip, at a point midway between $f\left(c_k\right)$ and $g\left(c_k\right)$, so we can average those function values to compute:

$$\overline{y}_k \approx \frac{f\left(c_k\right) + g\left(c_k\right)}{2}$$

The moment about the x-axis for this strip is thus:

$$\rho\left[f\left(c_k\right) - g\left(c_k\right)\right]\Delta x_k \cdot \left[\frac{f\left(c_k\right) + g\left(c_k\right)}{2}\right] = \frac{\rho}{2}\left[\left(f\left(c_k\right)\right)^2 - \left(g\left(c_k\right)\right)^2\right]\Delta x_k$$

Adding up the moments of all n strips yields:

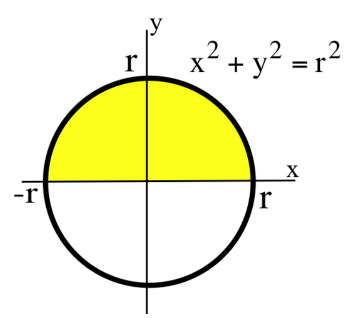
$$M_x = \sum_{k=1}^{n}\frac{\rho}{2}\left[\left(f\left(c_k\right)\right)^2 - \left(g\left(c_k\right)\right)^2\right]\Delta x_k \longrightarrow \int_a^b \frac{\rho}{2}\left[\left(f\left(x\right)\right)^2 - \left(g\left(x\right)\right)^2\right]dx$$

The y-coordinate of the centroid of the plate is therefore:

$$\overline{y} = \frac{M_x}{m} = \frac{\rho\int_a^b \frac{1}{2}\left[\left(f\left(x\right)\right)^2 - \left(g\left(x\right)\right)^2\right]dx}{\rho\int_a^b\left[f(x) - g(x)\right]dx} = \frac{\int_a^b \frac{1}{2}\left[\left(f\left(x\right)\right)^2 - \left(g\left(x\right)\right)^2\right]dx}{\int_a^b\left[f(x) - g(x)\right]dx}$$

If the bottom boundary of $\mathcal{R}$ is the x-axis, then $g(x) = 0$ and the previous formulas simplify to:

$$M_x = \frac{\rho}{2}\int_a^b x\left[f(x)\right]^2 dx \quad \text{and} \quad \overline{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\int_a^b f(x)\,dx}$$



Example 5. Find the y-coordinate of the centroid of the region $\mathcal{R}$ bounded below by the x-axis and above by the top half of a circle of radius r centered at the origin (see margin).

Solution. An equation for the circle is $x^2 + y^2 = r^2$ so the top half is given by $f(x) = y = \sqrt{r^2 - x^2}$, and $g(x) = 0$. The mass of the region is:

$$m = \int_{-r}^{r}\rho\sqrt{r^2 - x^2}\,dx = \rho\int_{-r}^{r}\sqrt{r^2 - x^2}\,dx = \rho\cdot\left[\text{area of } \mathcal{R}\right] = \rho\cdot\frac{\pi r^2}{2}$$

The moment of $\mathcal{R}$ about the y-axis is:

$$M_y = \int_{-r}^{r}\rho x\cdot\sqrt{r^2 - x^2}\,dx = \left[-\frac{\rho}{3}\left(r^2 - x^2\right)^{\frac{3}{2}}\right]_{x=-r}^{x=r} = 0$$

Could you have guessed this result merely by looking at the region?

so $\overline{x} = 0$.

The moment of $\mathcal{R}$ about the x-axis is:

$$M_x = \int_{-r}^{r} \frac{\rho}{2} \cdot \left[\sqrt{r^2 - x^2}\right]^2 dx = \frac{\rho}{2} \int_{-r}^{r} \left[r^2 - x^2\right] dx$$
$$= \frac{\rho}{2} \left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r} = \frac{\rho}{2} \cdot \frac{4}{3} r^3 = \frac{2\rho}{3} r^3$$

Could you have guessed that centroid would be located a bit less than halfway above the bottom edge of the semicircle, merely by looking at the region?

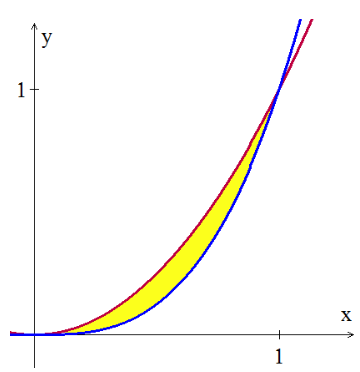
so $\overline{y} = \dfrac{\frac{2\rho}{3} r^3}{\frac{\rho\pi}{2} r^2} = \dfrac{4}{3\pi} r \approx 0.4244r$. ◀

Practice 5. Show that the centroid of a triangular region with vertices $(0,0)$, $(0,h)$ and $(b,0)$ is located at $(\overline{x}, \overline{y}) = \left(\frac{b}{3}, \frac{h}{3}\right)$.

The following table summarizes and compares formulas for computing moments and centers of mass for a system of point-masses in a plane (using sums) and for a region in a plane (using integrals). The integral formulas appear in a form for calculating moments of a region $\mathcal{R}$ bounded by the graphs of two functions, $f(x)$ and $g(x)$, and two vertical lines, $x = a$ and $x = b$, where $f(x) \geq g(x)$ for $a \leq x \leq b$.

	point-masses in plane	**region $\mathcal{R}$ between f and g**
total mass:	$m = \sum_{k=1}^{n} m_k$	$m = \int_a^b \rho\,[f(x) - g(x)]\, dx = \rho \cdot \text{Area}\,(\mathcal{R})$
moment about y-axis ($x = 0$):	$M_y = \sum_{k=1}^{n} m_k \cdot x_k$	$M_y = \int_a^b \rho x \cdot [f(x) - g(x)]\, dx$
moment about x-axis ($y = 0$):	$M_x = \sum_{k=1}^{n} m_k \cdot y_k$	$M_x = \int_a^b \frac{\rho}{2} \left[(f(x))^2 - (g(x))^2\right] dx$
center of mass (ρ constant):	$\overline{x} = \frac{M_y}{m}$, $\overline{y} = \frac{M_x}{m}$	$\overline{x} = \frac{M_y}{m}$, $\overline{y} = \frac{M_x}{m}$

With the knowledge of Riemann sums you have developed, you should be able to set up integrals to compute masses and moments for regions bounded by curves of the form $x = g(y)$, and deal with situations where the density of a thin plate is a function of x or y.



While the integral formulas above are often useful, it is important that you understand the process used to obtain these formulas in order to compute moments and centroids of more general regions.

Example 6. Find the centroid of the region $\mathcal{R}$ bounded by the graphs of $y = x^2$ and $y = x^3$.

Solution. The curves intersect where $x^2 = x^3 \Rightarrow x^2 - x^3 = 0 \Rightarrow x^2(1 - x) = 0 \Rightarrow x = 0$ or $x = 1$. A graph (see margin) helps confirm that $x^2 \geq x^3$ on $[0,1]$. If the density of $\mathcal{R}$ is ρ then the mass of $\mathcal{R}$ is:

$$m = \int_0^1 \rho \left[x^2 - x^3\right] dx = \rho \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{\rho}{12}$$

The moment of $\mathcal{R}$ about the y-axis is:

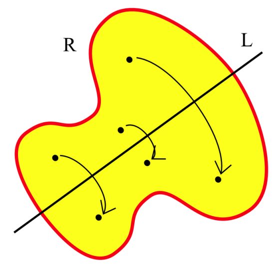
$$M_y = \rho \int_0^1 x \left[x^2 - x^3\right] dx = \rho \int_0^1 \left[x^3 - x^4\right] dx = \rho \left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1 = \frac{\rho}{20}$$

And the moment of $\mathcal{R}$ about the x-axis is:

$$M_x = \frac{\rho}{2}\int_0^1 \left[\left(x^2\right)^2 - \left(x^3\right)^2\right]\,dx = \frac{\rho}{2}\left[\frac{1}{5}x^5 - \frac{1}{7}x^7\right]_0^1 = \frac{\rho}{35}$$

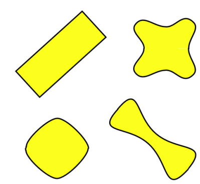
so $\overline{x} = \dfrac{M_y}{m} = \dfrac{\frac{\rho}{20}}{\frac{\rho}{12}} = \dfrac{3}{5}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{\frac{\rho}{35}}{\frac{\rho}{12}} = \dfrac{12}{35}$. Plotting the point $\left(\frac{3}{5}, \frac{12}{35}\right) \approx (0.60, 0.34)$ along with $\mathcal{R}$ confirms that it sits inside $\mathcal{R}$ (just barely) and appears to be a reasonable candidate for the centroid. ◀

Symmetry



Symmetry is a very powerful geometric concept that can simplify many mathematical and physical problems, including the task of finding centroids of regions. For some regions, we can use symmetry alone to determine the centroid. Geometrically, a region $\mathcal{R}$ is **symmetric about a line L** if, when $\mathcal{R}$ is folded along L, each point of $\mathcal{R}$ on one side of the fold matches up with exactly one point of $\mathcal{R}$ on the other side of the fold (see margin).

Example 7. Sketch two lines of symmetry for each region shown in the margin figure.



Solution. See solution to Practice 6. ◀

A very useful fact about symmetric regions is that the centroid $(\overline{x}, \overline{y})$ of a symmetric region must lie on every line of symmetry of the region. If a region has two different lines of symmetry, then the centroid must lie on each of them, so the centroid must be located at the point where the lines of symmetry intersect.

In Example 5, the half-disk was symmetric with respect to the y-axis, so we could have avoided setting up and evaluating the M_y integral by noticing that $(\overline{x}, \overline{y})$ must be located on the y-axis (the line $x = 0$) and concluding that $\overline{x} = 0$.

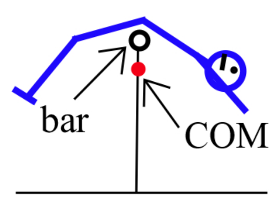
Practice 6. Locate the centroid of each region in Example 7.

Work

In a uniform gravitational field, the **center of gravity** of an object is located at the same point as its center of mass, and the work done to lift an object is the product of the object's weight and the distance that the center of gravity of the object is raised:

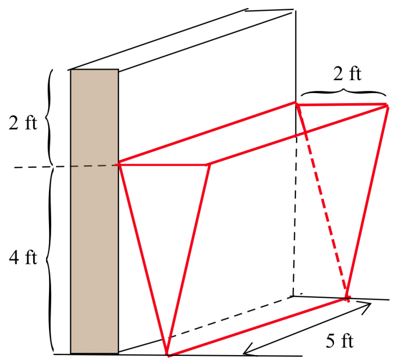
$$\text{work} = (\text{object's weight})\,(\text{distance object's center of gravity is raised})$$

In the high jump, this explains the effectiveness of the "Fosbury Flop," a technique where the jumper assumes an inverted **U** position while going over the bar (see margin): the jumper's body goes over the bar while the jumper's center of gravity goes under it, allowing the jumper to clear a higher bar with no additional upward thrust.



If you know the center of gravity of an object being lifted, some work problems become much easier.

We've already solved this problem (as Example 5 in Section 5.4) but here we try a new approach using centroids.



Example 8. The trough shown in the margin is filled with a liquid weighing 70 pounds per cubic foot. How much work is done pumping the liquid over the wall next to the trough?

Solution. This is a 3-D problem, but symmetry tells us the centroid of the liquid must be at a point 2.5 feet from either end of the trough, and 1 foot away from the wall. The vertical coordinate of the centroid will be the same as the centroid of the trough's triangular end region. Using the result of Practice 5, we can conclude that the centroid of the triangle is at a height of $\frac{2}{3} \cdot 4 = \frac{8}{3}$. The weight of the liquid is:

$$(\text{density}) \cdot (\text{volume}) = \left(70 \ \frac{\text{lb}}{\text{ft}^3}\right) \cdot \frac{1}{2} (5 \text{ ft}) \cdot (2 \text{ ft}) \cdot (4 \text{ ft}) = 1400 \text{ lbs}$$

and the distance the center of gravity must be moved is $6 - \dfrac{8}{3} = \dfrac{10}{3}$ ft so the total work required is:

$$(1400 \text{ lbs}) \cdot \left(\frac{10}{3} \text{ ft}\right) = \frac{14000}{3} \text{ ft-lbs} \approx 4666.7 \text{ ft-lbs}$$

which agrees with the answer obtained in Section 5.4. ◀

Theorems of Pappus

Pappus, the last of the great Greek geometers, flourished during the first half of the fourth century.

Two theorems due to Pappus of Alexandria can make some volume and surface area calculations relatively easy.

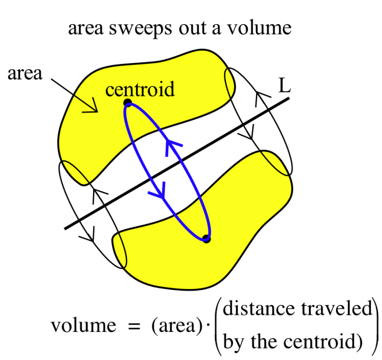
Theorem of Pappus: Volume of Revolution

If a plane region $\mathcal{R}$ with area A and centroid $(\bar{x}, \bar{y})$ is revolved around a line L in the plane that does not pass through $\mathcal{R}$

then the volume swept out by one revolution of $\mathcal{R}$ is the product of A and the distance traveled by the centroid.

Touching the boundary is OK.

The distance from the centroid to the line will be the radius of the circle swept out by the centroid, so the distance traveled by the centroid is 2π times this radius. When L is the x-axis, the volume of the solid is $A \cdot 2\pi\bar{y}$; when L is the y-axis, the volume of the solid is $A \cdot 2\pi\bar{x}$.



Example 9. Find the volume swept out when the region $\mathcal{R}$ bounded by the graphs of $y = x^2$ and $y = x^3$ is revolved around the line $x = 2$.

Solution. From Example 6, we know the area of $\mathcal{R}$ is $\frac{1}{12}$ and its centroid is $\left(\frac{3}{5}, \frac{12}{35}\right)$. The distance from this point to the line $x = 2$ is $2 - \frac{3}{5} = \frac{7}{5}$, so the distance traveled by the centroid is $2\pi \cdot \dfrac{7}{5} = \dfrac{14\pi}{5}$. The volume of the solid of revolution is therefore $\dfrac{1}{12} \cdot \dfrac{14\pi}{5} = \dfrac{7\pi}{30}$. ◀

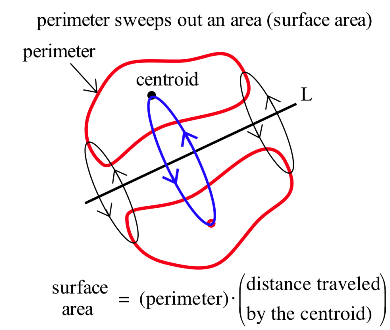
Theorem of Pappus: Surface Area of Revolution

If a plane region $\mathcal{R}$ with perimeter P and centroid $(\bar{x}, \bar{y})$ is revolved around a line L in the plane that does not pass through $\mathcal{R}$

then the surface area swept out by one revolution of $\mathcal{R}$ is the product of P and the distance traveled by the centroid.

Touching the boundary is OK.

When L is the x-axis, the surface area of the solid is $P \cdot 2\pi\bar{y}$; when L is the y-axis, the surface area is $P \cdot 2\pi\bar{x}$.



Example 10. Find the surface area of the solid swept out when the square region $\mathcal{R}$ with vertices at $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ is revolved around the line $y = 3$.

Solution. By symmetry, the centroid of the square is $(0,0)$ and its distance from $y = 3$ is 3. The perimeter of the square is $4\sqrt{2}$, so the surface area of the solid of revolution is $4\sqrt{2} \cdot 2\pi \cdot 3 = 24\pi\sqrt{3}$. ◀

5.6 Problems

1. (a) Find the total mass and the center of mass for a system consisting of the three point-masses in the table below left.
   (b) Where should you locate a new object with mass 8 so the new system has its center of mass at $x = 5$?
   (c) What mass should you put at $x = 10$ so the original system plus the new mass has its center of mass at $x = 6$?

m_k	2	5	5
x_k	4	2	6

m_k	5	3	2	6
x_k	1	7	5	5

2. (a) Find the total mass and the center of mass for a system consisting of the four point-masses in the table above right.
   (b) Where should you locate a new object with mass 10 so the new system has its center of mass at $x = 6$?
   (c) What mass should you put at $x = 14$ so the original system plus the new mass has its center of mass at $x = 6$?

3. (a) Find the total mass and the center of mass for a system consisting of the three point-masses in the table below.
   (b) Where should you locate a new object with mass 10 so the new system has its center of mass at $(5,2)$?

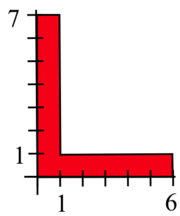
m_k	2	5	5
x_k	4	2	6
y_k	3	4	2

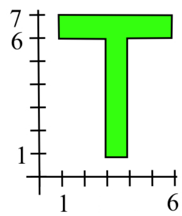
4. (a) Find the total mass and the center of mass for a system consisting of the four point-masses in the table below.
   (b) Where should you locate a new object with mass 12 so the new system has its center of mass at $(3,5)$?

m_k	5	3	2	6
x_k	1	7	5	5
y_k	4	7	0	8

In Problems 5–10, divide the plate shown into rectangles and semicircles, calculate the mass, moments and centers of mass of each piece, then find the center of mass of the plate. Assume the density of the plate is $\rho = 1$. Plot the location of the center of mass for each shape. (Refer to Example 5 for centroids of semicircular regions.)

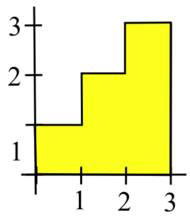
5. Use the figure below left.

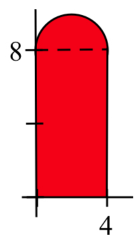




6. Use the figure above right.

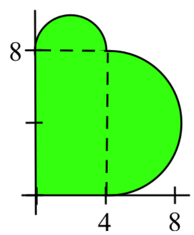
7. Use the figure below left.

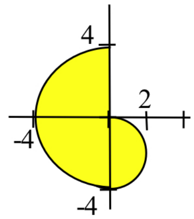




8. Use the figure above right.

9. Use the figure below left.





10. Use the figure above right.

In Problems 11–26, sketch the region bounded by the the given curves and find the centroid of each region (use technology to evaluate integrals, if necessary). Plot the location of the centroid on your sketch of the region.

11. $y = x$, the x-axis, $x = 3$

12. $y = x^2$, the x-axis, $x = -2$, $x = 2$

13. $y = x^2$, $y = 4$

14. $y = \sin(x)$, the x-axis, the y-axis, $x = \pi$

15. $y = 4 - x^2$ and the x-axis for $-2 \le x \le 2$

16. $y = x^2$, $y = x$

17. $y = 9 - x$, $y = 3$, $x = 0$, $x = 3$

18. $y = \sqrt{1 - x^2}$, the x-axis, $x = 0$, $x = 1$

19. $y = \sqrt{x}$, the x-axis, $x = 9$

20. $y = \ln(x)$, the x-axis, $x = e$

21. $y = e^x$, $y = e$ and the y-axis

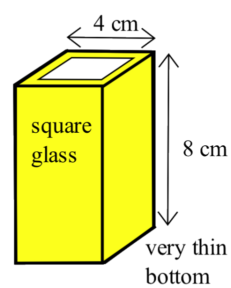
22. $y = x^2$ and $y = 2x$

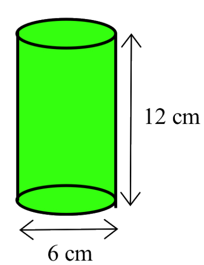
23. An empty box in the shape of a cube measuring 1 foot on each side weighs 10 pounds. By symmetry, we know its center of mass is 6 inches above its bottom. When the box is full of a liquid with density 60 lb/ft^3, the center of mass of the box-liquid system is again (due to symmetry) 6 inches above the bottom of the box.

   (a) Find the height of the center of mass of the box-liquid system as a function of h, the height of water in the box.

   (b) To what height should you fill the box so that the box-liquid system has the lowest center of gravity (and the greatest stability)?

24. The empty glass shown below left has a mass of 100 g when empty. Find the height of the center of mass of the glass-water system as a function of the height of water in the glass.



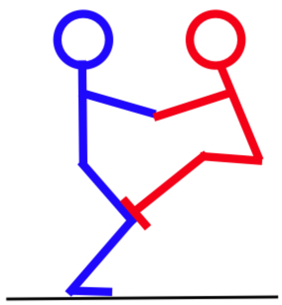


25. The empty soda can shown above right has a mass of 15 g when empty and 400 g when full of soda. Find the height of the center of mass of the can-soda system as a function of the height of the soda in the can.

26. Give a practical set of directions someone could actually use to find the height of the center of gravity of their body with their arms at their sides. How will the height of the center of gravity change if they lift their arms?

27. Try the following experiment. Stand straight with your back and heels against a wall. Slowly raise one leg, keeping it straight, in front of you. What happened? Why?

28. Why can't two dancers stand in the position shown below?



29. If a shape has exactly two lines of symmetry, the lines can meet at right angles. Must they meet at right angles?

30. Sketch regions with exactly two lines of symmetry, exactly three lines of symmetry, and exactly four lines of symmetry.

31. A rectangular box is filled to a depth of 4 feet with 300 pounds of water. How much work is done pumping the water to a point 10 feet above the bottom of the box?

32. A cylinder is filled to a depth of 2 feet with 40 pounds of water. How much work is done pumping the water to a point 7 feet above the bottom of the cylinder?

33. A sphere of radius 2 m is filled with water. How much work is done pumping the water to a point 3 m above the top of the sphere?

34. A sphere of radius 2 feet is filled with water. How much work is done pumping the water to a point 5 feet above the top of the sphere?

35. The center of a square region with sides of length 2 cm is located at the point $(3,4)$. Find the volume swept out when the square region is rotated:
    (a) about the x-axis.
    (b) about the y-axis.
    (c) about the line $y = 6$
    (d) about the line $x = 6$
    (e) about the line $2x + 3y = 6$

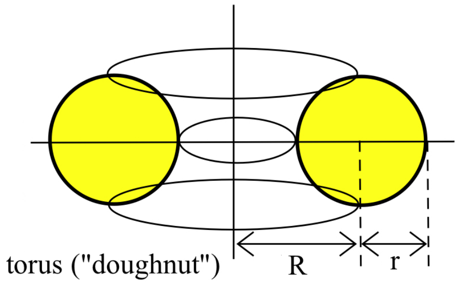
36. The lower left corner of a rectangular region with an 8-inch base and a 4-inch height is located at the point $(3,5)$. Find the volume swept out when the rectangular region is rotated:
    (a) about the x-axis.
    (b) about the y-axis.
    (c) about the line $y = x + 5$

37. The center of a square region with sides of length 2 cm is located at the point $(3,4)$. Find the surface area swept out when the square region is rotated:
    (a) about the x-axis.
    (b) about the y-axis.
    (c) about the line $y = 6$
    (d) about the line $x = 6$
    (e) about the line $2x + 3y = 6$

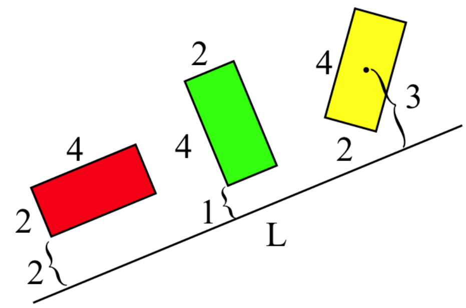
38. The lower left corner of a rectangular region with an 8-inch base and a 4-inch height is located at the point $(3,5)$. Find the surface area swept out when the rectangular region is rotated:
    (a) about the x-axis.
    (b) about the y-axis.
    (c) about the line $y = x + 5$

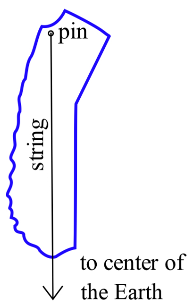
39. Find the volume and surface area swept out when the region inside the circle $(x-3)^2 + (y-5)^2 = 4$ is rotated:
    (a) about the x-axis.
    (b) about the y-axis.
    (c) about the line $y = 9$
    (d) about the line $x = 6$
    (e) about the line $2x + 3y = 6$

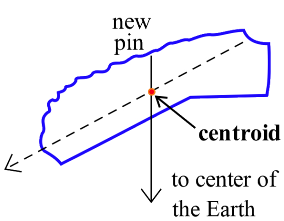
40. Find the volume and surface area swept out when the center of a circle with radius r and center $(R,0)$ is rotated about the y-axis (see below).



41. Find the volumes and surface areas swept out when the rectangles shown below are rotated about the line L. (Measurements are in feet.)







Physically Approximating Centroids of Regions

You can approximate the location of a centroid of a region experimentally, even if the region—such as a state or country—is not described by a formula.

Cut the shape out of a piece of some uniformly thick material, such as paper or cardboard, and pin an edge to a wall. The shape will pivot about the pin until its center of mass is directly below the pin (see margin) so the center of mass of the shape must lie directly below the pin, on the line connecting the pin with the center of mass of Earth. Repeat the process using a different point near the edge of the shape to find a different line. The center of mass also lies on the new line, so you can conclude that the centroid of the shape is located where the two lines intersect (see margin). It is a good idea to pick a third point near the edge and plot a third line to check that this third line also passes through the point of intersection of the first two lines.

You can experimentally approximate the "population center" of a region by attaching masses proportional to the populations of the cities and then repeating the "pin" process with this weighted model. The point on the new model where the lines intersect is the approximate "population center" of the region.

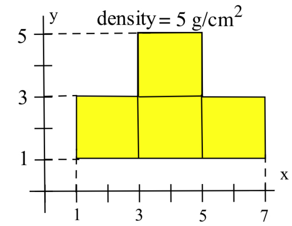
42. Determine the centroid of your state.

43. Which state would result in the easiest centroid problem? The most difficult centroid problem?

5.6 Practice Answers

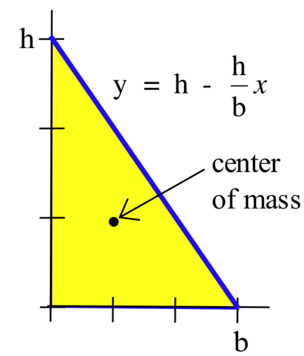
1. $m = 1+5+3 = 9$; $M_0 = (1)(6)+(5)(-2)+(3)(4) = 8$; $\overline{x} = \dfrac{M_0}{m} = \dfrac{8}{9}$; the three point-masses will balance on a fulcrum located at $\overline{x} = \frac{8}{9}$.

2. $m = 2+3+1+5+3 = 14$
$M_y = (2)(-3)+(3)(4)+(1)(6)+(5)(-2)+(3)(4) = 14$
$M_x = (2)(4)+(3)(-7)+(1)(-2)+(5)(1)+(3)(-6) = -28$
$\overline{x} = \dfrac{M_y}{m} = \dfrac{14}{14} = 1$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{-28}{14} = -2$
The five point-masses balance at the point $(1,-2)$.



3. There are several ways to break the region into "easy" pieces — one way is to consider the four 2 cm-by-2 cm squares. The center of mass of each square is located at the center of the square (at $(2,2)$, $(4,2)$, $(6,2)$ and $(4,4)$), and each square has mass $\left(4\text{ cm}^2\right)\left(5\,\frac{\text{g}}{\text{cm}^2}\right) = 20$ g so: $m = 4\,(20\,\text{g}) = 80\,\text{g}$, $M_y = 2(20)+4(20)+6(20)+4(20) = 320\,\text{g-cm}$ and $M_x = 2(20)+2(20)+2(20)+4(20) = 200$ g-cm. Therefore $\overline{x} = \dfrac{M_y}{m} = \dfrac{320\text{ g-cm}}{80\text{ g}} = 4$ cm and $\overline{y} = \dfrac{M_x}{m} = \dfrac{200\text{ g-cm}}{80\text{ g}} = 2.5$ cm so the center of mass is located at $(4, 2.5)$.

4. For simplicity, let $\rho = 1$. Then the mass is $m = \displaystyle\int_0^2 x^2\,dx = \frac{8}{3}$ while $M_y = \displaystyle\int_0^2 x\cdot x^2\,dx = \int_0^2 x^3\,dx = 4$ so $\overline{x} = \dfrac{4}{\frac{8}{3}} = \dfrac{3}{2} = 1.5$.



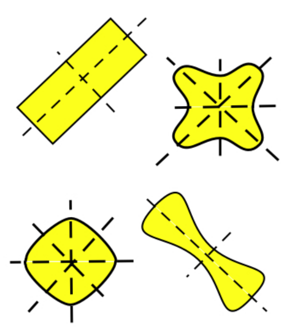
5. The triangular region appears in the margin. Here $f(x) = h - \dfrac{h}{b}x$ for $0 \le x \le b$ and $g(x) = 0$. The "mass" is just the area of the triangle, so $m = \frac{1}{2}\cdot b\cdot h$ while:

$$M_y = \int_0^b x\left[h - \frac{h}{b}x\right]\,dx = \int_0^b \left[hx - \frac{h}{b}x^2\right]\,dx = \left[\frac{h}{2}x^2 - \frac{h}{3b}x^3\right]_0^b = \frac{b^2h}{6}$$

and:

$$M_x = \int_0^b \frac{1}{2}\left[h - \frac{h}{b}x\right]^2\,dx = \left[\frac{1}{6}\left(-\frac{b}{h}\right)\left(h - \frac{h}{b}x\right)^3\right]_0^b = 0 + \frac{b}{6h}\cdot h^3 = \frac{bh^2}{6}$$

So $(\overline{x},\overline{y}) = \left(\dfrac{\frac{b^2h}{6}}{\frac{bh}{2}}, \dfrac{\frac{bh^2}{6}}{\frac{bh}{2}}\right) = \left(\dfrac{b}{3}, \dfrac{h}{3}\right)$.



6. The centroid of each region is located at the point where the lines of symmetry intersect (see margin figure).

5.7 *Improper Integrals*

In Section 5.4, we computed the work required to lift a payload of mass m from the surface of a moon of mass M and radius R to a height H above the surface of the moon:

$$\int_R^{R+H} \frac{GMm}{x^2}\,dx = \left[-\frac{GMm}{x}\right]_R^{R+H} = \frac{GMm}{R} - \frac{GMm}{R+H}$$

Notice that as the height H grows very large, the second term in this answer becomes very small and the total work approaches $\frac{GMm}{R}$. We can write:

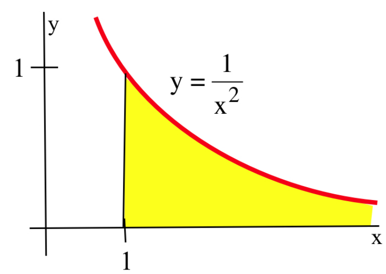
$$\lim_{H\to\infty}\left[\frac{GMm}{R} - \frac{GMm}{R+H}\right] = \frac{GMm}{R}$$

Here we're taking a limit of an expression that arose as the value of a definite integral, so we can also write:

$$\frac{GMm}{R} = \lim_{H\to\infty}\left[\frac{GMm}{R} - \frac{GMm}{R+H}\right] = \lim_{H\to\infty}\int_R^{R+H} \frac{GMm}{x^2}\,dx$$

We could write this last integral, at least informally, as:

$$\int_R^{\infty} \frac{GMm}{x^2}\,dx$$



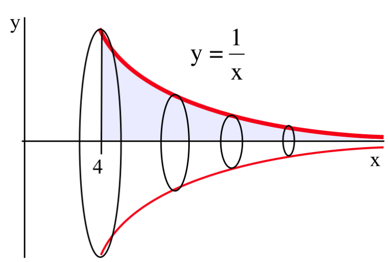
We call this new type of integral an **improper integral** because the interval of integration is infinite, violating an assumption we made when originally developing the definite integral $\int_a^b f(x)\,dx$ using Riemann sums that the length of the interval of integration, $[a,b]$, was finite.

Example 1. Represent the area of the infinite region between $f(x) = \frac{1}{x^2}$ and the x-axis for $x \geq 1$ (see margin) as an improper integral.

Solution. We can represent the area of region (which has infinite length) as:

$$\int_1^{\infty} \frac{1}{x^2}\,dx$$

We don't yet know whether this area is finite or infinite. ◀



Practice 1. Represent the volume swept out when the infinite region between $f(x) = \frac{1}{x}$ and the x-axis for $x \geq 4$ is revolved about the x-axis (see margin) using an improper integral.

General Strategy for Improper Integrals

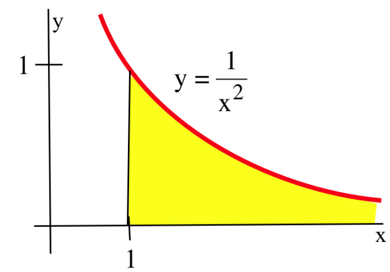
In the lifting-a-payload example above, we defined our first improper integral as the limit of a "proper" integral over a finite interval as the length of the interval became larger and larger.

Our general approach to evaluate improper integrals over infinitely long intervals—as well as another type of improper integral introduced later in this section—will mimic this strategy: Shrink the interval of integration so you have a (proper) definite integral you can evaluate, then let the interval grow to approach the desired interval of integration. The value of the improper integral will be the limiting value of the (proper) definite integrals as the intervals grow to the interval you want, provided that this limit exists.

Infinitely Long Intervals of Integration

To evaluate an improper integral on an infinitely long interval:

- replace the infinitely long interval with a finite interval
- evaluate the integral on the finite interval
- let the finite interval grow longer and longer, approaching the original infinitely long interval



Example 2. Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$ (see margin).

Solution. The interval $[1, \infty)$ is infinitely long, but we can evaluate the integral on finite intervals such as $[1, 2]$, $[1, 10]$, $[1, 1000]$ and, more generally, $[1, M]$ where M is some massive positive number:

$$\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = \left[-\frac{1}{2}\right] - \left[-\frac{1}{1}\right] = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\int_1^{10} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{10} = \left[-\frac{1}{10}\right] - \left[-\frac{1}{1}\right] = 1 - \frac{1}{10} = \frac{9}{10}$$

$$\int_1^{1000} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{1000} = \left[-\frac{1}{1000}\right] - \left[-\frac{1}{1}\right] = 1 - \frac{1}{1000} = \frac{999}{1000}$$

and, more generally, $\int_1^M \frac{1}{x^2}\,dx = 1 - \frac{1}{M}$ so:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{M\to\infty} \int_1^M \frac{1}{x^2}\,dx = \lim_{M\to\infty} \left[1 - \frac{1}{M}\right] = 1$$

The value of the improper integral is 1. ◀

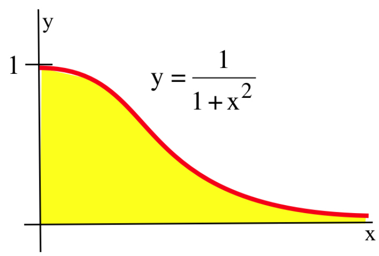
We say that the improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ in the Example 2 "is **convergent**" and that it "**converges to** 1."

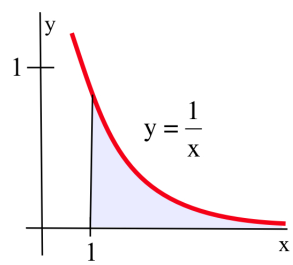
Furthermore, from Example 1, we know that this improper integral represents the area of an infinitely long region. We now have an example—which you may find highly counterintuitive—of a region with infinite length but finite area.

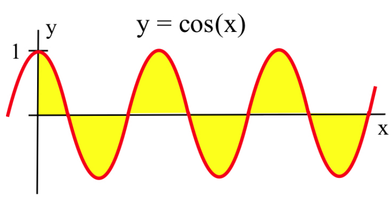
Not all improper integrals converge, however.

Example 3. Evaluate each improper integral. (See margin for graphical interpretations of these integrals as areas of unbounded regions.)

$$\text{(a)} \int_0^\infty \frac{1}{1+x^2}\,dx \qquad \text{(b)} \int_1^\infty \frac{1}{x}\,dx \qquad \text{(c)} \int_0^\infty \cos(x)\,dx$$







Solution. (a) Replacing the upper limit of the improper integral with a massive positive number M:

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \int_0^M \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \Big[\arctan(x)\Big]_0^M$$
$$= \lim_{M\to\infty} \Big[\arctan(M) - 0\Big] = \frac{\pi}{2}$$

so the improper integral is convergent and converges to $\dfrac{\pi}{2}$.

(b) Replacing the upper limit of the improper integral with a massive positive number M:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{M\to\infty} \int_1^M \frac{1}{x}\,dx = \lim_{M\to\infty} \Big[\ln(x)\Big]_1^M = \lim_{M\to\infty} \ln(M) = \infty$$

Because this limit diverges, we say the improper integral **is divergent** or that it **diverges**.

(c) Once again replacing ∞ with M in the upper limit of the integral:

$$\lim_{M\to\infty} \int_0^M \cos(x)\,dx = \lim_{M\to\infty} \Big[\sin(x)\Big]_0^M = \lim_{M\to\infty} \sin(M)$$

As M grows without bound, the values of $\sin(M)$ oscillate between -1 and 1, never approaching a single value, so the limit does not exist; we say that this improper integral diverges. ◀

Practice 2. Evaluate: (a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ (b) $\displaystyle\int_0^\infty \sin(x)\,dx$

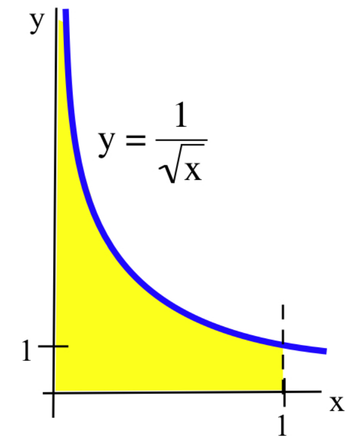
Definition: For any integrable function $f(x)$ defined for all $x \geq a$ and any integrable function $g(x)$ defined for all $x \leq b$:

$$\int_a^\infty f(x)\,dx = \lim_{M\to\infty} \int_a^M f(x)\,dx$$
$$\int_{-\infty}^b g(x)\,dx = \lim_{N\to-\infty} \int_N^b g(x)\,dx$$

If the limit in question exists and is finite, we say that the corresponding improper integral **converges** or **is convergent** and define the value of the improper integral to be the value of the limit. If the limit in question does not exist, we say that the corresponding improper integral **diverges** or **is divergent**.

Functions Undefined at an Endpoint of the Interval of Integration

Consider the graph of $\frac{1}{\sqrt{x}}$ on the interval $(0,1]$ (see margin) and compare this region to the graph from Example 2. It appears we can generate the new region by reflecting the old region across $y = x$ and adding a rectangle (of area 1) at the bottom, so we might reasonably assume that the integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is a finite number. This integral is over a finite interval, $[0,1]$, but we have a new problem: the integrand is undefined at $x = 0$, one of the endpoints of the interval of integration. This violates another assumption we made when developing the definition of a definite integral as a limit of Riemann sums.



If the function you want to integrate is unbounded at one of the endpoints of an interval of finite length, as in this situation, you can shrink the interval of integration so that the function is bounded at both endpoints of the new, smaller interval, then evaluate the integral over the smaller interval, and finally let the smaller interval grow to approach the original interval.

Example 4. Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

Solution. The function $\dfrac{1}{\sqrt{x}}$ is not defined at $x = 0$, the lower endpoint of integration, but the function is bounded on intervals such as $[0.36, 1]$, $[0.09, 1]$ and, more generally, on the interval $[c, 1]$ for any $c > 0$:

$$\int_{0.36}^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_{0.36}^1 = 2\sqrt{1} - 2\sqrt{0.36} = 2 - 1.2 = 0.8$$
$$\int_{0.09}^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_{0.09}^1 = 2\sqrt{1} - 2\sqrt{0.09} = 2 - 0.6 = 1.4$$

and, in general:

$$\int_c^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_c^1 = 2\sqrt{1} - 2\sqrt{c} = 2 - 2\sqrt{c}$$

so, taking the limit as c decreases toward 0:

$$\lim_{c\to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} \left[2 - 2\sqrt{c}\right] = 2$$

which is what you should have expected based on the graph. ◄

A region of area 1 plus a rectangle of area 1 should have an area of $1 + 1 = 2$.

Definition: For any function $f(x)$ defined and continuous on $(a, b]$ and any function $g(x)$ defined and continuous on $[a, b)$:

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$
$$\int_a^b g(x)\,dx = \lim_{c\to b^-} \int_a^c g(x)\,dx$$

If the limit exists, we say the integral **converges** and define the value of the integral to be the value of the limit. If the limit does not exist, we say that the integral **diverges**.

Practice 3. Show that (a) $\displaystyle\int_1^{10} \frac{1}{\sqrt{10-x}}\,dx = 6$ and (b) $\displaystyle\int_0^1 \frac{1}{x}\,dx$ diverges.

If an integrand is unbounded at one or more points *inside* the interval of integration, you can split the original improper integral into two or more improper integrals over subintervals where the integrand is unbounded at only one endpoint of each subinterval.

See Problems 22–26 for practice with integrals of this type.

Testing for Convergence: The P-Test and the Comparison Test

Sometimes we care only whether or not an improper integral converges. We now consider two methods for testing the convergence of an improper integral. Neither method gives you the actual value of the integral, but each enables you to determine whether or not certain improper integrals converge. The **Comparison Test for Integrals** enables you to determine the convergence (or divergence) of certain integrals by comparing them with other (easier) integrals. The **P-Test** involves special cases often used with the Comparison Test for Integrals.

P-Test for integrals: For any $a > 0$, the improper integral $\displaystyle\int_a^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. It is easiest to consider three cases rather than two: $p = 1$, $p > 1$ and $p < 1$. If $p = 1$ then:

$$\int_a^{\infty} \frac{1}{x^p}\,dx = \int_a^{\infty} \frac{1}{x}\,dx = \lim_{M\to\infty} \int_a^M \frac{1}{x}\,dx = \lim_{M\to\infty} \Big[\ln\left(|x|\right)\Big]_a^M$$
$$= \lim_{M\to\infty} \Big[\ln(M) - \ln(a)\Big] = \infty$$

so the improper integral diverges. For the other two cases, $p \neq 1$, so:

$$\lim_{M\to\infty} \int_a^M x^{-p}\,dx = \lim_{M\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^M = \lim_{M\to\infty} \left[\frac{M^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right]$$

If $p > 1$, then $1 - p < 0$ so $\displaystyle\lim_{M\to\infty} \frac{M^{1-p}}{1-p} = 0$ and:

$$\int_a^{\infty} \frac{1}{x^p}\,dx = \lim_{M\to\infty} \left[\frac{M^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}\right] = \frac{a^{1-p}}{p-1}$$

which is a finite number, so the improper integral converges. If $p < 1$, then $1 - p > 0$ so $\displaystyle\lim_{M\to\infty} \frac{M^{1-p}}{1-p} = \infty$ and:

$$\int_a^{\infty} \frac{1}{x^p}\,dx = \lim_{M\to\infty} \left[\frac{M^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1}\right] = \infty$$

so the improper integral diverges. □

Example 5. Determine the convergence or divergence of each integral.

$$\text{(a)} \int_5^\infty \frac{1}{x^2}\,dx \qquad \text{(b)} \int_1^\infty \frac{1}{\sqrt{x}}\,dx \qquad \text{(c)} \int_1^8 \frac{1}{\sqrt[3]{x}}\,dx$$

Solution. (a) The integral matches the form required by the P-Test with $p = 2 > 1$, so the improper integral converges. The P-Test does *not* tell us the value of the integral.

(b) The integral matches the form required by the P-Test with $p = \frac{1}{2} < 1$, so the improper integral diverges.

(c) This is not an improper integral, so the P-Test does not apply, but:

$$\int_1^8 \frac{1}{\sqrt[3]{x}}\,dx = \int_1^8 x^{-\frac{1}{3}}\,dx = \left[\frac{3}{2}x^{\frac{2}{3}}\right]_1^8 = \frac{3}{2}\left[8^{\frac{2}{3}} - 1^{\frac{2}{3}}\right] = \frac{3}{2}\left[4-1\right] = \frac{9}{2}$$

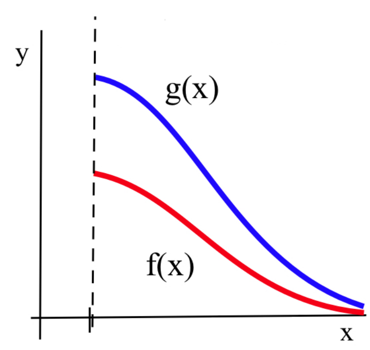
so the value of the integral is 4.5. ◀

The following **Comparison Test** enables us to determine the convergence or divergence of an improper integral of a positive function by comparing this function with functions whose improper integrals we already know converge or diverge.

Comparison Test for Integrals of Positive Functions: Suppose $f(x)$ and $g(x)$ are defined and integrable for all $x \geq a$ with $0 \leq f(x) \leq g(x)$. Then:

- $\displaystyle\int_a^\infty g(x)\,dx$ converges $\Rightarrow$ $\displaystyle\int_a^\infty f(x)\,dx$ converges.
- $\displaystyle\int_a^\infty f(x)\,dx$ diverges $\Rightarrow$ $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

The proof involves a straightforward application of the definition of an improper integral and various facts about limits, but the graph in the margin provides a geometrically intuitive way of understanding why these results must hold. If $\int_a^\infty g(x)\,dx$ converges, then the area under the graph of $g(x)$ is finite, so the (smaller) area under the graph of $f(x)$ must also be finite, and $\int_a^\infty f(x)\,dx$ must converge as well. If $\int_a^\infty f(x)\,dx$ diverges, then the area under the graph of $f(x)$ is infinite, so the (bigger) area under the graph of $g(x)$ must also be infinite, and $\int_a^\infty g(x)\,dx$ must also diverge.



Just as important as understanding what this Comparison Test *does* tell us is realizing what the Comparison Test does *not* tell us. If $\int_a^\infty g(x)\,dx$ diverges, or if $\int_a^\infty f(x)\,dx$ converges, the Comparison Test *tells us absolutely nothing* about the convergence or divergence of the other integral. Geometrically, if $\int_a^\infty g(x)\,dx$ diverges, then the area under the graph of $g(x)$ is infinite, but the (smaller) area under the graph of $f(x)$ could be either finite or infinite, so we can't conclude anything about the convergence or divergence of $\int_a^\infty f(x)\,dx$. Likewise, if $\int_a^\infty f(x)\,dx$ converges, then the area under the graph of $f(x)$ is finite, but the (bigger) area under the graph of $g(x)$ could be either finite or infinite, so we can't conclude anything about the convergence or divergence of $\int_a^\infty g(x)\,dx$.

Example 6. Determine whether each of these integrals is convergent or divergent by comparing it with an appropriate integral that you already know converges or diverges.

(a) $\displaystyle\int_1^\infty \frac{7}{x^3+5}\,dx$ (b) $\displaystyle\int_1^\infty \frac{3+\sin(x)}{x^2}\,dx$ (c) $\displaystyle\int_6^\infty \frac{9}{\sqrt{x-5}}\,dx$

Solution. (a) We know that $5 > 0$ and $x \geq 1$ so:

$$x^3+5 > x^3 \;\Rightarrow\; 0 < \frac{1}{x^3+5} < \frac{1}{x^3} \;\Rightarrow\; 0 < \frac{7}{x^3+5} < \frac{7}{x^3}$$

We also know, by the P-Test with $p = 3 > 1$, that $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ converges, so $\displaystyle\int_1^\infty \frac{7}{x^3}\,dx = 7 \cdot \int_1^\infty \frac{1}{x^3}\,dx$ also converges. By the Comparison Test, the smaller integral $\displaystyle\int_1^\infty \frac{7}{x^3+5}\,dx$ must converge as well.

The numerator of the original integrand is constant and the dominant term in the denominator of that integrand is x^3, so it should make sense to compare the original integrand with $\frac{1}{x^3}$.

(b) We know that $-1 \leq \sin(x) \leq 1$, so:

$$2 \leq 3+\sin(x) \leq 4 \;\Rightarrow\; 0 < \frac{3+\sin(x)}{x^2} \leq \frac{4}{x^2} = 4 \cdot \frac{1}{x^2}$$

By the P-Test with $p = 2 > 1$, $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, so $\displaystyle\int_1^\infty \frac{4}{x^2}\,dx = 4 \cdot \int_1^\infty \frac{1}{x^2}\,dx$ also converges. By the Comparison Test, the smaller integral $\displaystyle\int_1^\infty \frac{3+\sin(x)}{x^2}\,dx$ must converge as well.

The numerator of the original integrand fluctuates between 2 and 4 while the dominant (and only) term in its denominator is x^2, so it should make sense to compare the original integrand with $\frac{1}{x^2}$.

(c) We know that $\sqrt{u}$ is an increasing function, so:

$$x-5 < x \;\Rightarrow\; \sqrt{x-5} < \sqrt{x} \;\Rightarrow\; \frac{1}{\sqrt{x-5}} > \frac{1}{\sqrt{x}}$$

By the P-Test with $p = \frac{1}{2} < 1$, $\displaystyle\int_6^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, so the bigger integral $\displaystyle\int_6^\infty \frac{1}{\sqrt{x-5}}\,dx$ must also diverge. ◀

5.7 Problems

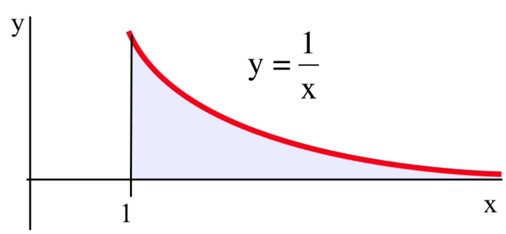
In Problems 1–26, evaluate each improper integral, or show why it diverges.

1. $\int_{10}^{\infty} \frac{1}{x^3}\,dx$
2. $\int_{e}^{\infty} \frac{5}{x \cdot [\ln(x)]^2}\,dx$
3. $\int_{\sqrt{3}}^{\infty} \frac{1}{1+x^2}\,dx$
4. $\int_{1}^{\infty} \frac{2}{e^x}\,dx$
5. $\int_{e}^{\infty} \frac{5}{x \cdot \ln(x)}\,dx$
6. $\int_{0}^{\infty} \frac{x}{1+x^2}\,dx$
7. $\int_{3}^{\infty} \frac{1}{x-2}\,dx$
8. $\int_{3}^{\infty} \frac{1}{(x-2)^2}\,dx$
9. $\int_{3}^{\infty} \frac{1}{(x-2)^3}\,dx$
10. $\int_{3}^{\infty} \frac{1}{x+2}\,dx$
11. $\int_{3}^{\infty} \frac{1}{(x+2)^2}\,dx$
12. $\int_{3}^{\infty} \frac{1}{(x+2)^3}\,dx$
13. $\int_{0}^{4} \frac{1}{\sqrt{x}}\,dx$
14. $\int_{0}^{8} \frac{1}{\sqrt[3]{x}}\,dx$
15. $\int_{0}^{16} \frac{1}{\sqrt[4]{x}}\,dx$
16. $\int_{0}^{2} \frac{1}{\sqrt{2-x}}\,dx$
17. $\int_{0}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$
18. $\int_{0}^{2} \frac{3x^2}{\sqrt{8-x^3}}\,dx$
19. $\int_{-2}^{\infty} \sin(x)\,dx$
20. $\int_{\pi}^{\infty} \sin(x)\,dx$
21. $\int_{0}^{\frac{\pi}{2}} \tan(x)\,dx$
22. $\int_{0}^{3} \frac{1}{x-2}\,dx$
23. $\int_{0}^{\pi} \tan(x)\,dx$
24. $\int_{3}^{\infty} \frac{1}{x\sqrt{x}}\,dx$
25. $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$
26. $\int_{-\infty}^{0} \frac{1}{x^2}\,dx$

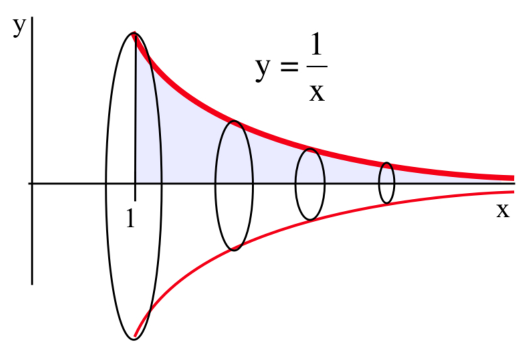
In Problems 27–44, determine whether each improper integral converges or diverges, but do not evaluate the integral.

27. $\int_{1}^{\infty} \frac{1}{x^5}\,dx$
28. $\int_{2}^{\infty} \frac{1}{\sqrt[5]{x}}\,dx$
29. $\int_{3}^{\infty} \frac{1}{\sqrt[5]{x^6}}\,dx$
30. $\int_{4}^{\infty} \frac{1}{\sqrt[5]{x^4}}\,dx$
31. $\int_{5}^{\infty} \frac{1}{\sqrt{x\sqrt[3]{x}}}\,dx$
32. $\int_{6}^{\infty} x^{-\frac{4}{7}}\,dx$
33. $\int_{7}^{\infty} x^{-\frac{7}{4}}\,dx$
34. $\int_{8}^{\infty} \frac{1}{1+x^2}\,dx$
35. $\int_{3}^{\infty} \frac{1}{x^2+5}\,dx$
36. $\int_{4}^{\infty} \frac{7}{x^2+5}\,dx$
37. $\int_{5}^{\infty} \frac{1}{x^3+x}\,dx$
38. $\int_{6}^{\infty} \frac{1}{x-2}\,dx$
39. $\int_{e}^{\infty} \frac{7}{x+\ln(x)}\,dx$
40. $\int_{2}^{\infty} \frac{1}{x^2-1}\,dx$
41. $\int_{\pi}^{\infty} \frac{1+\cos(x)}{x^2}\,dx$
42. $\int_{0}^{\infty} \frac{x^4}{x^6+1}\,dx$
43. $\int_{0}^{\infty} \frac{x^4}{x^5+1}\,dx$
44. $\int_{0}^{\infty} \frac{\sqrt{x}}{x^2+1}\,dx$

45. Example 3(b) showed that $\int_{1}^{M} \frac{1}{x}\,dx$ grew arbitrarily large as M grew arbitrarily large, so no finite amount of paint would cover the region bounded by the x-axis and the graph of $f(x) = \frac{1}{x}$ for $x > 1$:

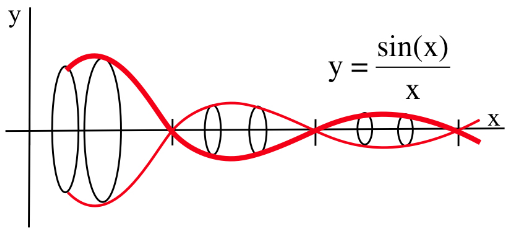


Show that the volume of the solid obtained when the region graphed above is revolved about the x-axis:

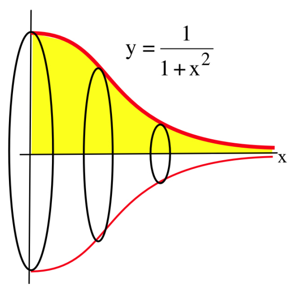


is finite, so the 3-dimensional trumpet-shaped region can be filled with a finite amount of paint. Does this present a contradiction?

46. Determine whether or not the volume of the solid obtained by revolving the region between the x-axis and the graph of $f(x) = \dfrac{\sin(x)}{x}$ for for $x \geq 1$ (see below) about the x-axis is finite.

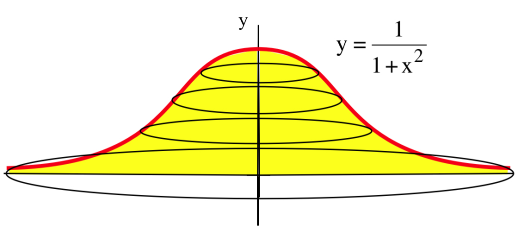


47. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = \dfrac{1}{x^2+1}$ (see below) is revolved about the x-axis.



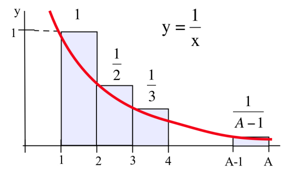
48. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = e^{-x}$ is revolved about the x-axis.

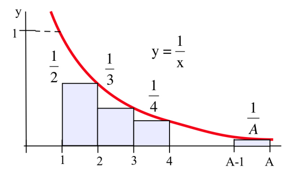
49. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = \dfrac{1}{x^2+1}$ (see below) is revolved about the y-axis.



50. Compute the volume of the solid obtained when the region in the first quadrant between the positive x-axis and the graph of $f(x) = e^{-x}$ is revolved about the y-axis.

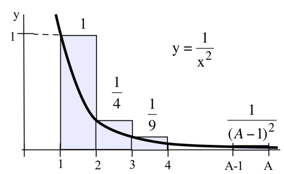
51. Use the figure below left to help determine which is larger: $\displaystyle\int_1^A \frac{1}{x}\,dx$ or $\displaystyle\sum_{k=1}^{A-1} \frac{1}{k}$.

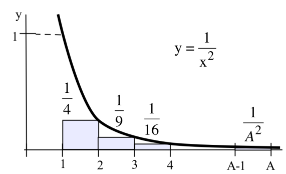




52. Use the figure above right to help determine which is larger: $\displaystyle\int_1^A \frac{1}{x}\,dx$ or $\displaystyle\sum_{k=2}^{A} \frac{1}{k}$.

53. Use the figure below left to help determine which is larger: $\displaystyle\int_1^A \frac{1}{x^2}\,dx$ or $\displaystyle\sum_{k=1}^{A-1} \frac{1}{k^2}$.





54. Use the figure above right to help determine which is larger: $\displaystyle\int_1^A \frac{1}{x^2}\,dx$ or $\displaystyle\sum_{k=2}^{A} \frac{1}{k^2}$.

The **Laplace transform** of a function $f(t)$ is defined using an improper integral involving a parameter s:

$$F(s) = \int_0^\infty e^{-st} \cdot f(t)\,dt$$

Laplace transforms are often used to solve differential equations.

55. Compute the Laplace transform of the constant function $f(t) = 1$.

56. Compute the Laplace transform of $f(t) = e^{4t}$.

57. Define a function $g(t)$ by:

$$g(t) = \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}$$

Compute the Laplace transform of $g(t)$.

58. Define a function $h(t)$ by:

$$h(t) = \begin{cases} 1 & \text{if } t < 3 \\ 0 & \text{if } t \geq 3 \end{cases}$$

Compute the Laplace transform of $h(t)$.

59. Devise a "Q-Test" to determine whether $\int_0^b \frac{1}{x^q}\,dx$ converges or diverges for any number $b > 0$.

60. Use the result of the previous problem to test the convergence of $\int_0^e \frac{1}{\sqrt[3]{x}}\,dx$ and $\int_0^\pi \frac{1}{x \cdot \sqrt[3]{x}}\,dx$.

5.7 *Practice Answers*

1. $\int_4^\infty \pi \cdot \left(\frac{1}{x}\right)^2\,dx = \pi \int_4^\infty \frac{1}{x^2}\,dx$

2. (a) $\int_1^\infty \frac{1}{x^3}\,dx = \lim_{M\to\infty} \int_1^M x^{-3}\,dx = \lim_{M\to\infty} \left[-\frac{1}{2}x^{-2}\right]_1^M = \lim_{M\to\infty} \left[-\frac{1}{2}\cdot\frac{1}{M^2} + \frac{1}{2}\right] = \frac{1}{2}$

   (b) Replacing ∞ with M in the upper limit of the integral:

$$\begin{aligned}\int_0^\infty \sin(x)\,dx &= \lim_{M\to\infty} \int_0^M \sin(x)\,dx = \lim_{M\to\infty} \Big[-\cos(x)\Big]_0^M \\ &= \lim_{M\to\infty} \Big[-\cos(M) + 1\Big] = \lim_{M\to\infty} [1 - \cos(M)\end{aligned}$$

   This limit does not exist (the values of $1 - \cos(M)$ oscillate between 0 and 2 and never approach any fixed number) so the improper integral diverges.

3. (a) The integral is improper at its upper limit, where $x = 10$, so:

$$\begin{aligned}\int_1^{10} \frac{1}{\sqrt{10-x}}\,dx &= \lim_{c\to 10^-} \int_1^c (10-x)^{-\frac{1}{2}}\,dx = \lim_{c\to 10^-} \left[-2\sqrt{10-x}\right]_1^c \\ &= \lim_{c\to 10^-} \left[-2\sqrt{10-c} + 2\sqrt{9}\right] = 0 + 2\cdot 3 = 6\end{aligned}$$

   (b) The integral is improper at its lower limit, where $x = 0$, so:

$$\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \Big[\ln(|x|)\Big]_c^1 = \lim_{c\to 0^+} [\ln(1) - \ln(c)] = \infty$$

   so the integral diverges.

5.8 Additional Applications

This section introduces two additional applications of integrals that once again illustrate the process of going from an applied problem to a Riemann sum and on to a definite integral. A third application does not follow this process: it uses the idea of "area" to model an election and to qualitatively understand why certain election outcomes occur.

The main point of this section is to demonstrate the power of definite integrals to solve a wide variety of applied problems. Each of these new applications is treated more briefly than those in the previous sections. These are far from the only applications that could be included here. By now, however, you should have developed enough of an understanding of the Riemann-sum process so that when you encounter other applications (in physics, engineering, biology, statistics, probability, economics, computer graphics ...) you will be able to use that process to set up an integral to compute or approximate a desired quantity.

Fluid Pressures and Forces

In physics, **pressure** is defined as force per unit of area. The **hydrostatic pressure** on an object immersed in a fluid (such as water) is the product of the density of that fluid and the depth of the object:

$$\text{pressure} = (\text{density})(\text{depth})$$

Fluids exert pressure in all possible directions, and the forces due to this pressure act on solid objects in a direction perpendicular to the object.

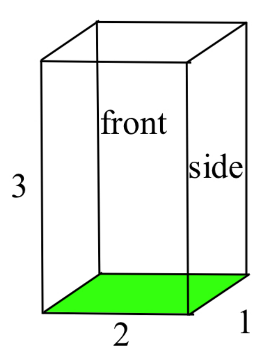
The total **hydrostatic force** applied against an immersed object is the sum of the hydrostatic forces against each part of the object.

If an entire object is at the same depth, we can determine the total hydrostatic force against that (necessarily flat) object simply by multiplying the density of the fluid times the depth of the object times the object's area. If the unit of density is "pounds per cubic foot" and depth is measured in "feet," then the unit of pressure is "pounds per square foot," a measure of force per unit of area. If pressure, with the units "pounds per square foot," is multiplied by an area with units "square feet," the result is a force, measured in "pounds."

In the metric system, the standard unit of force is a **pascal** (abbreviated "Pa"):

$$1\,\text{Pa} = 1\,\frac{\text{N}}{\text{m}}$$

and named after Blaise Pascal (1623–1662), a French mathematician, physicist, inventor, writer and philosopher.



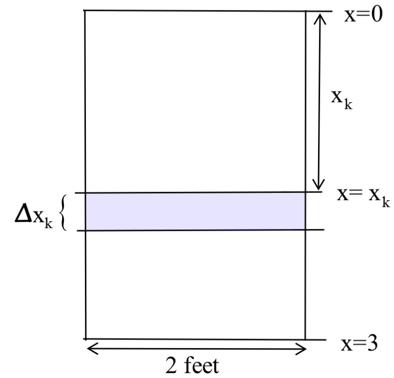
Example 1. Find the total hydrostatic force against the bottom of the freshwater aquarium shown in the margin.

Solution. Water's desity is $62.5\,\frac{\text{lb}}{\text{ft}^3}$, so the total hydrostatic force is:

$$(\text{density}) \cdot (\text{depth}) \cdot (\text{area}) = \left(62.5\,\frac{\text{lb}}{\text{ft}^3}\right) \cdot (3\,\text{ft}) \cdot \left(2\,\text{ft}^2\right)$$

or 375 lbs. Finding the total hydrostatic force against the *front* of the aquarium is a very different problem, because different parts of that front face are located at different depths and subject to different pressures. ◀

To compute the force against the *front* of the aquarium, we can partition it into n thin horizontal slices (see margin) and focus on one of them. Because the slice is very thin, every part of the k-th slice is at (almost) the same depth, so every part of that slice is subject to (almost) the same pressure. We can approximate the total hydrostatic force against the slice at the depth x_k as:



$$(\text{density}) \cdot (\text{depth}) \cdot (\text{area}) = \left(62.5\,\frac{\text{lb}}{\text{ft}^3}\right) \cdot (x_k\,\text{ft}) \cdot (2\,\text{ft})\,(\Delta x_k\,\text{ft})$$

or $125x_k \cdot \Delta x_k$ lbs. The total hydrostatic force against the front is the sum of the forces against each slice:

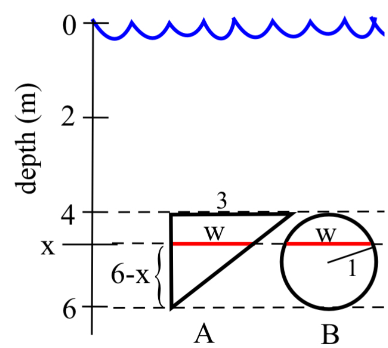
$$\text{total hydrostatic force} \approx \sum_{k=0}^{n} 125x_k \cdot \Delta x_k$$

which is a Riemann sum. The limit of this Riemann sum as the slices get thinner ($\Delta x_k \to 0$) is a definite integral:

$$\sum_{k=0}^{n} 125x_k \cdot \Delta x_k \quad\longrightarrow\quad \int_{x=0}^{x=3} 125x\,dx = \left[62.5x^2\right]_{x=0}^{x=3} = 562.5\,\text{lbs}$$

Practice 1. Find the total hydrostatic force against one side of the aquarium and the total force against the entire aquarium.

Example 2. Find the total hydrostatic force against viewing windows A and B in the freshwater aquarium shown in the margin.



Solution. For window A, using similar triangles, the width w of a slice at depth x m satisfies:

$$\frac{w}{6-x} = \frac{3}{2} \;\Rightarrow\; w = \frac{3}{2}(6-x) = 9 - \frac{3}{2}x$$

so the area of a slice of height Δx_k m at depth x_k m is $\left(9 - \frac{3}{2}x_k\right)\Delta x_k\,\text{m}^2$. The density of water is $1000\,\frac{\text{kg}}{\text{m}^3}$. Multiplying this density by area (with units m^2) would give kg per m, but pressure is measured in N per m, so we need to multiply by the acceleration due to gravity, $g \approx 9.81\,\frac{\text{m}}{\text{sec}^2}$. The hydrostatic force applied to the k-th slice is thus:

$$1000(9.81)x_k\left(9 - \frac{3}{2}x_k\right)\Delta x_k$$

and the total hydrostatic force applied to the window is therefore:

$$\int_{x=4}^{x=6} 9810\left[9x - \frac{3}{2}x^2\right]dx = 9810\left[\frac{9}{2}x^2 - \frac{1}{2}x^3\right]_4^6$$
$$= 9810\left[(162-108)-(72-32)\right] = 137340\,\text{N}$$

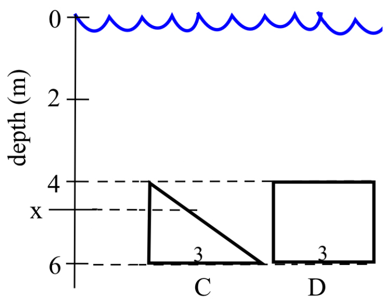
For window B, applying the Pythagorean Theorem yields:

$$(5-x)^2 + \left(\frac{w}{2}\right)^2 = 1 \;\Rightarrow\; w = 2\sqrt{1-(5-x)^2}$$

The total hydrostatic force is thus:

$$\int_{x=4}^{x=6} 1000(9.81)x \cdot 2\sqrt{1-(5-x)^2}\,dx$$

which (using technology) is approximately 154,095 N. ◀



Practice 2. Find the total hydrostatic force against viewing windows C and D of the freshwater aquarium shown in the margin.

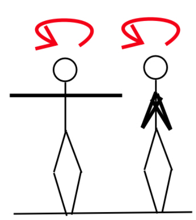
Because the total force at even moderate depths is so large, underwater windows are made of thick glass or plastic and strongly secured to their frames. Similarly, the bottom of a dam is much thicker than the top in order to withstand the greater force against the bottom.

Kinetic Energy

Physicists define the **kinetic energy** (energy of motion) of an object with mass m and velocity v to be:

$$\text{KE} = \frac{1}{2} m \cdot v^2$$

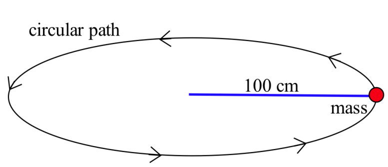
The greater the mass of an object or the faster it is moves, the greater its kinetic energy. If every part of the object has the same velocity, computing its kinetic energy becomes relatively easy.



Sometimes, however, different parts of an object move with different velocities. For example, if an ice skater is spinning with an angular velocity of 2 revolutions per second, her arms travel further in one second (have a greater *linear* velocity) when they are extended than when drawn in close to her body (see margin). So the ice skater, spinning at 2 revolutions per second, has greater kinetic energy when her arms are extended. Similarly, the tip of a rotating propeller (or the barrel of a swinging baseball bat) has a greater linear velocity than other parts of the propeller (or the bat's handle).

If the units of mass are kg and the units of velocity are m/sec^2, then:

$$\text{KE} = \frac{1}{2}(m\,\text{kg}) \cdot \left(v\,\frac{\text{m}}{\text{sec}}\right)^2 = \frac{1}{2}mv^2\ \text{kg} \cdot \text{m} \cdot \frac{\text{m}}{\text{sec}^2}$$



so the units of kinetic energy are N-m, or Joules, the same as work. Similarly, if the units of mass are g and the units of velocity are cm/sec^2, then the units of kinetic energy are dyn-cm, or ergs.

Example 3. A point-mass of 1 gram at the end of a (massless) 100-cm string rotates at a rate of 2 revolutions per second (see margin).

(a) Find the kinetic energy of the point-mass.

(b) Find its kinetic energy if the string is 200 cm long.

Solution. (a) In one second, the mass travels twice around a circle with radius 100 cm so it travels $2 \cdot (2\pi \cdot 100) = 400\pi$ cm. Its velocity is thus $v = 400\pi$ cm/sec, and:

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(1\text{ g}) \cdot \left(400\pi\ \frac{\text{cm}}{\text{sec}}\right)^2 = 80000\pi^2 \text{ ergs}$$

or about 0.079 J.

(b) If the string is 200 cm long, then the velocity is $2 \cdot (2\pi \cdot 400) = 800\pi$ cm/sec and:

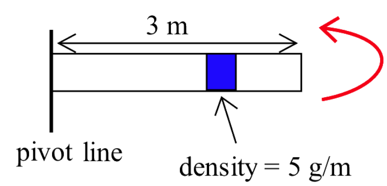
$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}(1\text{ g}) \cdot \left(800\pi\ \frac{\text{cm}}{\text{sec}}\right)^2 = 320000\pi^2 \text{ ergs}$$

or about 0.316 J. ◀

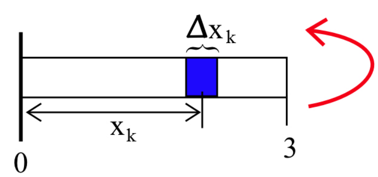
When the length of the string doubles, the velocity doubles and the kinetic energy quadruples.

Practice 3. A 1-gram point-mass at the end of a 2-meter (massless) string rotates at a rate of 4 revolutions per second. Find the kinetic energy of the point mass.

If different parts of a rotating object are different distances from the axis of rotation, then those parts have different linear velocities, and it becomes more difficult to calculate the total kinetic energy of the object. By now the method should seem very familiar: partition the object into small pieces, approximate the kinetic energy of each piece, and add the kinetic energies of the small pieces (a Riemann sum) to approximate the total kinetic energy of the object. The limit of the Riemann sum as the pieces get smaller is a definite integral.



Example 4. The density of a narrow bar (see margin) is 5 grams per meter of length. Find the kinetic energy of the 3-meter-long bar when it rotates at a rate of 2 revolutions per second.



Solution. Partition the bar (see margin) into n pieces so that the mass of the k-th piece is:

$$m_k \approx (\text{length}) \cdot (\text{density}) = (\Delta x_k\ \text{m})\left(5\ \frac{\text{g}}{\text{m}}\right) = 5 \cdot \Delta x_k \text{ g}$$

During one second, the k-th piece, located at a distance of x_k m from the pivot line, will make two revolutions, traveling approximately:

$$2\left(2\pi\left[\text{radius}\right]\right) = 4\pi\Big[100x_k\ \text{cm}\Big] = 400\pi x_k \text{ cm}$$

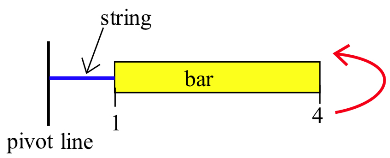
so $v_k \approx 400\pi x_k$ cm/sec. The kinetic energy of the k-th piece is:

$$\frac{1}{2}m_k \cdot v_k^2 \approx \frac{1}{2}(5\Delta x_k\ \text{g})\left(400\pi x_k\ \frac{\text{cm}}{\text{sec}}\right)^2 = 400000\pi^2 x_k^2 \text{ ergs}$$

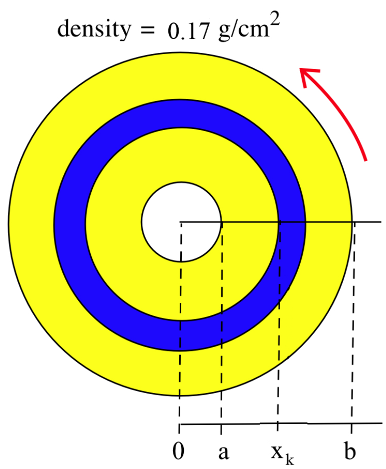
and the total kinetic energy of the rotating bar is therefore:

$$\sum_{k=1}^{n} 400000\pi^2 x_k^2 \cdot \Delta x_k \longrightarrow \int_{x=0}^{x=3} 400000\pi^2 x^2\,dx = 400000\pi^2 \left[\frac{1}{3}x^3\right]_0^3$$

which equals $3600000\pi^2$ ergs, or about 3.55 J. ◀



Practice 4. Find the kinetic energy of the bar in the previous Example if it rotates at 2 revolutions per second at the end of a 100-centimeter (massless) string (see margin).



Example 5. Find the kinetic energy of the thin, flat object with density 0.17 g/cm^2 shown in the margin when it rotates at 45 revolutions per minute.

Solution. We can partition the object along one radial line and form n annular "slices" each Δx cm wide. Then the "slice" between x_k and $x_k + \Delta x$ is a thin annulus (a disk with a smaller disk removed from its center) with area:

$$\begin{aligned}\pi\left(x_k + \Delta x\right)^2 - \pi\left(x_k\right)^2 &= \pi\left[x_k^2 + 2x_k\Delta x + (\Delta x)^2 - x_k^2\right] \\ &= 2\pi x_k \Delta x + \pi\left(\Delta x\right)^2 \approx 2\pi x_k \Delta x\end{aligned}$$

The "slices" that give rise to the Riemann sum in this problem are—unlike most examples we have seen previously—*not* rectangles. We also use here the notion that if Δx is small, then $(\Delta x)^2$ is *very* small, so we can essentially ignore it in our approximation of area.

and mass $(0.17)2\pi x_k \Delta x$. During one revolution, a point on this slice travels approximately $2\pi x_k$ cm and 45 rev/min is equivalent to $\frac{3}{4}$ rev/sec, so the linear velocity of the point is $2\pi x_k \cdot \frac{3}{4} = \frac{3}{2}\pi x_k$ cm/sec. The kinetic energy of this slice is therefore:

$$\left((0.17)2\pi x_k \Delta x\right)\left(\frac{3}{2}\pi x_k\right)^2 = \frac{9}{2}(0.17)\pi^2 x_k^3 \Delta x$$

so the total kinetic energy of the object is:

$$\sum_{k=1}^{n} \frac{9}{2}(0.17)\pi^2 x_k^3 \cdot \Delta x \longrightarrow \int_a^b \frac{9}{2}(0.17)\pi^2 x^3\,dx$$

Evaluating this integral yields:

$$\frac{9}{2}(0.17)\pi^2\left[\frac{1}{4}x^4\right]_a^b = \frac{9}{8}(0.17)\pi^2\left[b^4 - a^4\right]$$

Because b is raised to the fourth power, a small increase in the value of b (if $b > 1$) leads to a large increase in the object's kinetic energy.

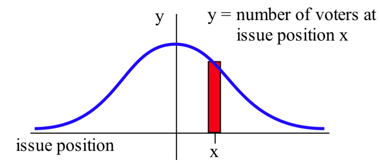
If $a = 0.75$ in ≈ 1.905 cm and $b = 3.75$ in ≈ 9.525 cm, the total mass of the object is 42 g and its total kinetic energy is about 15,512 ergs. ◀

In the not-so-distant past your grandparents (and perhaps even your parents) used such objects to listen to music—and each one only held two songs!

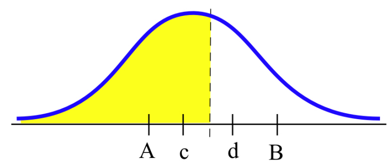
Areas and Elections

The previous applications in this chapter have used definite integrals to determine areas, volumes, pressures and energies precisely. But exactness and numerical precision are not the same as "understanding," and sometimes we can gain insight and understanding simply by determining which of two areas or integrals is larger. One situation of this type involves models of elections.

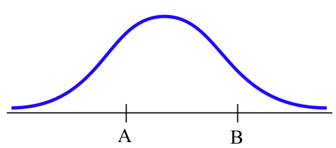
Suppose the voters of a state have been surveyed about their positions on a single issue, with their responses recorded on a quantitative scale. The distribution of voters who place themselves at each position on this issue appears in the margin. Suppose also that each voter casts his or her vote for the candidate whose position on this issue is closest to his or her position.

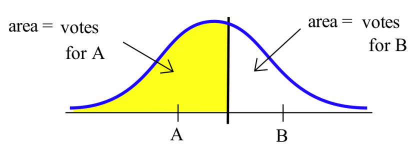


If two candidates have taken the positions labeled A and B, then a voter at position c votes for the candidate at A because A is closer to c than B is to c. Similarly, a voter at position d votes for the candidate at B. The total votes for the candidate at A in this election is represented by the shaded area under the curve, and the candidate with the larger number of votes — the larger area — wins the election. In this illustration, the candidate at A wins.



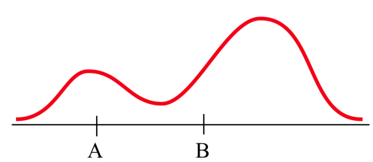
Example 6. The distribution of voters on an issue appears below left. If these voters decide between candidates on the basis of that single issue, which candidate will win the election?



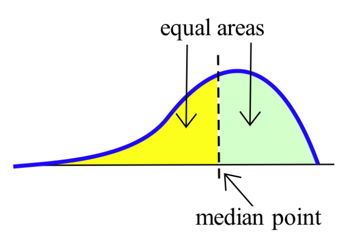


Solution. The figure above right illustrates that A corresponds to a larger area (more votes) than B: A will win. ◀

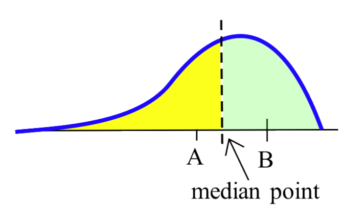
Practice 5. In an election between candidates with positions A and B in the margin figure, who will win?

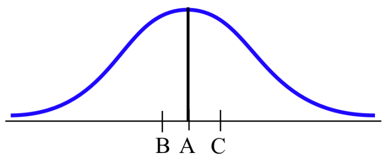


If voters behave as described and if the election is between two candidates, then we can give the candidates some advice. The best position for a candidate is at the "median point," the location that divides the voters into two equal-sized (equal-area) groups so that half of the voters are on one side of the median point and half are on the other side (see margin). A candidate at the median point gets more votes than a candidate at any other point. (Why?)



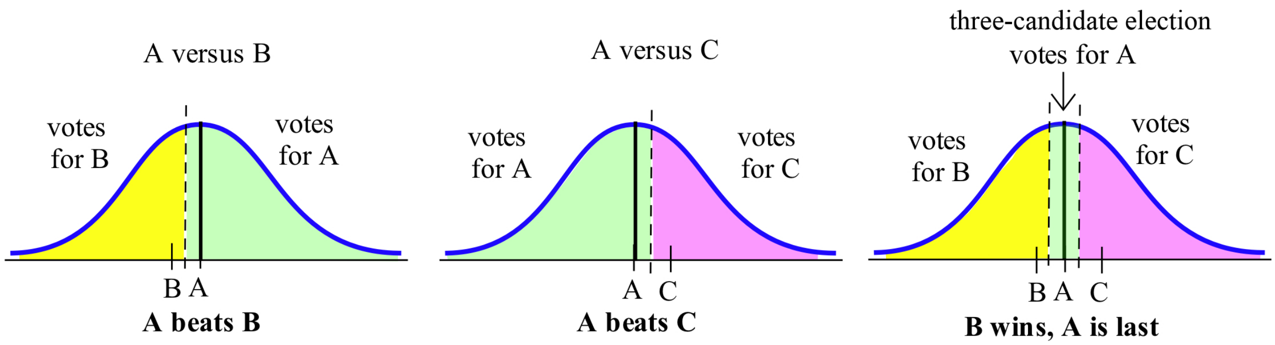
If two candidates have positions on opposite sides of the median point (see margin), then a candidate can get more votes by moving a bit toward the median point. This "move toward the middle ground"





commonly occurs in elections as candidates attempt to sell themselves as "moderates" and their opponents as "extremists."

If more than two candidates are running in an election, the situation changes dramatically. A candidate at the median position, the unbeatable place in a two-candidate election can even get the fewest votes. If the margin figure represents the distribution of voters on the single issue in the election, then candidate A would beat B in an election just between A and B (below left) and A would beat C in an election just between A and C (below center). But in an election among all three candidates, A would get the fewest votes (below right).



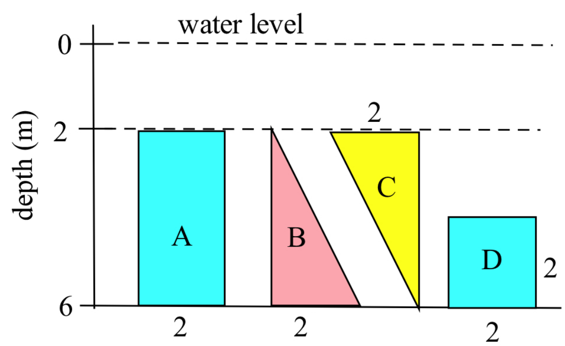
This type of situation really does occur. It leads to the political saying about a primary election with many candidates and a general election between the final nominees from two parties: "extremists can win primaries, but moderates are elected to office."

The previous discussion of elections and areas is greatly oversimplified. Most elections involve several issues of different importance to different voters, and the views of the voters are seldom completely known before the election. Many candidates take "fuzzy" positions on issues. And it is not even certain that real voters vote for the candidate with the "closest" position: perhaps they don't vote at all unless some candidate is "close enough" to their position. But this very simple model of elections can still help us understand how and why some things happen in elections. It is also a starting place for building more sophisticated models to help understand more complicated election situations and to test assumptions about how voters really do make voting decisions.

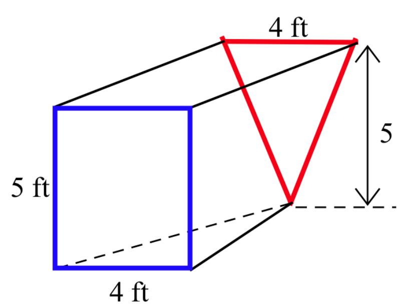
5.8 Problems

In Problems 1–5, use ρ for the density of the fluid in the given container.

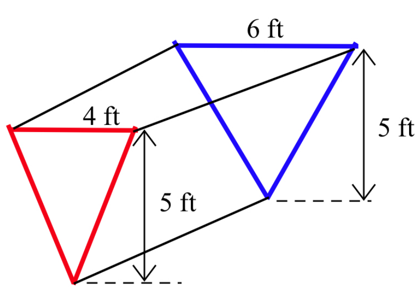
1. Calculate the force against windows A and B in the figure below.



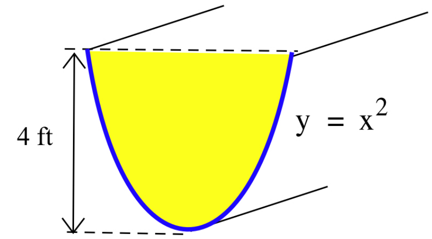
2. Calculate the force against windows C and D in the figure above.
3. Calculate the total force against each end of the tank shown below. How does the total force against the ends of the tank change if the length of the tank is doubled?



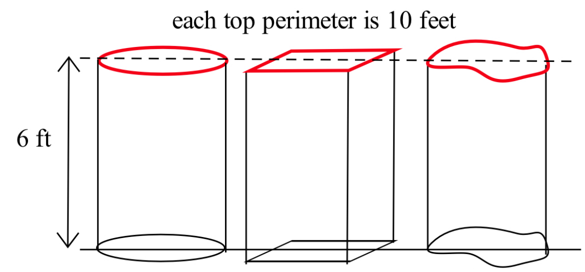
4. Calculate the total force against each end of the tank shown below.



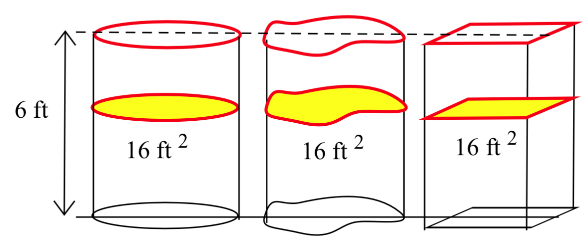
5. Calculate the total force against the end of the tank shown below.



6. The three tanks shown below are all 6 feet tall and the top perimeter of each tank is 10 feet. Which tank has the greatest total force against its sides?

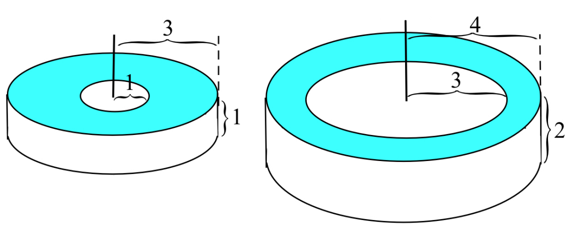


7. The three tanks shown below are all 6 feet tall and the cross-sectional area of each tank is 16 ft^2. Which tank has the greatest total force against its sides?

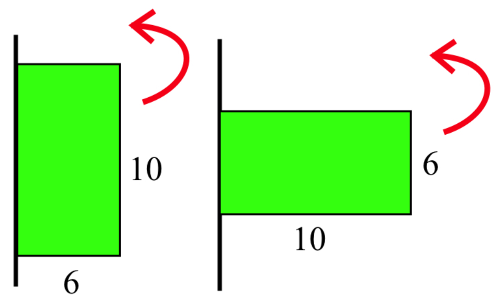


8. Calculate the total force against the bottom 2 feet of the sides of a tank with a square 40-foot by 40-foot base that is filled with water (a) to a depth of 30 feet. (b) to a depth of 35 feet.
9. Calculate the total force against the bottom 2 feet of the side of a cylindrical tank with a radius of 20 feet that is filled with water (a) to a depth of 30 feet. (b) to a depth of 35 feet.

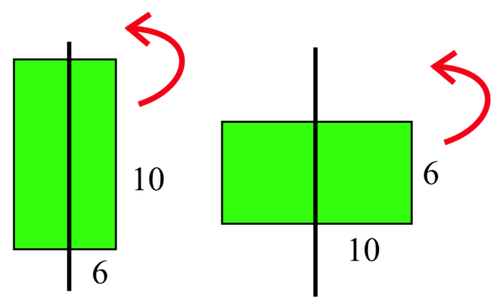
10. Calculate the total force against the side and bottom of a cylindrical aluminum soda can with diameter 6 cm and height 12 cm if it is filled with 385 g of soda. (Assume the can has been opened so carbonization is not a factor.)
11. Find the kinetic energy of a 20-gram object rotating at 3 revolutions per second at the end of (a) a 15-cm (massless) string and (b) a 20-cm string.
12. Each centimeter of a metal bar has a mass of 3 grams. Calculate the kinetic energy of the 50-centimeter bar if it is rotating at a rate of 2 revolutions per second about one of its ends.
13. Each centimeter of a metal bar has a mass of 3 grams. Calculate the kinetic energy of the 50-centimeter bar if it is rotating at a rate of 2 revolutions per second at the end of a 10-cm cable.
14. Calculate the kinetic energy of a 20-gram meter stick if it is rotating at a rate of 1 revolution per second about one of its ends.
15. Calculate the kinetic energy of a 20-gram meter stick if it is rotating at a rate of 1 revolution per second about its center point.
16. A flat, circular plate is made from material that has a density of 2 grams per cubic centimeter. The plate is 5 centimeters thick, has a radius of 30 centimeters and is rotating about its center at a rate of 2 revolutions per second. (a) Calculate its kinetic energy. (b) Find the radius of a plate that would have twice the kinetic energy of the first plate, assuming the density, thickness and rotation rate are the same.
17. Each "washer" in the figure below is made from material with density of 1 gram per cm^3, and each is rotating about its center at a rate of 3 revolutions per second. Calculate the kinetic energy of each washer (dimensions are in cm).



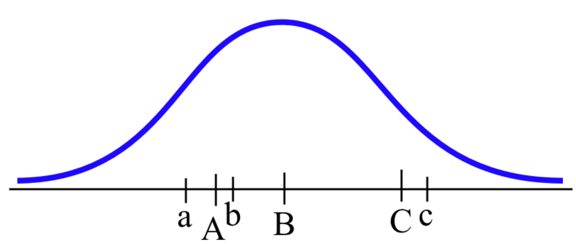
18. The rectangular plate shown below is 1 cm thick, 10 cm long and 6 cm wide and is made of a material with a density of 3 grams per cm^3. Calculate the kinetic energy of of the plate if it is rotating at a rate of 2 revolutions per second (a) about its 10-cm side and (b) about its 6-cm side.



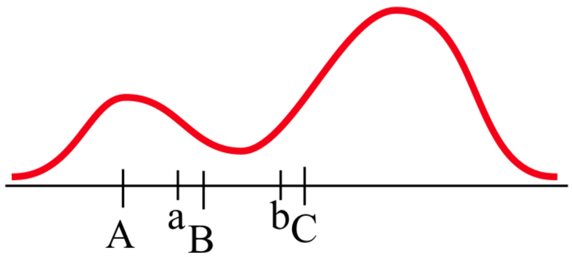
19. Calculate the kinetic energy of the plate in Problem 18 if it is rotating at a rate of 2 revolutions per second about a vertical line through the center of the plate, as shown below left.



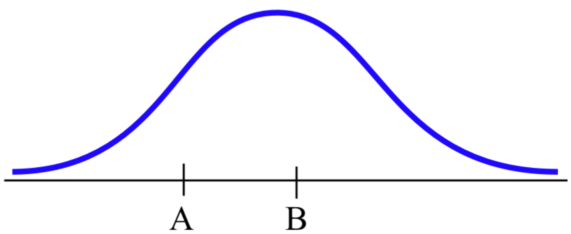
20. Calculate the kinetic energy of the plate in Problem 18 if it is rotating at a rate of 2 revolutions per second about a vertical line through the center of the plate, as shown above right.
21. For the voter distribution shown below, which candidates would the voters at positions a, b and c vote for?



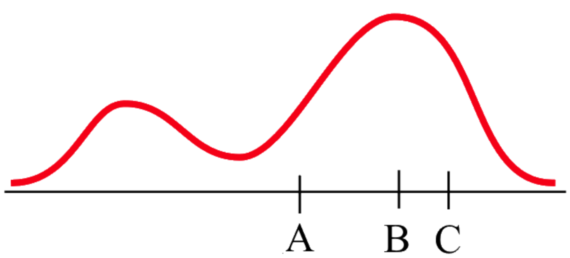
22. For the voter distribution shown below, which candidates would the voters at positions a, b and c vote for?



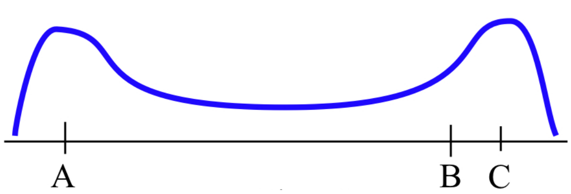
23. Shade the region representing votes for candidate A in the distribution shown below. Which candidate wins?



24. Shade the region representing votes for candidate A in the distribution shown below. Which candidate wins?

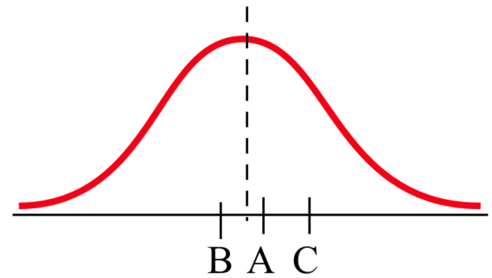


25. Refer to the voter distribution shown below.



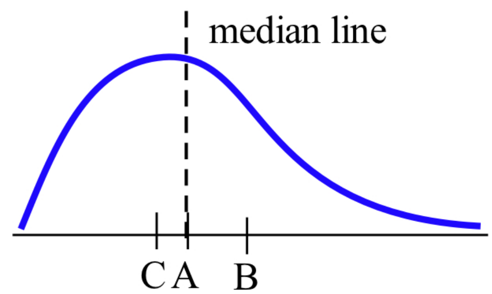
(a) Which candidate wins?
(b) If candidate B withdraws before the election, which candidate will win?
(c) If candidate B stays in the election but C withdraws, then who wins?

26. Refer to the voter distribution shown below.



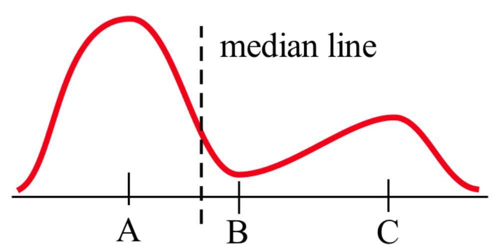
(a) Which candidate wins?
(b) If candidate B withdraws before the election, which candidate will win?
(c) If candidate B stays in the election but C withdraws, then who wins?

27. Refer to the voter distribution shown below.



(a) If the election is between A and B, who wins?
(b) If the election is between A and C, who wins?
(c) If the election is among A, B and C, who wins?

28. Refer to the voter distribution shown below.



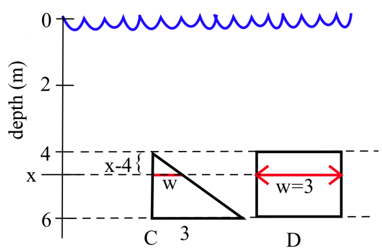
(a) If the election is between A and B, who wins?
(b) If the election is between A and C, who wins?
(c) If the election is among A, B and C, who wins?

29. Sketch a distribution for a two-issue election.

5.8 Practice Answers

1. The reasoning for a side of the aquarium is exactly the same as for the front, except a side is 1 foot long instead of 2, so the force is half of that against the front: 281.25 lbs. The total force against all sides (and the bottom) is:

$$2(281.25) + 2(562.5) + 375 = 2062.5 \text{ lbs}$$



2. For window C, using similar triangles (see margin), the width w of a slice at depth x m satisfies:

$$\frac{w}{x-4} = \frac{3}{2} \Rightarrow w = \frac{3}{2}(x-4) = \frac{3}{2}x - 6$$

so the area of a slice of height Δx_k m at depth x_k m is $\left(\frac{3}{2}x_k - 6\right)\Delta x_k \text{ m}^2$. The density of water is $1000 \frac{\text{kg}}{\text{m}^3}$ and $g \approx 9.81 \frac{\text{m}}{\text{sec}^2}$, so the hydrostatic force applied to the k-th slice is:

$$1000(9.81)x_k\left(\frac{3}{2}x_k - 6\right)\Delta x_k$$

and the total hydrostatic force applied to the window is therefore:

$$\int_{x=4}^{x=6} 9810\left[\frac{3}{2}x^2 - 6x\right] dx = 9810\left[\frac{1}{2}x^3 - 3x^2\right]_4^6$$
$$= 9810\Big[(108-108) - (32-48)\Big] = 156960\,\text{N}$$

Windows A (from Example 2) and C (with a horizontal flip) fit together to form window D, so it is encouraging that the sum of the total hydrostatic forces against A and C is $137340 + 156960 = 294300$ N, the total hydrostatic force against window D.

For window D, the width is 3 at all depths, so the total hydrostatic force against the window is:

$$\int_{x=4}^{x=6} 1000(9.81)x \cdot 3\,dx = 14715x^2\Big|_4^6 = 294300 \text{ N}$$

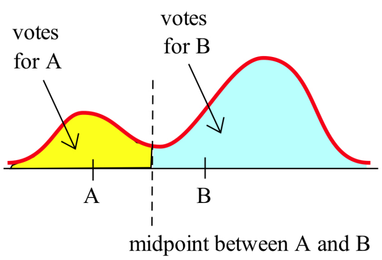
3. The object travels $2\pi\,(2 \text{ m}) = 4\pi$ m during one revolution, so during the 1 second it takes to make 4 revolutions, the object travels 16π m; its velocity is thus $v = 1600\pi \frac{\text{cm}}{\text{sec}}$ and its kinetic energy is:

$$\frac{1}{2}m \cdot v^2 = \frac{1}{2}(1 \text{ g})\left(1600\pi \frac{\text{cm}}{\text{sec}}\right)^2 = 1280000\pi^2 \text{ ergs} \approx 12633094 \text{ ergs}$$

4. Everything remains the same as in Example 4, except for the endpoints of integration:

$$\int_{x=1}^{x=4} 400000\pi^2 x^2\,dx = 400000\pi^2\left[\frac{1}{3}x^3\right]_1^4 = 8400000\pi^2 \text{ ergs}$$

which is approximately 82,904,677 ergs, or 8.29 J.



5. The shaded regions in the margin figure show the total votes for each candidate: B wins.