# 6 Differential Equations

This chapter introduces you to differential equations, a major field in applied and theoretical mathematics that provides useful tools for engineers, scientists and others studying changing phenomena.

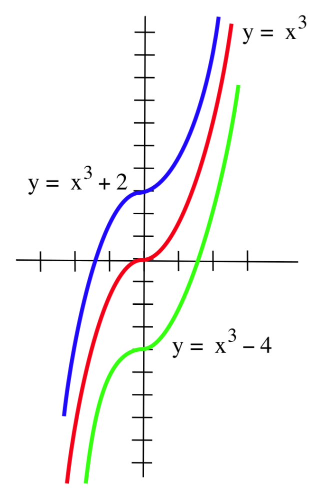
Physical laws of motion, heat and electricity can be expressed using differential equations. The growth of a population, the changing gene frequencies in that population, and the spread of a disease can be described by differential equations. Economic and social models use differential equations, and the earliest examples of "chaos" came from differential equations used for modeling atmospheric behavior. Some scientists even assert that the main purpose of a calculus course should be to teach people to understand and solve differential equations.

The purpose of this chapter is to introduce some basic ideas, vocabulary and techniques for differential equations and to explore additional applied problems that can be solved using calculus. Applications in this chapter include exponential population growth, calculating how long a population takes to double in size, radioactive decay and its use for dating ancient objects and detecting fraud, describing the motion of an object, and chemical mixtures and rates of reaction.

This chapter merely provides an introduction. In the near future you may very well take one or more classes solely devoted to solving differential equations.

## 6.1 Introduction to Differential Equations

Algebraic equations involve constants and variables, and solutions of algebraic equations typically involve numbers. For example, $x = 3$ and $x = -2$ are solutions of the algebraic equation $x^2 = x + 6$. **Differential equations** contain derivatives (or differentials) of functions and solutions of differential equations are functions. The differential equation $y' = 3x^2$ has infinitely many solutions, and two of those solutions are the functions $y = x^3 + 2$ and $y = x^3 - 4$ (see margin).




You have already solved lots of differential equations: every time you found an antiderivative of a function $f(x)$, you solved the differential equation $y' = f(x)$ to get a solution $y$. You have also used differential equations in applications. Areas, volumes, work and motion problems all involved integration and finding antiderivatives, so they all involved solving a differential equation. The differential equation $y' = f(x)$, however, is just the beginning. Other applications generate differential equations that may involve higher-order derivatives of $y$ (such as $y''$) and functions of $y$ as well as $x$.

*Checking Solutions of Differential Equations*

Whether a differential equation is easy or difficult to solve, it is important to be able to check that a possible solution actually satisfies the differential equation. A possible solution of an algebraic equation can be checked by putting the solution into the equation to see if it results in true statment: $x = 3$ is a solution of $5x + 1 = 16$ because $5(3) + 1 = 16$ is true; $x = 4$ is not a solution, because $5(4) + 1 \neq 16$.

Similarly, a solution of a differential equation can be checked by substituting the function (and its appropriate derivatives) into the original equation to see if the result is true: $y = x^2$ is a solution of $xy' = 2y$ because $y = x^2 \Rightarrow y' = 2x$ and $x \cdot 2x = 2 \cdot x^2$ is a true statement for all values of $x$.

**Example 1.** Check that (a) $y = x^2 + 5$ is a solution of $y'' + y = x^2 + 7$ and (b) $y = x + \dfrac{5}{x}$ is a solution of $y' + \dfrac{y}{x} = 2$.

**Solution.** (a) $y = x^2 + 5 \Rightarrow y' = 2x \Rightarrow y'' = 2$. Substituting these functions for $y$ and $y''$ into the left side of the differential equation $y'' + y = x^2 + 7$ yields:

$$y'' + y = (2) + (x^2 + 5) = x^2 + 7$$

so $y = x^2 + 5$ is a solution of the differential equation.

(b) $y = x + \dfrac{5}{x} \Rightarrow y' = 1 - \dfrac{5}{x^2}$. Substituting these functions for $y$ and $y'$ into the left side of the differential equation $y' + \dfrac{y}{x} = 2$, we have:

$$y' + \frac{y}{x} = \left[1 - \frac{5}{x^2}\right] + \frac{1}{x}\left[x + \frac{5}{x}\right] = 1 - \frac{5}{x^2} + 1 + \frac{5}{x^2} = 2$$

which matches the right side of the original differential equation. ◀

**Practice 1.** Check that (a) $y = 2x + 6$ is a solution of $y - 3y' = 2x$ and (b) $y = e^{3x}$, $y = 5e^{3x}$ and $y = Ae^{3x}$ (where $A$ is any constant) are all solutions of $y'' - 2y' - 3y = 0$.

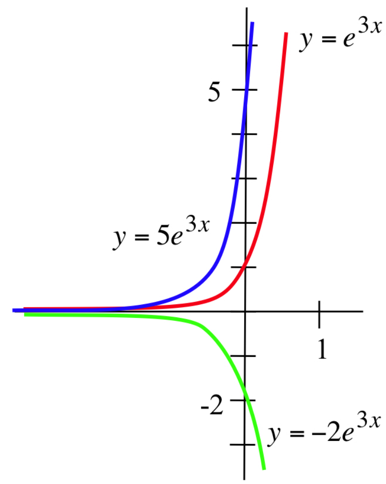
A solution of a differential equation with the **initial condition** $y(x_0) = y_0$ is a function that satisfies the differential equation as well as the initial condition. To check the solution of an **initial value problem** (or **IVP**), we must check that a solution function satisfies both the equation and the initial condition.

**Example 2.** Which of the given functions is a solution of the initial value problem $y' = 3y$, $y(0) = 5$?

(a) $y = e^{3x}$ (b) $y = 5e^{3x}$ (c) $y = -2e^{3x}$

**Solution.** All three functions satisfy the differential equation, but only one of them satisfies the initial condition that $y(0) = 5$. If $y = e^{3x}$, then $y(0) = e^{3(0)} = 1 \neq 5$ so $y = e^{3x}$ does not satisfy the initial condition (see margin). If $y = 5e^{3x}$, then $y(0) = 5e^{3(0)} = 5$ so $y = 5e^{3x}$ does satisfy the initial condition. If $y = -2e^{3x}$, then $y(0) = -2e^{3(0)} = -2 \neq 5$ so $y = e^{3x}$ does not satisfy the initial condition. ◀

You should check that they do.



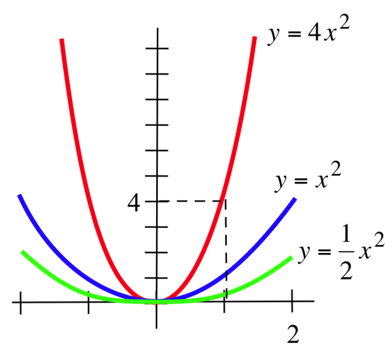

**Practice 2.** Which function is a solution of the initial value problem $y'' + 9y = 0$, $y(0) = 2$?

(a) $y = \sin(3x)$ (b) $y = 2\sin(3x)$ (c) $y = 2\cos(3x)$

*Finding the Value of the Constant*

Differential equations usually have many solutions, typically a whole "family" of them, with each solution in the family satisfying a different initial condition. To find which solution of a differential equation also satisfies a given initial condition of the form $y(x_0) = y_0$, we replace $x$ and $y$ in an equation describing the solution family with the values $x_0$ and $y_0$, then algebraically solve for the value of an unknown constant.

**Example 3.** For every value of $C$, the function $y = Cx^2$ is a solution of $xy' = 2y$ (see margin). Find the value of $C$ so that $y(5) = 50$.




**Solution.** Substituting the initial condition $x = 5$ and $y = 50$ into the solution $y = Cx^2$:

$$50 = C(5^2) \;\Rightarrow\; C = \frac{50}{25} = 2$$

so the function $y = 2x^2$ satisfies both the differential equation and the initial condition. ◀

**Practice 3.** For every value of $C$, the function $y = e^{2x} + C$ is a solution of $y' = 2e^{2x}$. Find the value of $C$ so that $y(0) = 7$.

*Types of Differential Equations*

In Chapter 14, we will begin studying **partial derivatives** of functions of more than one variable, which can appear in differential equations called **partial differential equations** (or **PDE**s). Because of this, we will call differential equations involving ordinary derivatives, such as $\frac{dy}{dx}$, $\frac{dy}{dt}$ or $\frac{d^2y}{dt^2}$, **ordinary differential equations** (or **ODE**s).

## 6.1 Problems

In Problems 1–10, check that the function $y$ is a solution of the given differential equation.

1. $y' + 3y = 6$; $y = e^{-3x} + 2$
2. $y' - 2y = 8$; $y = e^{2x} - 4$
3. $y'' - y' + y = x^2$; $y = x^2 + 2x$
4. $3y'' + y' + y = x^2 - 4x$; $y = x^2 - 6x$
5. $xy' - 3y = x^2$; $y = 7x^3 - x^2$
6. $xy'' - y' = 3$; $y = x^2 - 3x + 5$
7. $y' + y = e^x$; $y = \frac{1}{2}e^x + 2e^{-x}$
8. $y'' + 25y = 0$; $y = \sin(5x) + 2\cos(5x)$
9. $y' = -\dfrac{x}{y}$; $y = \sqrt{7 - x^2}$
10. $y' = x - y$; $y = x - 1 + 2e^{-x}$

In Problems 11–20, check that the function $y$ is a solution of the given initial value problem.

11. $y' = 6x^2 - 3$, $y(1) = 2$; $y = 2x^3 - 3x + 3$
12. $y' = 6x + 4$, $y(2) = 3$; $y = 3x^2 + 4x - 17$
13. $y' = 2\cos(2x)$, $y(0) = 1$; $y = \sin(2x) + 1$
14. $y' = 1 + 6\sin(2x)$, $y(0) = 2$; $y = x - 3\cos(2x) + 5$
15. $y' = 5y$, $y(0) = 7$; $y = 7e^{5x}$
16. $y' = -2y$, $y(0) = 3$; $y = 3e^{-2x}$
17. $xy' = -y$, $y(1) = -4$; $y = -\dfrac{4}{x}$
18. $y \cdot y' = -x$, $y(0) = 3$; $y = \sqrt{9 - x^2}$
19. $y' = \dfrac{5}{x}$, $y(e) = 3$; $y = 5\ln(x) - 2$
20. $y' + y = e^x$, $y(0) = 5$; $y = \frac{1}{2}e^x + \frac{9}{2}e^{-x}$

In Problems 21–30, find the value of the constant $C$ so a function from the given family of solutions satisfies the given initial value problem.

21. $y' = 2x$, $y(3) = 7$; $y = x^2 + C$
22. $y' = 3x^2 - 5$, $y(1) = 2$; $y = x^3 - 5x + C$
23. $y' = 3y$, $y(0) = 5$; $y = Ce^{3x}$
24. $y' = -2y$, $y(0) = 3$; $y = Ce^{-2x}$
25. $y' = 6\cos(3x)$, $y(0) = 4$; $y = 2\sin(3x) + C$
26. $y' = 3 - 2\sin(2x)$, $y(0) = 1$; $y = 3x + \cos(2x) + C$
27. $y' = \dfrac{1}{x}$, $y(e) = 2$; $y = \ln(x) + C$
28. $y' = \dfrac{1}{x^2}$, $y(1) = 3$; $y = -\frac{1}{x} + C$
29. $y' = -\dfrac{y}{x}$, $y(2) = 10$; $y = -\frac{C}{x}$
30. $y' = -\frac{x}{y}$, $y(3) = 4$; $y = \sqrt{C - x^2}$

In Problems 31–40, find the find the function $y$ that satisfies the given initial value problem.

31. $y' = 4x^2 - x$, $y(1) = 7$
32. $y' = x - e^x$, $y(0) = 3$
33. $y' = \dfrac{3}{x}$, $y(1) = 2$
34. $xy' = 1$, $y(e) = 7$
35. $y' = 6e^{2x}$, $y(0) = 1$
36. $y' = 36(3x - 2)^2$, $y(1) = 8$
37. $y' = x \cdot \sin(x^2)$, $y(0) = 3$
38. $y' = \dfrac{6}{x^2}$, $y(1) = 2$
39. $xy' = 6x^3 - 10x^2$, $y(2) = 5$
40. $x^2y' = 6x^3 - 1$, $y(1) = 10$
41. Show that if $y = f(x)$ and $y = g(x)$ are both solutions to $y' + 5y = 0$, then $y = 3 \cdot f(x)$, $y = 7 \cdot g(x)$, $y = f(x) + g(x)$ and $y = A \cdot f(x) + B \cdot g(x)$ are solutions for any constants $A$ and $B$.
42. Show that if $f(x)$ and $g(x)$ are both solutions to $y'' + 2y' - 3y = 0$, then so are $y = 3 \cdot f(x)$, $y = 7 \cdot g(x)$, $y = f(x) + g(x)$ and $y = A \cdot f(x) + B \cdot g(x)$ for any constants $A$ and $B$.
43. Show that $y = \sin(x) + x$ and $y = \cos(x) + x$ are both solutions of $y'' + y = x$. Are $y = 3[\sin(x) + x]$ and $y = [\sin(x) + x] + [\cos(x) + x]$ solutions of $y'' + y = x$?
44. Show that $y = e^{3x} - 2$ and $y = 5e^{3x} - 2$ are both solutions of $y' - 3y = 6$. Are $y = 7\left[e^{3x} - 2\right]$ and $y = \left[e^{3x} - 2\right] + \left[5e^{3x} - 2\right]$ also solutions?

45. The ODE $\frac{dy}{dt} = A - By$ (where $A$ and $B$ are positive constants) describes the concentration $y$ of glucose in a person's blood at time $t$. Check that $y = \frac{A}{B} - C \cdot e^{-Bt}$ is a solution of the ODE for any value of the constant $C$.

46. The ODE $\frac{dy}{dt} = Ay$ (where $A$ is a positive constant) is used to model "exponential" growth and decay. Check that $y = C \cdot e^{At}$ is a solution of the differential equation for any value of the constant $C$.

47. The ODE $L \cdot \frac{dI}{dt} + RI = E$ (where $L$, $R$ and $E$ are positive constants) describes the current $I(t)$ in an electrical circuit. Show that $I = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ is a solution of the ODE.

48. The ODE $m \cdot y'' + C \cdot y = 0$ (where $C$ is a positive constant) describes the position $y$ of an object hung from a spring as it moves up and down. Show that $y = A \cdot \sin(\omega t) + B \cdot \cos(\omega t)$ with $\omega = \sqrt{\frac{C}{m}}$ is a solution of the ODE for all values of the constants $A$ and $B$.

## *6.1 Practice Answers*

1. (a) $y = 2x + 6 \Rightarrow y' = 2$; $y - 3y' = (2x + 6) - 3(2) = 2x$ (OK)

   (b) $y = e^{3x} \Rightarrow y' = 3e^{3x} \Rightarrow y'' = 9e^{3x}$ so:

   $$y'' - 2y' - 3y = 9e^{3x} - 2(3e^{3x}) - 3(e^{3x}) = 0 \quad \text{(OK)}$$

   $y = 5e^{3x} \Rightarrow y' = 15e^{3x} \Rightarrow y'' = 45e^{3x}$ so:

   $$y'' - 2y' - 3y = 45e^{3x} - 2(15e^{3x}) - 3(5e^{3x}) = 0 \quad \text{(OK)}$$

   $y = Ae^{3x} \Rightarrow y' = 3Ae^{3x} \Rightarrow y'' = 9Ae^{3x}$ so:

   $$y'' - 2y' - 3y = 9Ae^{3x} - 2(3Ae^{3x}) - 3(Ae^{3x}) = 0 \quad \text{(OK)}$$

2. We want $y'' + 9y = 0$ and $y(0) = 2$.

   (a) $y = \sin(3x) \Rightarrow y' = 3\cos(3x) \Rightarrow y'' = -9\sin(3x)$ so we have $y'' + 9y = -9\sin(3x) + 9 \cdot \sin(3x) = 0$ (OK) but checking the initial condition: $y(0) = \sin(0) = 0 \neq 2$.

   (b) $y = 2\sin(3x) \Rightarrow y' = 6\cos(3x) \Rightarrow y'' = -18\sin(3x)$ so we have $y'' + 9y = -18\sin(3x) + 9 \cdot 2\sin(3x) = 0$ (OK) but checking the initial condition: $y(0) = 2\sin(0) = 0 \neq 2$

   (c) $y = 2\cos(3x) \Rightarrow y' = -6\sin(3x) \Rightarrow y'' = -18\cos(3x)$ so $y'' + 9y = -18\cos(3x) + 9 \cdot 2\cos(3x) = 0$ (OK) and checking the initial condition: $y(0) = 2\cos(0) = 2$ (OK).

   Only $y = 2\cos(3x)$ satisfies both the ODE and the initial condition.

3. $y = e^{2x} + C \Rightarrow y' = 2e^{2x}$ (OK) so, plugging in the initial values:

   $$7 = y(0) = e^{2 \cdot 0} + C \Rightarrow 7 = 1 + C \Rightarrow C = 6$$

   so $y = e^{2x} + 6$.

## 6.2 *The Differential Equation $y' = f(x)$*

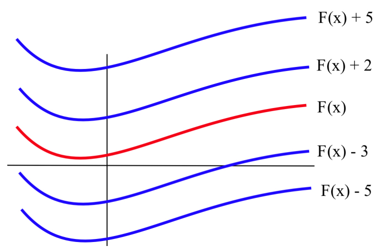
We'll use these concepts in later sections as we examine more complicated differential equations and their applications.

This section introduces some basic concepts and vocabulary of the study of ODEs as they apply to the familiar problem, $y' = f(x)$: the notion of a general solution of an ODE, the (possibly unique) solution to an IVP and the direction field of an ODE.

### *Solving $y' = f(x)$*

This is due to Corollary 2 to the Mean Value Theorem in Section 3.2.

The solution of the ODE $y' = f(x)$ is the collection of all antiderivatives of $f$: $y = \int f(x)\,dx$. If $y = F(x)$ is one antiderivative of $f$, then we have essentially found *all* antiderivatives of $f$ because any antiderivative of $f$ has the form $F(x) + C$, for some value of the constant $C$. If $F$ is one particular antiderivative of $f$, the collection of functions $F(x) + C$ is called the **general solution** of $y' = f(x)$. The general solution consists of a **one-parameter family** of functions (the parameter here is $C$).

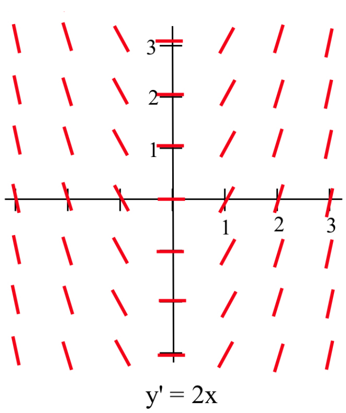



**Example 1.** Find the general solution of the ODE $y' = 2x + e^{3x}$.

**Solution.** $y = \int \left[2x + e^{3x}\right] dx = x^2 + \frac{1}{3}e^{3x} + C$. ◀

**Practice 1.** Find general solutions for $y' = x + \frac{3}{x+2}$ and $y' = \frac{6}{x^2+1}$.

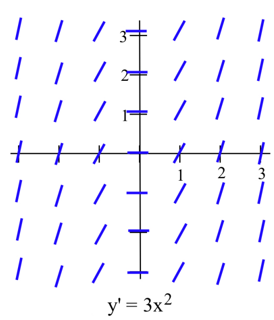
### *Direction Fields*

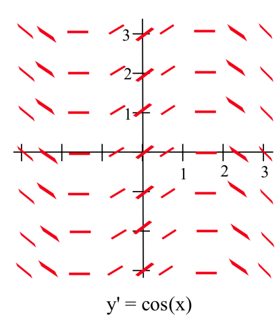



What are the solutions to $y' = 2x$? Can you "see" those solution curves in the direction field graphed above?

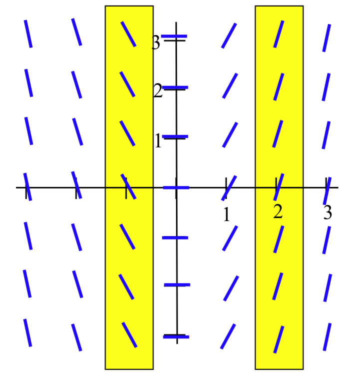
Geometrically, a derivative tells us the slope of the tangent line to a curve, so we can interpret the ODE $y' = f(x)$ as a geometric condition: at each point $(a, b)$ on the graph of the solution function $y$, the slope of the tangent line is $f(a)$. The ODE $y' = 2x$ says that at each point $(a, b)$ on the graph of $y$, the slope of the line tangent to the graph is $2a$: if the point $(5, 3)$ is on the graph of $y$, then the slope of the tangent line there is $2 \cdot 5 = 10$. We can present this information graphically as a **direction field**: a collection of short line segments through some sample points in the plane so that the slope of the segment through $(a, b)$ is $f(a)$. A direction field for $y' = 2x$ appears in the margin: at a point $(a, b)$, the slope is $2a$. Direction fields for $y' = 3x^2$ and $y' = \cos(x)$ appear below:

What are the solutions to $y' = 3x^2$? To $y' = \cos(x)$? Can you "see" those solution curves in the direction fields?

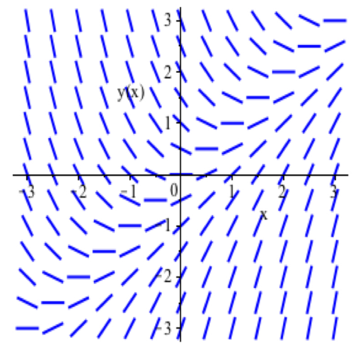


For any ODE of the form $y' = f(x)$, the values of $y'$ depend only on $x$, so along any vertical line (where $x$ is fixed) all the small line segments have the same $y'$, hence the same slope, and they are parallel (see margin). If $y'$ depends on both $x$ and $y$, then the slopes of the line segments will depend on both $x$ and $y$, and the slopes of the small line segments along a vertical line are not all the same. The second margin figure shows a direction field for $y' = x - y$, where $y'$ is a function of both $x$ and $y$. A direction field of an ODE $y' = g(x, y)$ is a collection of short line segments with slope $g(a, b)$ at the point $(a, b)$.



Direction field for y' = f(x)

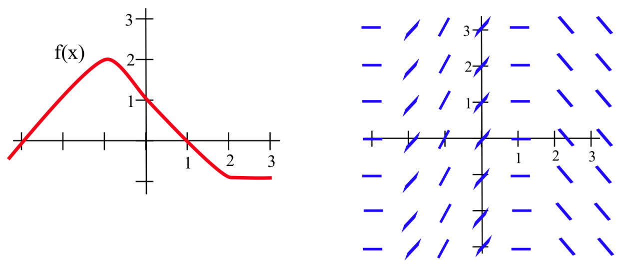


Direction field for y' = x - y

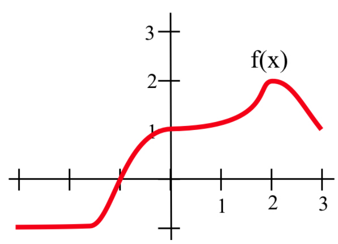
**Practice 2.** Construct direction fields for (a) $y' = x + 1$ and (b) $y' = x + y$ by sketching a short line segment with slope $y'$ at each point $(a, b)$ with integer coordinates from $-3$ to 3.

As you discovered in the previous Practice problem, direction fields are usually tedious to plot by hand, but computers (and some calculators) can plot them quickly. If you only have a graph of the function $f$ in the differential equation $y' = f(x)$, you can construct an approximate direction field using the information you have about $f$ from its graph.

**Example 2.** Construct a direction field for the differential equation $y' = f(x)$ for the $f$ given graphically below left.

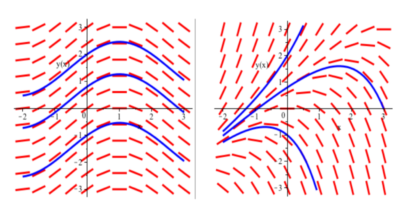



**Solution.** If $x = 0$ then $y' = f(0) = 1$, so at every point on the vertical line where $x = 0$ (the $y$-axis) the line segments of the direction field have slope $y' = 1$ (above right). Similarly, if $x = 1$ then $y' = f(1) = 0$, so the line segments of the direction field have slope $y' = 0$ at every point on the vertical line where $x = 1$. The small line segments along any vertical line are parallel. ◀



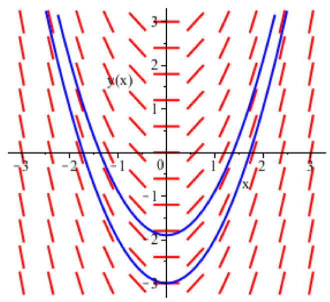

**Practice 3.** Construct a direction field for the differential equation $y' = f(x)$ for the $f$ given graphically in the margin.

Once you have a direction field for an ODE, you can sketch curves that have the appropriate tangent-line slopes so you can "see" the shapes of the solution curves even if you do not have formulas for them. (See margin.) These shapes can be useful for estimating which initial conditions lead to straight-line solutions or periodic solutions or solutions with other properties, and they can help us understand the behavior of machines and organisms in applied problems.



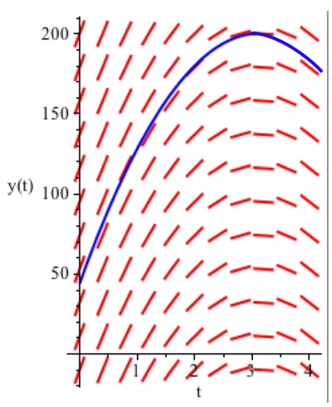
*Initial Value Problems*

An initial condition $y(x_0) = y_0$ specifies that the solution $y$ of the differential equation should go through the point $(x_0, y_0)$ in the plane. To solve a differential equation with an initial condition, you typically use integration to find the general solution (a family of solutions containing an arbitrary constant) and then you use algebra to find the one value for the constant so the solution satisfies the initial condition.




**Example 3.** Solve the differential equation $y' = 2x$ with the initial condition $y(2) = 1$.

**Solution.** The general solution is $y = \int 2x\,dx = x^2 + C$. Substituting $x_0 = 2$ and $y_0 = 1$ into the general solution: $1 = (2)^2 + C \Rightarrow C = -3$. So the solution we want is $y = x^2 - 3$. (A quick check verifies that $\left[x^2 - 3\right]' = 2x$ and $2^2 - 3 = 1$.) The margin shows a direction field for $y' = 2x$ and the solution curve that goes through $(2, 1)$, $y = x^2 - 3$, along with the solution of the ODE that satisfies $y(1) = -1$. ◀



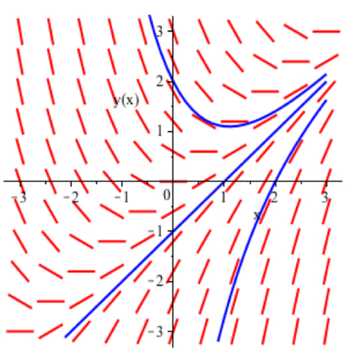

**Example 4.** If you toss a ball upward with an initial velocity of 100 feet per second, its height $y$ (in feet) at time $t$ (in seconds) satisfies the differential equation $y' = 100 - 32t$. Sketch the direction field for $y$ (for $0 \le t \le 4$) and then sketch the solution that satisfies the condition that the ball is 200 feet high after 3 seconds.

**Solution.** $y = \int [100 - 32t]\,dt = 100t - 16t^2 + C$; if $y(3) = 200$:

$$200 = 100(3) - 16(3)^2 + C \;\Rightarrow\; 200 = 156 + C \;\Rightarrow C = 44$$

The function we want is $y = 100t - 16t^2 + 44$. The direction field and the solution satisfying $y(3) = 200$ appear in the margin. ◀

**Practice 4.** Find the solution of $y' = 9x^2 - 6\sin(2x) + e^x$ that goes through the point $(0, 6)$.




**Example 5.** A direction field for $y' = x - y$ appears in the margin. Sketch the three solutions of the ODE $y' = x - y$ that satisfy the initial conditions $y(0) = 2$, $y(0) = -1$ and $y(1) = -2$.

**Solution.** See margin figure. ◀

*Existence and Uniqueness*

When solving IVPs, three questions often present themselves:

- Does a solution to the IVP exist?
- Is that solution unique?
- On what interval(s) is that solution valid?

We typically hope the answer the first two questions is "yes," but (as we will see in the next section) that will not always be true.

Surprisingly, for reasonably "nice" IVPs, we will eventually be able to answer these questions without actually finding the solution.

For IVPs of the form $y' = f(x)$, $y(x_0) = y_0$, where $f(x)$ is continuous on some interval $(a, b)$ with $a < x_0 < b$, the answer to the first two questions is "yes" and the answer to third question is "on the interval $(a, b)$, and possibly on some larger interval." In this situation, define:

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$$

The Fundamental Theorem of Calculus tells us that $y'(x) = f(x)$, so this $y(x)$ solves the ODE, and:

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(t)\,dt = y_0 + 0 = y_0$$

so it also satisfies the initial value condition. If $\tilde{y}(x)$ is any other solution to the IVP, then $\tilde{y}'(x) = f(x) = y'(x)$ so $\tilde{y}(x) = y(x) + C$, but we also know that $\tilde{y}(x_0) = y_0 = y(x_0)$:

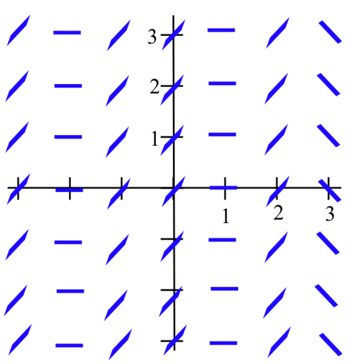
$$\tilde{y}(x_0) = y(x_0) + C \;\Rightarrow\; y_0 = y_0 + C \;\Rightarrow\; C = 0 \;\Rightarrow \tilde{y}(x) = y(x)$$

which tells us that $y(x)$ is the *only* solution. Finally, $f(x)$ is continuous on $(a, b)$, hence integrable on $(a, b)$, so the integral definition of $y(x)$ is defined (and solves the IVP) on $(a, b)$.

## *6.2 Problems*

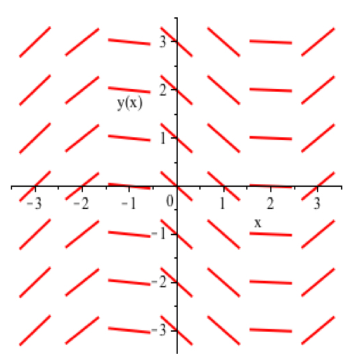
In Problems 1–6, use the given direction field to sketch the solutions of the underlying ODE that satisfy the given initial conditions.

1. See figure below; $y(0) = 1$, $y(1) = -2$ and $y(1) = 3$.



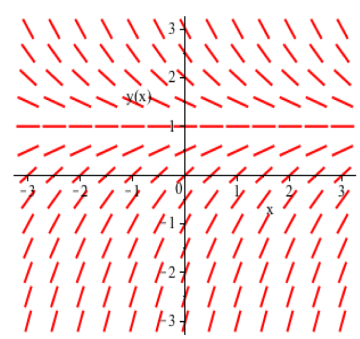

2. See figure above; $y(0) = 2$, $y(1) = -1$ and $y(1) = -2$.

3. See figure below; $y(-2) = 1$, $y(0) = 1$ and $y(2) = 1$.




4. See figure above; $y(-2) = -1$, $y(0) = -1$, $y(2) = -1$.

5. See figure below; $y(0) = -2$, $y(0) = 0$ and $y(0) = 2$.




6. See figure above; $y(2) = -2$, $y(2) = 0$ and $y(2) = 2$.

7. How do the three solutions in Problem 5 behave for large values of $x$?

8. How do the three solutions in Problem 6 behave for large values of $x$?

In Problems 9–14, (a) sketch a direction field for the given ODE and (b) without solving the ODE, sketch solutions that go through the points $(0, 1)$ and $(2, 0)$.

9. $y' = 2x$

10. $y' = 2 - x$

11. $y' = 2 + \sin(x)$

12. $y' = e^x$

13. $y' = 2x + y$

14. $y' = 2x - y$

In Problems 15–20, (a) find the family of functions that solve the given ODE, (b) find the member of the family that satisfies the given IVP and (c) report the interval on which the solution to the IVP is valid.

15. $y' = 2x - 3$, $y(1) = 4$

16. $y' = 1 - 2x$, $y(2) = -3$

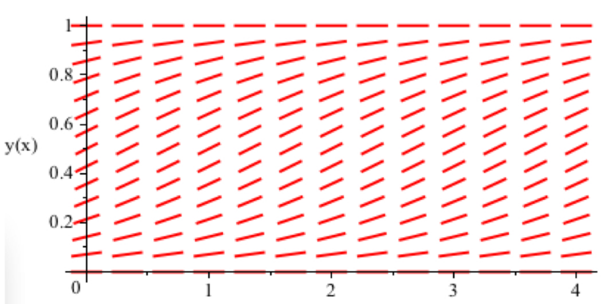
17. $y' = e^x + \cos(x)$, $y(0) = 7$

18. $y' = \sin(2x) - \cos(x)$, $y(0) = -5$

19. $y' = \dfrac{6}{2x+1} + \sqrt{x}$, $y(1) = 4$

20. $y' = \dfrac{e^x}{1+e^x}$, $y(0) = 0$

Problems 21–22 concern a direction field (shown below) that comes from an ODE called the **logistic equation**, $y' = y(1 - y)$, used to model the growth of a population in an environment with renewable but limited resources. (It is also used to describe the spread of a rumor or disease through a population.)

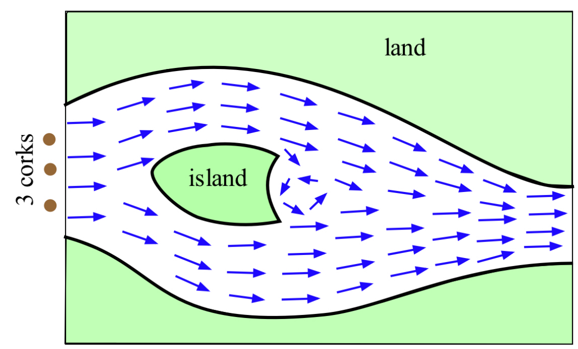



21. Sketch the solution that satisfies the initial condition $y(0) = 0.1$. What letter of the alphabet does this solution resemble?

22. Sketch several solutions that have different initial values for $y(0)$. What appears to happen to all of these solutions after a "long time" (for large values of $x$)?

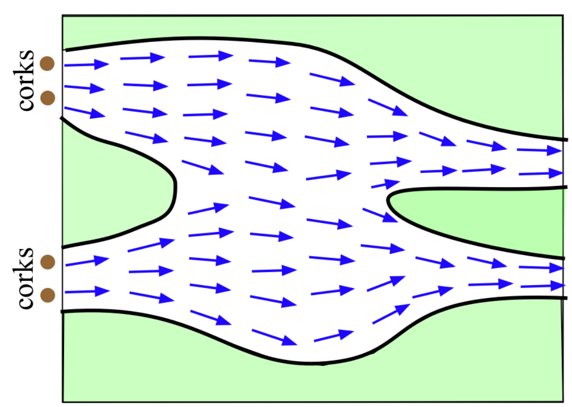
In Problems 23–24, the given figures show the direction of surface flow at different locations along a river. Sketch the paths small corks will follow if they are put into the river at the dots in each figure. (Because they indicate both the magnitude and the direction of flow, each diagram is called a **vector field**.) Notice that corks that start close to each other can drift far apart, and corks that start far apart can drift close together.

23. See figure below.




Surface flow along a river

24. See figure below.



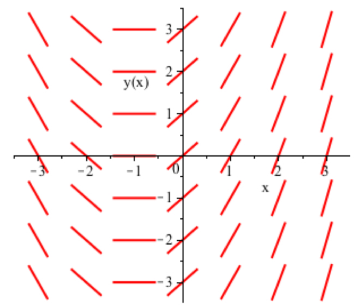

Surface flow along a river

*6.2 Practice Answers*

1. $y' = x + \dfrac{3}{x+2} \;\Rightarrow\; y = \displaystyle\int \left[ x + \frac{3}{x+2} \right] dx = \frac{1}{2}x^2 + 3\ln\left(|x+2|\right) + C$

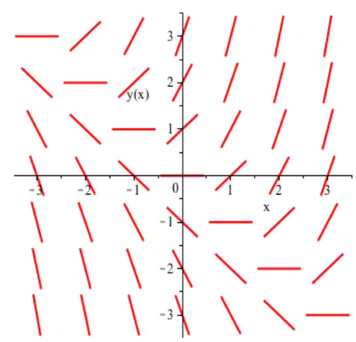
   $y' = \dfrac{6}{x^2+1} \;\Rightarrow\; y = \displaystyle\int \frac{6}{x^2+1}\, dx = 6\arctan(x) + C$

2. (a) If $y' = x + 1$ then the table below lists values of $y'$ for integer values of $x$ and $y$ from $-3$ to $3$ (notice that $y'$ does not depend on the value of $y$); the margin figure shows the direction field.



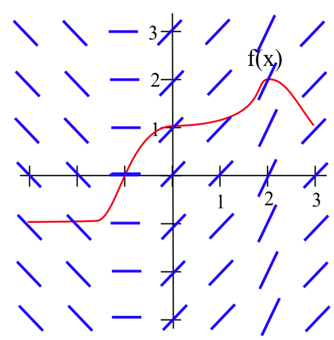

| | $x=-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y=3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $2$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $1$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $0$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $-1$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $-2$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |

   (b) If $y' = x + y$ then the table below lists values of $y'$ for integer values of $x$ and $y$ from $-3$ to $3$ (notice that $y'$ here *does* depend on both $x$ and $y$); the margin figure shows the direction field.




| | $x=-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y=3$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
| $2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
| $1$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $0$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
| $-1$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
| $-2$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ |
| $-3$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |

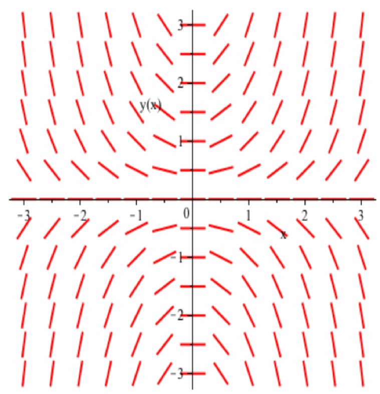
3. An approximate direction field for the ODE $y' = f(x)$ appears in the margin. (The function shown is $f(x)$, not a solution to the ODE.)



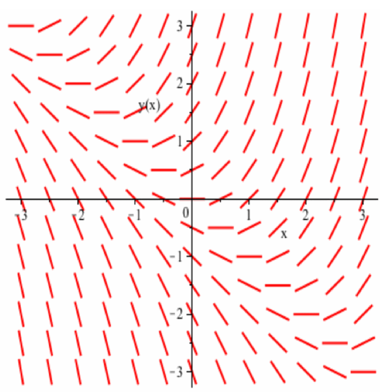

4. $y = \displaystyle\int \left[ 9x^2 - 6\sin(2x) + e^x \right] dx = 3x^3 + 3\cos(2x) + e^x + C$ so:

$$6 = y(0) = 3\cdot 0^3 + 3\cos(2\cdot 0) + e^0 + C = 0 + 3 + 1 + C \;\Rightarrow\; C = 2$$

   and therefore $y = 3x^3 + 3\cos(2x) + e^x + 2$.



Direction field for y' = xy



Direction field for y' = x+y

## 6.3 *Separable Equations*

So far, we have only learned how to solve ODEs of the form $y' = f(x)$ (which you already knew how to solve), where $y'$ depended only on $x$ and the slopes of the line segments of a corresponding direction field did not depend on the $y$-coordinate of the location of the line segment. In many situations, however, $y'$ depends on both $x$ and $y$, for example, $y' = xy$ (see first margin figure for a direction field) or $y' = x + y$ (see second margin figure for a direction field).

In Example 2 of Section 2.9, we considered the problem of finding a tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$. You can solve this problem geometrically. Or you can solve the equation for $y$ to get $y = \sqrt{25 - x^2}$ and then compute $\frac{dy}{dx}$. Or you can implicitly differentiate the equation $x^2 + y^2 = 25$, keeping in mind that $y$ is a function of $x$:

$$x^2 + y^2 = 25 \Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow 2y \cdot \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

The result is a differential equation (although we didn't call it that in Section 2.9) and putting $x = 3$ and $y = 4$ into the right side of that ODE yields $\frac{dy}{dx} = -\frac{3}{4}$, which gives us the slope of the line tangent to the circle at the point $(3, 4)$.

What if we start with $\frac{dy}{dx} = -\frac{x}{y}$ and try to solve this ODE? It is not of the form $y' = f(x)$, but we can try to solve it by beginning to reverse the steps in the implicit differentiation process above:

$$\frac{dy}{dx} = -\frac{x}{y} \quad \Rightarrow \quad y \cdot \frac{dy}{dx} = -x$$

Each side of this new ODE is a function of $x$ because $y$ and $\frac{dy}{dx}$ are functions of $x$, so we can integrate both sides of this equation to get:

Each indefinite integral yields a constant, but these are not necessarily equal.

$$\int y \cdot \frac{dy}{dx}\, dx = \int -x\, dx \Rightarrow \frac{1}{2}y^2 + C_1 = -\frac{1}{2}x^2 + C_2$$

The integration on the left-hand side of this equation uses the fact that:

$$\frac{d}{dx}\left[\frac{1}{2}y^2\right] = \frac{1}{2} \cdot 2y \cdot \frac{dy}{dx} = y \cdot \frac{dy}{dx}$$

but this is the same result as if we had "cancelled" the $dx$ in the original left-hand integral:

$$\int y \cdot \frac{dy}{dx}\, dx = \int y\, dy = \frac{1}{2}y^2 + C_1$$

so in our first steps above we could have instead rewritten the ODE using differentials:

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y \cdot \frac{dy}{dx} = -x \Rightarrow y\, dy = -x\, dx \Rightarrow \int y\, dy = \int -x\, dx$$

and then integrated both sides. (We will follow this procedure as we solve similar ODEs from now on.) We can now move both constants of integration to the same side of the equation:

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + [C_2 - C_1] \;\Rightarrow\; \frac{1}{2}y^2 = -\frac{1}{2}x^2 + K$$

where $K$ is another arbitrary constant. Because we can *always* move the left-hand constant to the right side of the equation and combine it with the right-hand constant, in the future we will use a single constant on the right side when integrating both sides of the differential equation.

At this stage we can write:

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + K \;\Rightarrow\; y^2 = -x^2 + 2K \;\Rightarrow x^2 + y^2 = C$$

where $C$ is yet another arbitrary constant. We recognize this as an equation for a circle centered at the origin and we call a solution of this type an **implicit solution** to the ODE because we have not explicitly solved for $y$. If we know our solution curve passes through $(3, 4)$, we can put this information into our solution equation to get:

$$3^2 + 4^2 = C \;\Rightarrow\; C = 25 \;\Rightarrow\; x^2 + y^2 = 25$$

Solving for $y$, we get $y = \sqrt{25 - x^2}$, an **explicit solution** to the ODE.

We first get $y = \pm\sqrt{25 - x^2}$ but knowing that $y(3) = 4$ tells us to use the $+$ symbol rather than the $-$ symbol.

*Separable Equations*

In the ODE we solved above, we were able to "separate" the variables so that only $y$ appeared on one side and only $x$ appeared on the other, allowing us to integrate each side separately to solve the ODE.

**Definition**: A differential equation is called **separable** if we can separate the variables algebraically so that the equation has the form:

$$g(y) \cdot y' = f(x)$$

**Example 1.** If possible, separate the variables in each ODE by writing the given differential equation in the form $g(y) \cdot y' = f(x)$: (a) $y' = xy$ (b) $xy' = \dfrac{y+1}{x}$ (c) $y' = \dfrac{1 + \sin(x)}{y^2 + y}$ (d) $y' = y$ (e) $y' = x + y$

**Solution.** (a) Divide each side of $y' = xy$ by $y$ (to do so we must assume $y \neq 0$) to get $\dfrac{1}{y} \cdot y' = x$ so $g(y) = \dfrac{1}{y}$ and $f(x) = x$.

(b) Divide each side by $x(y + 1)$ (to do so we must assume $x \neq 0$ and $y \neq -1$) to get $\dfrac{1}{y+1} \cdot y' = \dfrac{1}{x^2}$ so $g(y) = \dfrac{1}{y+1}$ and $f(x) = \dfrac{1}{x^2}$.

(c) Multiply each side by $y^2+y$ to get $\left(y^2+y\right)\cdot y' = 1+\sin(x)$ so $g(y)=y^2+y$ and $f(x)=1+\sin(x)$.

(d) Divide each side by $y$ (to do so we must assume $y \neq 0$) to get $\frac{1}{y}\cdot y' = 1$ so $g(y)=\frac{1}{y}$ and $f(x)=1$.

(e) We cannot write this ODE in the form $g(y)\cdot y' = f(x)$.
The first four ODEs are separable, but the last one is not. ◀

**Practice 1.** If possible, separate the variables in each ODE by writing the given differential equation in the form $g(y)\cdot y' = f(x)$:

(a) $y' = x^3(y-5)$ (b) $y' = \dfrac{3}{2x + x\cdot\sin(y+2)}$

*Solving Separable ODEs*

To solve a separable differential equation, we will follow the steps outlined above when solving $\frac{dy}{dx} = -\frac{x}{y}$:

- Use algebra to separate the variables in the ODE: $g(y)\cdot y' = f(x)$
- Put into an equivalent form using differentials: $g(y)\,dy = f(x)\,dx$
- Integrate each side of the equation: $\int g(y)\,dy = \int f(x)\,dx$
- Find antiderivatives, if possible: $G(y) = F(x) + C$
- If given an initial condition $(x_0, y_0)$, find $C$: $C = G(y_0) - F(x_0)$
- If possible, solve explicitly for $y$.

**Example 2.** Find the general solution of $\frac{1}{x}\cdot y' = \frac{x}{2y}$.

**Solution.** Multiply each side by $2xy$ to get $2y\cdot y' = x^2$, so the ODE is separable and we can translate it into differential form:

$$2y\cdot\frac{dy}{dx} = x^2 \quad\Rightarrow\quad 2y\,dy = x^2\,dx$$

Integrating each side, we get:

$$\int 2y\,dy = \int x^2\,dx \quad\Rightarrow\quad y^2 = \frac{1}{3}x^3 + C$$

which is an implicit form of the general solution. Solving for $y$, we get:

$$y = \pm\sqrt{\frac{1}{3}x^3 + C}$$

which is an explicit form of the general solution. ◀

**Example 3.** Find the solution of $y' = \dfrac{6x+1}{2y}$ that satisfies $y(2) = 3$.

**Solution.** We can rewrite this ODE as $2y \cdot y' = 6x + 1$, so it is separable. Next rewrite using differentials:

$$2y \cdot \frac{dy}{dx} = 6x + 1 \quad \Rightarrow \quad 2y\,dy = (6x+1)\ dx$$

and then integrate each side:

$$\int 2y\,dy = \int (6x+1)\ dx \quad \Rightarrow \quad y^2 = 3x^2 + x + C$$

Putting $x = 2$ and $y = 3$ into the general solution $y^2 = 3x^2 + x + C$:

In an initial value problem, it is often easier to solve for $C$ immediately after finding the antiderivatives.

$$3^2 = 3 \cdot 2^2 + 2 + C \quad \Rightarrow \quad 9 = 12 + 2 + C \quad \Rightarrow \quad C = -5$$

So $y^2 = 3x^2 + x - 5 \quad \Rightarrow \quad y = \pm\sqrt{3x^2 + x - 5}$. Because $y(2) = 3$, we need the $+$ value of the square root: $y = \sqrt{3x^2 + x - 5}$. ◀

**Practice 2.** Find the general solution of $y' = \dfrac{1 - \sin(x)}{3y^2}$ and then find the solution that passes through $(0, 2)$.

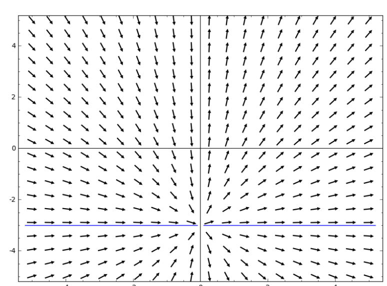
Sometimes algebra is the hardest part of solving an ODE, and often logarithms crop up when solving separable equations.

**Example 4.** Solve $x \cdot y' = y + 3$.

**Solution.** To write the ODE in the form $g(y) \cdot y' = f(x)$ we can divide by $x$ and by $y + 3$ (so we must assume that $x \neq 0$ and $y \neq -3$):

Study the direction field for $x \cdot y' = y + 3$ shown below. What do you notice about its behavior near the lines $x = 0$ and $y = -3$?

$$x \cdot \frac{dy}{dx} = y + 3 \ \Rightarrow \ \frac{1}{y+3}\frac{dy}{dx} = \frac{1}{x} \ \Rightarrow \ \frac{1}{y+3}\,dy = \frac{1}{x}\,dx$$



Now integrate both sides:

$$\int \frac{1}{y+3}\,dy = \int \frac{1}{x}\,dx \quad \Rightarrow \quad \ln\left(|y+3|\right) = \ln\left(|x|\right) + C$$

which is an implicit form of the general solution. To explicitly solve for $y$, recall that $e^{\ln(a)} = a$ so:

$$e^{\ln(|y+3|)} = e^{\ln(|x|) + C} = e^{\ln(|x|)} \cdot e^C \quad \Rightarrow \quad |y+3| = e^C \cdot |x|$$

Removing the absolute value signs we get:

$$y + 3 = \pm e^C \cdot x \quad \Rightarrow \quad y = \pm e^C x - 3 \quad \Rightarrow \quad y = Ax - 3$$

where $A$ is any nonzero constant. In the first step above, we had to assume that $y \neq -3$. You can check that the constant function $y = -3$ is also a solution to the ODE: $x \cdot [-3]' = -3 + 3$. So $y = Ax - 3$ is a solution of the ODE even when $A = 0$, and the general solution of the ODE is $y = Ax - 3$ for any value of $A$. ◀

The domain of any particular solution of the form $y = Ax - 3$ is either $(-\infty, 0)$ or $(0, \infty)$, which is related to the assumption that $x \neq 0$ in our solution. Putting $x = 0$ into the original ODE yields $0 = y + 3 \Rightarrow y - 3$, so the only solution for which $x = 0$ is the single point $(0, -3)$ rather than a function defined on an open interval.

### *Existence and Uniqueness*

In Section 6.2, we saw that any IVP of the form $y' = f(x)$, $y(x_0) = y_0$ had a unique solution: that is, at least one solution to the IVP exists, and this solution is the only possible solution. Unfortunately, for other types of IVPs things can be more complicated.

Can you see why?

The ODE $y^2 + (y')^2 = -1$ has *no* solution, no matter what initial value condition we might specify. The ODE $y^2 + (y')^2 = 0$ has only one solution (the constant solution $y = 0$) so the IVP $y^2 + (y')^2 = 0$, $y(0) = 0$ has one unique solution, but the IVP $y^2 + (y')^2 = 0$, $y(0) = 5$ has none.

Even separable equations can behave in unexpected ways. Consider the IVP $y' = 3y^{\frac{2}{3}}$, $y(0) = 0$. The ODE is separable (if $y \neq 0$):

$$\frac{dy}{dx} = 3y^{\frac{2}{3}} \quad \Rightarrow \quad \frac{1}{3}y^{-\frac{2}{3}}\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{1}{3}y^{-\frac{2}{3}}\,dy = dx$$

Integrating both sides:

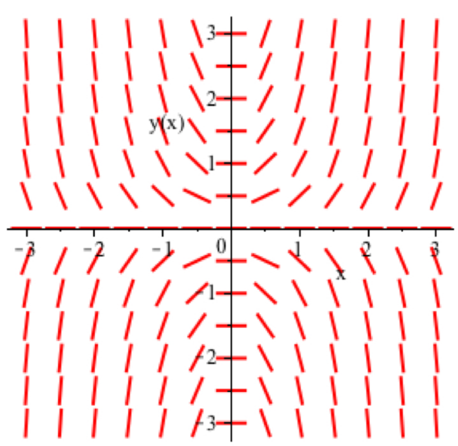
$$\int \frac{1}{3}y^{-\frac{2}{3}}\,dy = \int 1\,dx \quad \Rightarrow \quad y^{\frac{1}{3}} = x + C \quad \Rightarrow \quad y = (x+C)^3$$

In a course on differential equations, you will learn about a theorem that specifies restrictions on a first-order IVP that will guarantee a unique solution.

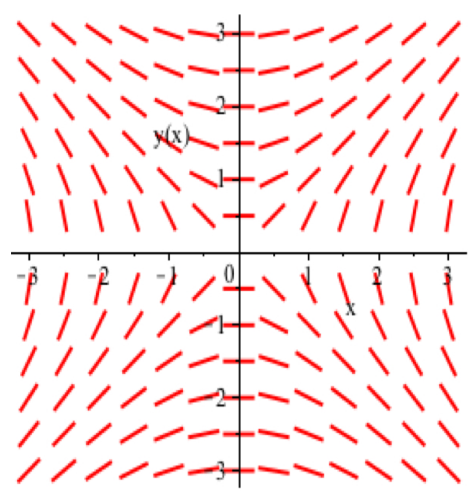
Using the initial condition, $0 = y(0) = (0+C)^3 = C^3 \Rightarrow C = 0$ so $y = x^3$ solves the IVP. But we had to assume that $y \neq 0$; checking the constant function $y = 0$, $[0]' = 3 \cdot 0^{\frac{2}{3}}$ and $y(0) = 0$, so $y(x) = 0$ also satisfies the IVP, giving the IVP *two* solutions: $y(x) = 0$ and $y(x) = x^3$.

## *6.3 Problems*

1. The direction field of the separable ODE $y' = 2xy$ appears below. Sketch solutions of the ODE that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$ and $y(1) = 2$.




2. The direction field of the separable ODE $y' = x/y$ appears below. Sketch solutions of the ODE that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$ and $y(1) = 2$.



In Problems 3–10, solve the separable ODE.

3. $y' = 2xy$
4. $y' = \dfrac{x}{y}$
5. $\left(1+x^2\right) \cdot y' = 3$
6. $xy' = y + 3$
7. $y' \cdot \cos(x) = e^y$
8. $y' = x^2 y + 3y$
9. $y' = 4y$
10. $y' = 5(2-y)$

In Problems 11–18, solve the separable ODEs subject to the given initial conditions.

11. $y' = 2xy$ for $y(0) = 3$, $y(0) = 5$ and $y(1) = 2$.
12. $y' = \dfrac{x}{y}$ for $y(0) = 3$, $y(0) = 5$ and $y(1) = 2$.
13. $y' = 3y$ for $y(0) = 4$, $y(0) = 7$ and $y(1) = 3$.
14. $y' = -2y$ for $y(0) = 4$, $y(0) = 7$ and $y(1) = 3$.
15. $y' = 5(2-y)$ for $y(0) = 5$ and $y(0) = -3$.
16. $y' = 7(1-y)$ for $y(0) = 4$ and $y(0) = -2$.
17. $\left(1+x^2\right) \cdot y' = 3$ for $y(1) = 4$ and $y(0) = 2$.
18. $xy' = y + 3$ for $y(1) = 20$.
19. For $xy' = y + 3$, can $y(0) = 2$?
20. Find all solutions to the IVP $y' = \sqrt[3]{y}$, $y(0) = 0$.

## *6.3 Practice Answers*

1. (a) If $y \neq 5$, divide the ODE $y' = x^3(y-5)$ by $y-5$ to get $\dfrac{1}{y-5} \cdot \dfrac{dy}{dx} = x^3$ so $g(y) = \dfrac{1}{y-5}$ and $f(x) = x^3$.

   (b) Factor $x$ out of the denominator on the right-hand side and multiply both sides of the ODE by $2 + \sin(y+2)$ to get:

$$\frac{dy}{dx} = \frac{3}{x\,[2+\sin(y+2)]} \quad \Rightarrow \quad [2+\sin(y+2)] \cdot \frac{dy}{dx} = \frac{3}{x}$$

   so $g(y) = 2 + \sin(y+2)$ and $f(x) = \dfrac{3}{x}$.

2. Multiply both sides of the ODE by $3y^2$ to get:

$$3y^2 \cdot \frac{dy}{dx} = 1 - \sin(x) \quad \Rightarrow \quad 3y^2\,dy = [1-\sin(x)]\,dx$$

   and integrate:

$$\int 3y^2\,dy = \int [1-\sin(x)]\,dx \quad \Rightarrow \quad y^3 = x + \cos(x) + C$$

   This is an implicit form of the general solution; an explicit general solution is $y = \sqrt[3]{x + \cos(x) + C}$.
   Using the initial condition $y(0) = 2$ with the implicit form of the general solution:

$$2^3 = 0 + \cos(0) + C \;\Rightarrow 8 = 1 + C \quad \Rightarrow \quad C = 7$$

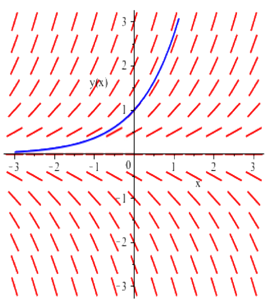
   so the solution to the IVP is $y = \sqrt[3]{x + \cos(x) + 7}$.

## 6.4 *Exponential Growth and Decay*

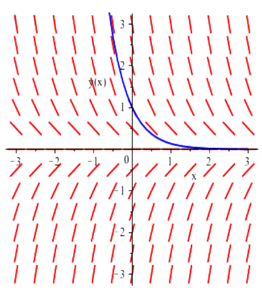
The separable differential equation $y' = ky$ is relatively simple to solve, but it can model a wealth of important situations, including population growth, radioactive decay and drug absorption in the bloodstream. In this section we will solve this ODE and explore some related applications.

### *The Differential Equation $y' = ky$*

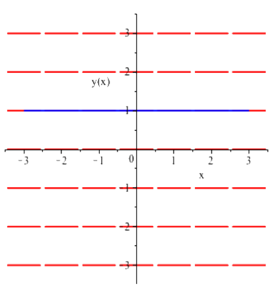
The differential equation $y' = ky$ says that the rate of change of a quantity $y$ is proportional to the value of $y$. The margin figures show direction fields for $y' = 1y$ (growth) and $y' = -2y$ (decay). The ODE $y' = ky$ can model the behavior of populations (the rate at which babies are born is proportional to the number of people currently in the population), radioactive decay (the rate at which atoms decay is proportional to the number of atoms present), the absorption of some medicines by our bodies, and many other situations. The solutions of $y' = ky$ will help us determine how long it takes a population to double in size, the age of some prehistoric artifacts, and even how often some medicines should be taken in order to maintain a safe and effective concentration of that medicine in a patient's body.



y ' = 1y, y(0)=1



y ' =-2y, y(0)=1



y ' =0y, y(0)=1

Direction fields for y ' = ky

$$\boxed{\text{If } y' = ky \text{ (for } y > 0\text{), then } y(t) = y(0) \cdot e^{kt}.}$$

*Proof.* The ODE $y' = ky$ is separable, so we can employ the method of Section 6.3 to solve it:

$$\frac{dy}{dt} = k \cdot y \;\Rightarrow\; \frac{1}{y} \cdot \frac{dy}{dt} = k \;\Rightarrow\; \frac{1}{y}\,dy = k\,dt \;\Rightarrow\; \int \frac{1}{y}\,dy = \int k\,dt$$

$$\Rightarrow\; \ln(|y|) = kt + C \;\Rightarrow\; e^{\ln(|y|)} = e^{kt+C}$$

$$\Rightarrow\; |y| = e^{kt} \cdot e^{C} \;\Rightarrow\; y = \pm e^{C} e^{kt}$$

Because we assumed that $y > 0$, we didn't need to worry about dividing by $y$, and we didn't really need the absolute values (or the $\pm$) in the solution above. But $y = 0$ is also a solution to $y' = ky$, so $y = Ae^{kt}$ solves $y' = ky$ for any value of $A$.

We've found an infinite family of solutions for $y' = ky$, but how do we know that we've found *all* solutions to that ODE? Let $f(t)$ be any solution to $y' = ky$ so that $f'(t) = k \cdot f(t)$. Then define another function $g(t) = f(t) \cdot e^{-kt}$ so that:

$$g(t) = \frac{f(t)}{e^{kt}} \;\Rightarrow\; g'(t) = \frac{e^{kt} \cdot f'(t) - f(t) \cdot e^{kt}}{\left[e^{kt}\right]^2} = \frac{e^{kt}\left[f'(t) - k \cdot f(t)\right]}{e^{2kt}}$$

The last expression in brackets is 0 (because $f'(t) = k \cdot f(t)$), so:

$$g'(t) = 0 \;\Rightarrow\; g(t) = C \;\Rightarrow\; \frac{f(t)}{e^{kt}} = C \;\Rightarrow\; f(t) = Ce^{kt}$$

We now know that any function of the form $y = Ae^{kt}$ solves $y' = ky$ *and* that any solution of $y' = ky$ must have the form $y = Ae^{kt}$.

Finally, putting $t = 0$ into the general solution:

$$y(0) = Ae^{k \cdot 0} = A \;\Rightarrow\; A = y(0) \;\Rightarrow y(t) = y(0) \cdot e^{kt}$$

which holds for any value of $y(0)$, but in particular for any $y(0) > 0$. □

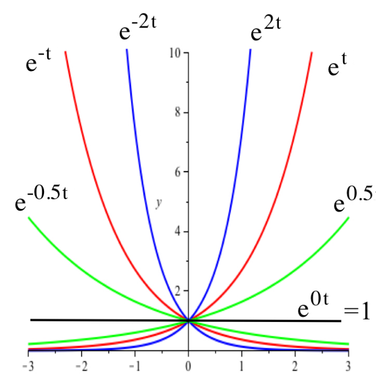
### *Exponential Growth*

A population of people, a chunk of radioactive material and the amount of money in a bank account can all share a common trait. In each situation, the rate at which an amount changes at a particular time is often proportional to the value of that amount at that time. For example:

- the number of births per year is proportional to the number of people in the population
- the number of atoms per hour that release a particle is proportional to the number of atoms present
- the number of dollars of interest per year added to a bank account is proportional to the amount of money in that bank account

These situations can all be modeled with the separable ODE solved above. Our focus in this section will be on using those equations and their solutions to answer questions about applied problems. The applications here all involve the rate of change of some quantity with respect to time, so the input variable will generally be time $t$ (instead of $x$). We might also write the output quantity as $f(t)$ (instead of $y$). The ODE $y' = ky$ then becomes $f'(t) = k \cdot f(t)$ and the solution $y = y_0 \cdot e^{kx}$ becomes $f(t) = f(0) \cdot e^{kt}$.

When $k > 0$, $f(t) = f(0) \cdot e^{kt}$ represents **exponential growth** and we call $k$ the **growth constant**. When $k < 0$, $f(t) = f(0) \cdot e^{kt}$ represents **exponential decay** and we call $k$ the **decay constant**. The margin figure shows the graphs of $f(t) = e^{kt}$ for several values of $k$.




**Example 1.** The number of bacteria on a Petri plate $t$ hours after an experiment starts is $2000 \cdot e^{0.0488t}$.

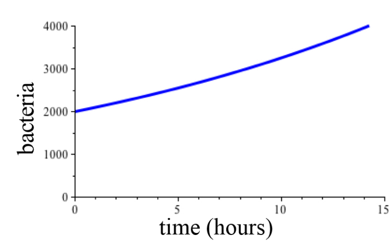
(a) How many bacteria are on the plate after one hour? Two hours?

(b) What is the percentage growth of the population from $t = 0$ to $t = 1$? From $t = 1$ to $t = 2$?

(c) How long does it take for the population to reach 3000? To double?

When you know the initial population $f(0)$ and the growth constant $k$, you can write an equation for $f(t)$, the population at any time $t$, and use it to answer questions about the population.

**Solution.** See margin figure for a graph of $f(t) = 2000 \cdot e^{0.0488t}$.



(a) $f(1) = 2000 \cdot e^{0.0488} \approx 2100$; $f(2) = 2000 \cdot e^{0.0976} \approx 2205$

(b) The percentage growth from $t = 0$ to $t = 1$ is:

$$\frac{f(1) - f(0)}{f(0)} = \frac{2100 - 2000}{2000} = \frac{100}{2000} = 0.05 = 5\%$$

The percentage growth from $t = 1$ to $t = 2$ is:

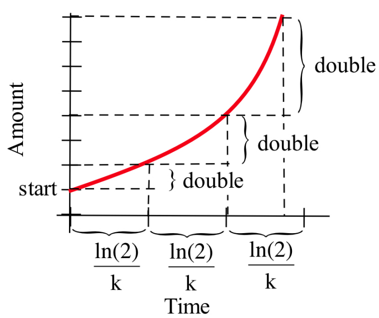
$$\frac{f(2) - f(1)}{f(1)} = \frac{2205 - 2100}{2100} = \frac{105}{2100} = 0.05 = 5\%$$

During the first hour, the population grows by 100 and during the second hour it grows by 105, but the percentage growth during each hour remains constant at 5%.

(c) We need the value of $T$ so that $3000 = f(T) = 2000 \cdot e^{0.0488T}$:

$$1.5 = e^{0.0488T} \Rightarrow \ln(1.5) = \ln\left(e^{0.0488T}\right) = 0.0488T$$
$$\Rightarrow T = \frac{\ln(1.5)}{0.0488} \approx 8.31 \text{ hours}$$

The original population is 2000, so the doubled population is 4000 and the doubling time is $\frac{\ln(2)}{0.0488} \approx 14.2$ hours. ◀



Constant doubling times

When we know the growth constant $k$, the doubling time is simple to find (as in the preceding Example). If $f(t) = f(0) \cdot e^{kt}$ then the doubling time is the time $t_d$ so that:

$$2f(0) = f(0) \cdot e^{kt_d} \Rightarrow 2 = e^{kt_d} \Rightarrow \ln(2) = kt_d \Rightarrow t_d = \frac{\ln(2)}{k}$$

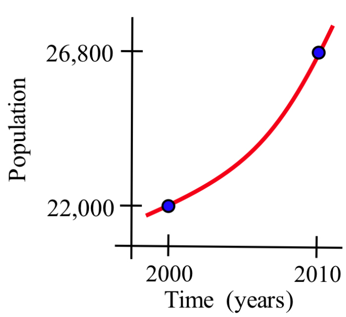
An important aspect of exponential growth is that the doubling time depends only on the growth constant $k$.

**Practice 1.** Use the information from the previous Example to:

(a) determine the population at $t = 5$.

(b) find how long it takes for the population to reach 5,000. To triple.

If you do not know the value of the growth constant $k$, your first step will typically be to use other information to find it.




**Example 2.** The population of a community was 22,000 in 2000 and 26,800 in 2010. Assuming that the community maintains the same rate of exponential growth (see margin figure):

(a) Find a formula for the population $t$ years after 2000.

(b) Find the annual percentage rate of growth of the community.

**Solution.** Let $t$ represent the number of years since 2000, so the year 2000 corresponds to $t = 0$ and the year 2010 corresponds to $t = 10$. Then $f(0) = 22000$, $f(10) = 26800$ and $f(t) = 22000 \cdot e^{kt}$.

(a) To find the value for $k$:

$$26800 = f(10) = 22000 \cdot e^{k(10)} \Rightarrow 1.218 = e^{10k} \Rightarrow \ln(1.218) = 10k$$
$$\Rightarrow k = 0.1 \ln(1.218) \approx 0.0197$$

so $f(t) \approx 22000 \cdot e^{0.0197t}$.

(b) $f(0) = 22000$ and $f(1) \approx 22000 \cdot e^{(0.0197)1} \approx 22438$ so the annual percentage increase was

$$\frac{f(1) - f(0)}{f(0)} = \frac{438}{22000} \approx 0.0199 = 1.99\%$$

during the first year. ◀

**Practice 2.** A scientist released 12,000 free neutrons into a material. Two seconds later, the material contained 18,000 free neutrons. If the number of free neutrons grows exponentially:

(a) Find a formula for the number of neutrons present $t$ seconds after the beginning of the experiment.

(b) Find the doubling time for the number of free neutrons.

Compound interest provides another example of exponential growth.

**Example 3.** How long does it take \$1,000 to double when invested in a savings account with interest compounded continuously at an annual rate of 5%? At an effective annual rate of return of 5% (compounded continuously)?

**Solution.** If the interest is compounded continuously at an annual rate of 5% and $A(t)$ is the amount of money in the account $t$ years after the initial deposit of \$1,000, then:

$$A'(t) = 0.05A(t),\ A(0) = 1000 \quad \Rightarrow \quad A(t) = 1000e^{0.05t}$$

and the doubling time is $t_d = \frac{\ln(2)}{0.05} \approx 13.86$ years.

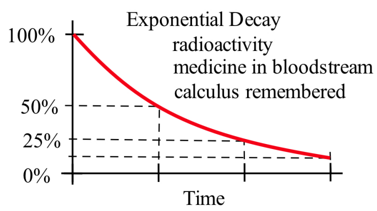
If the effective annual rate of return is 5%, then $A(1) = 1000 + 0.05(1000) = 1.05(1000) = 1050$ so:

$$1050 = 1000e^{k \cdot 1} \quad \Rightarrow \quad 1.05 = e^k \quad \Rightarrow \quad k = \ln(1.05) \approx 0.0488$$

hence the doubling time is $t_d = \frac{\ln(2)}{\ln(1.05)} \approx 14.21$ years. (An effective annual return of 5% corresponds to a continuously compounded annual interest rate of 4.88%.) ◀

**Practice 3.** How long does it take an investment to double if the effective annual rate of return is 2%?

*Exponential Decay*




Exponential decay occurs when the rate of loss of something is proportional to the amount present. One example of exponential decay is radioactive decay: the number of atoms of a radioactive substance that "decay" (split into nonradioactive atoms and release particles) during a short time interval is proportional to the number of radioactive atoms present at that time. Exponential decay (see margin) also models how quickly some medicines are absorbed from the bloodstream — and even how quickly you forget calculus concepts.

| element | half-life |
|---|---|
| iodine-131 | 8.07 days |
| strontium-90 | 29 years |
| argon-39 | 265 years |
| carbon-14 | 5700 years |
| plutonium-239 | 24400 years |
| uranium-238 | $4.51 \times 10^5$ years |
| uranium-234 | $2.47 \times 10^9$ years |

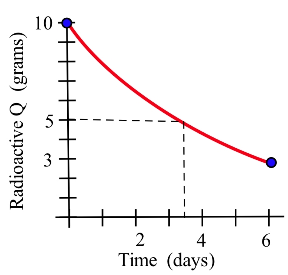
Exponential decay calculations are similar to those for growth, but the value of $k$ is negative and we talk about "**half-life**," the time for half of the material to decay or be absorbed, instead of the doubling time. The margin table shows the half-lives of some isotopes.

**Example 4.** You started an experiment with 10 g of a radioactive substance, but after 6 days of decay only 3 g remained.

(a) Find a formula for the amount of radioactive material present $t$ days after beginning the experiment.

(b) Find the half-life for the radioactive substance.

**Solution.** Let $f(t)$ represent the amount of the radioactive substance present after $t$ days. Then $f(t) = 10e^{kt}$ (see margin figure).




(a) $3 = f(6) = 10e^{6k} \Rightarrow 0.3 = e^{6k} \Rightarrow \ln(0.3) = 6k$ so $k = \frac{1}{6}\ln(0.3) \approx -0.2007$ and $f(t) = 10e^{-0.2007t}$.

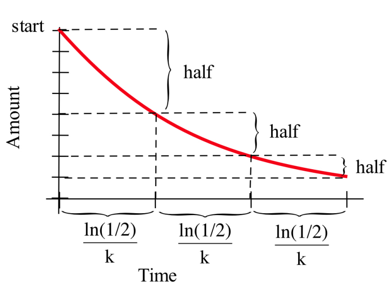
(b) The half-life $t_h$ is the time required for half of the material to decay, so we need to solve $5 = 10e^{-0.2007t_h}$ for $t_h$:

$$0.5 = e^{-0.2007t_h} \Rightarrow \ln(0.5) = -0.2007t_h \Rightarrow t_h = \frac{\ln(0.5)}{-0.2007} \approx 3.5 \text{ days}$$

Note that $t_h = \dfrac{\ln(0.5)}{k}$, which will hold true generally. ◀

When you know the decay constant $k$, the half-life $t_h$ is simple to find, as in the preceding Example:

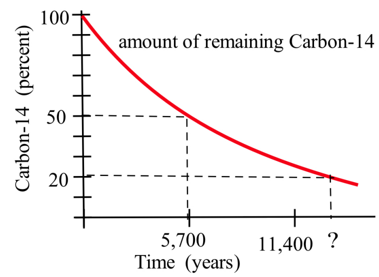
$$t_h = \frac{\ln(0.5)}{k} = \frac{\ln\left(\frac{1}{2}\right)}{k} = \frac{-\ln(2)}{k}$$




The half-life depends only on the decay constant $k$ (see margin figure).

If you know the half-life of a substance and you know how much of the substance is present in a sample now, you can determine how much was present at some past time or determine how long ago the sample contained a particular amount of the substance.

Scientists use radioactive **carbon-14**, with a half-life of about 5,700 years, in this way to estimate how long ago plants and animals lived. A living plant continually exchanges carbon-14 and ordinary carbon with the atmosphere so that the ratio of carbon-14 to non-radioactive carbon remains relatively constant. But once the plant dies, this exchange stops. The ordinary carbon remains in the material, but the carbon-14 decays, so the ratio of carbon-14 to ordinary carbon decreases at a known rate. By measuring the ratio of carbon-14 to ordinary carbon in a sample of plant tissue, scientists can determine how long ago the plant died and obtain an estimate for the age of the sample.




**Example 5.** The amount of carbon-14 in plant fiber of a woven basket is 20% of the amount present in a living plant (see margin figure). Estimate the age of the basket.

**Solution.** Let $f(t)$ represent the relative amount of carbon-14 in a sample with age $t$ years. Because the half-life of carbon-14 is 5,700 years:

$$t_h = \frac{-\ln(2)}{k} \Rightarrow 5700 = \frac{-\ln(2)}{k} \Rightarrow k = \frac{-\ln(2)}{5700} \approx -0.0001216$$

so that $f(t) = f(0) \cdot e^{-0.0001216t}$. Because 20% of the carbon-14 remains in our sample, we want the value of $T$ so that:

$$\begin{aligned} 0.20f(0) = f(T) &\Rightarrow 0.20f(0) = f(0) \cdot e^{-0.0001216T} \\ &\Rightarrow 0.20 = e^{-0.0001216T} \Rightarrow \ln(0.20) = -0.0001216T \\ &\Rightarrow T = \frac{\ln(0.2)}{-0.0001216} \approx 13235 \end{aligned}$$

We can conclude that the basket was made from a plant that died about 13,200 years ago. (Does that mean the basket was made then?) ◀

This dating method is very sensitive to small changes in the measured amount of carbon-14.

**Practice 4.** The half-life of an isotope is 8 days. Write a formula for the amount of the isotope present $t$ days after you begin an experiment with 10 mg of the isotope.

The rate at which many medicines are absorbed from the blood is proportional to the concentration of the medicine in the blood: the higher the concentration in the blood, the faster it is absorbed.

**Example 6.** Suppose a certain medicine has an absorption (decay) constant of $-0.17$ (determined experimentally) and that the lowest "effective"concentration of the medicine is 0.3 mg/l (milligrams of medicine per liter of blood). If a patient who has 5 liters of blood is injected with 20 mg of the medicine, how long will the medicine be effective?

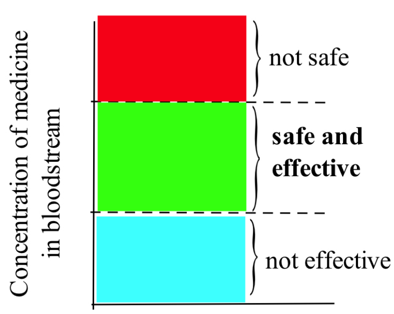
**Solution.** because the patient is starting with 20 mg of the medicine in 5 liters of blood, the initial concentration is $\frac{20 \text{ mg}}{5 \text{ L}} = 4$ mg/L. The amount of medicine in the blood $t$ hours later is thus $f(t) = 4e^{-0.17t}$ and we want to find $T$ so that $f(T) = 0.3$ mg/L

$$0.3 = 4e^{-0.17T} \Rightarrow \frac{0.3}{4} = e^{-0.17T} \Rightarrow \ln\left(\frac{0.3}{4}\right) = -0.17T \Rightarrow T \approx 15.2$$

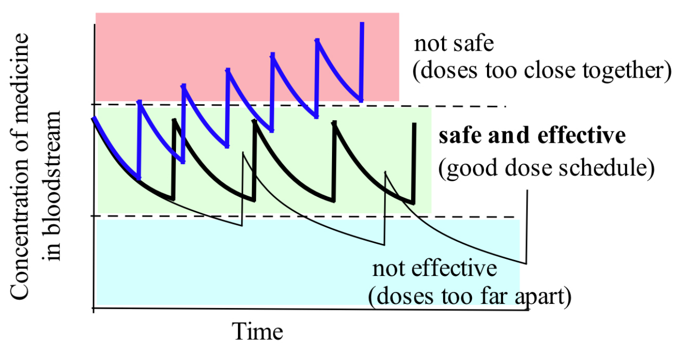
so a patient should receive a new does of the medicine about 15 hours after the first dose. ◀

**Practice 5.** Should the amount of the second dose in the preceding Example be the same as the initial dose?

Many medicines have a "safe and effective" interval of concentrations (see figure below), so the goal of a schedule for taking the medicine is to keep the concentration near the middle of that range. Taking doses too close together in time can result in an overdose, while taking them too far apart is eventually ineffective.



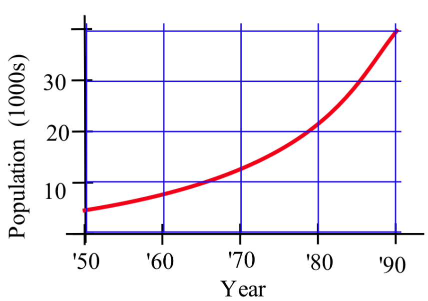
Safe and effective region



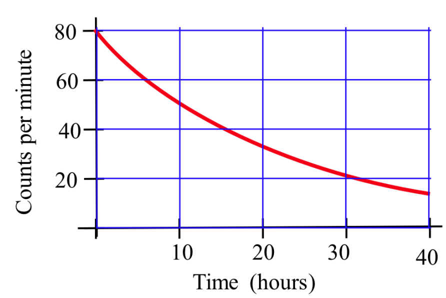
Safe and effective dosing schedule

## 6.4 Problems

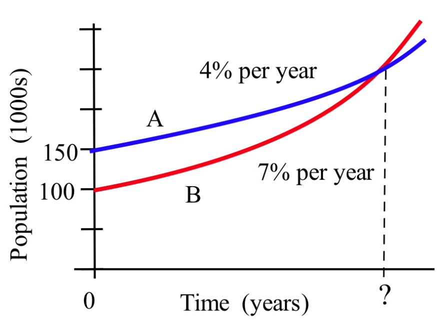
1. How long did it take the population of a city (see below) to double from 10,000 to 20,000? From 15,000 to 30,000? Approximate the doubling time.



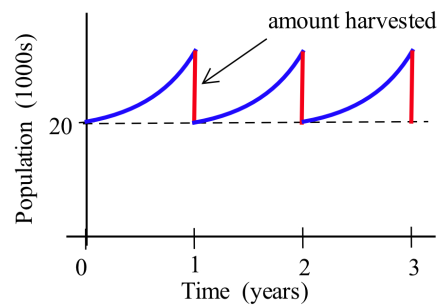

2. How long did it take the counts for a radioactive material to decay from 80 per minute to 40? From 60 to 30? From 40 to 20? What is the half-life?



3. The population of a community in 1990 was 48,000 people and in 2010 it was 64,000 people.
   (a) Write a formula for the population of the community $t$ years after 1990.
   (b) Estimate the population in the year 2020.
   (c) When will the population be 100,000?
   (d) What is the doubling time of the population?
4. Repeat Problem 3 if the population of another community was was 40,000 people in 1990 and 60,000 people in 2010.
5. A terrific investment pays interest at an effective annual rate of 15%. How long will it take for a \$5,000 investment to double? To triple?
6. You have invested \$3,000 for 10 years at an effective annual rate of 7.5% and a friend has invested the same amount invested at an effective annual interest rate of 7.75%. Your friend will get back how much more money than you at the end of 10 years? At the end of 20 years?
7. Find a formula for the population in Problem 1.
8. Each bacterium of a certain species splits into two bacteria at the end of each minute. If we start with a few bacteria in a bowl at 3:00 p.m. and the bowl is full of bacteria at 4:30 p.m., when was the bowl half full? (Calculus is not required.)
9. A newscaster reports that the world's population is now doubling every 50 years. What annual rate of growth results in a 50-year doubling time?
10. Group A has a population of 150,000 and an annual growth rate of 4%; group B has a population of 100,000 and an annual growth rate of 7% (see figure below). After how many years will the two groups be the same size?

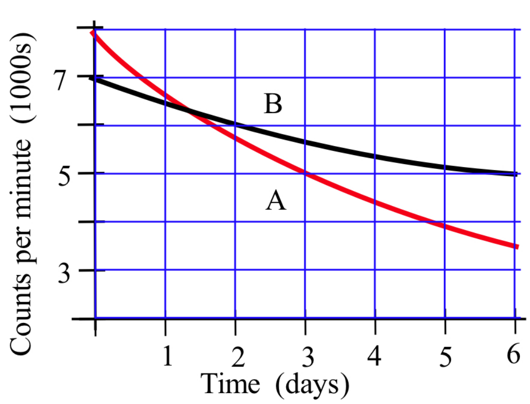



11. Group A has a population of 600,000 and an annual growth rate of 3%; group B has a population of 400,000 and an annual growth rate of 6%. After how many years will the two groups be the same size?
12. The unregulated population of fish in a certain lake grows by 30% per year under optimum conditions. A fish census reveals there are approximately 20,000 fish in the lake. How many fish can be harvested (see figure below) at the end of each year in order to maintain a stable population? (This is an example of calculating the yield for a "renewable resource." In practice, more sophisticated calculations also take into account the distribution of species, ages and genders.)



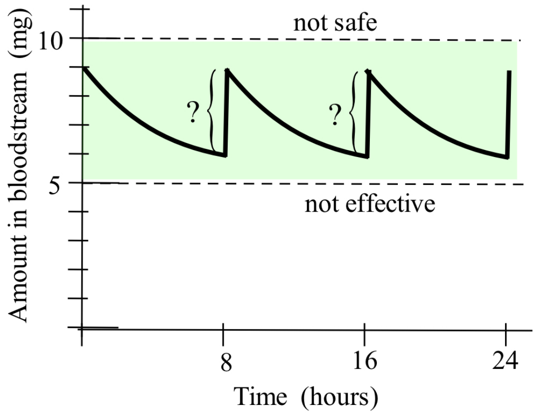

13. The annual exponential growth constant for a population of snails is $k = 0.14$. Currently you have 8,000 snails.
   (a) Determine the size of the population over the next 20 months if you harvest 2,000 snails every 2 months.
   (b) What happens if you harvest 3,000 snails every 2 months?
   (c) How many can we harvest every 2 months in order to maintain a stable population?
14. An exponential function $f(t) = Ae^{kt}$ has constant doubling time, but some non-exponential functions also have constant doubling times.
   (a) Show that the exponential function $f(t) = 2^t = e^{\ln(2)t}$ has a constant doubling time of 1. (Show that $f(t+1) = 2f(t)$.)
   (b) Graph $g(t) = 2^t\,[1 + A\sin(2\pi t)]$ for $A = 0.5$ and $A = 1.5$. Show that $g$ has a constant doubling time 1 for any choice of $A$.

15. An experiment starts with 10 grams of a radioactive material and 14 days later 2 grams remain.
    (a) Find a formula for the amount of material remaining $t$ days after the experiment starts.
    (b) Find the half-life of the material.
    (c) When will 0.7 grams of the material remain?
16. You start with 8 mg of a radioactive substance and 10 days later determine that 6.3 mg remains.
    (a) Find a formula for the amount left $t$ days later.
    (b) Find the half-life of the substance.
    (c) When will 1 mg of the substance remain?
17. A Geiger counter initially recorded 187 counts per minute from a radioactive material, but 2 days later the count was down to 143 counts per minute. (The count per minute is proportional to the amount of radioactive material present.)
    (a) What is the half-life of the material?
    (b) When will the count be down to 20 counts per minute?
18. The initial Geiger counter measurement from a radioactive substance was 540 counts per minute, and a week later it was 500 counts per minute.
    (a) What is the half-life of the substance?
    (b) When will the count be down to 100 counts per minute?
19. Find a formula for the counts per minute for the radioactive material A in the figure below.




20. Find a formula for the counts per minute for the radioactive material B in the figure above.
21. Your friend plans to purchase a letter reputedly written by Isaac Newton (1642–1727), but an analysis of the paper shows that it contains 97.5% of the proportion of carbon-14 present in new paper of the same type. Can you be certain the letter is a forgery? If the age of the paper is consistent with time frame during which Newton lived, can you be certain the letter is genuine?
22. For several centuries, the Shroud of Turin was widely believed to be the shroud of Jesus. Three independent laboratories in England, Switzerland and the United States used carbon-14 dating on a few square centimeters of the cloth, and in 1988 they reported (*Science*, October 21, 1988) that the Shroud of Turin was probably made during the early 1300s and certainly after 1200 A.D.
    (a) If the Shroud was made in 1300 A.D., what percentage of the original carbon-14 was still present in 1988?
    (b) If the Shroud was made in 30 A.D., what percentage of the original carbon-14 was still present in 1988?
23. Half of a particular medicine is used up by the body every 6 hours, and the medicine is not effective if the concentration in the blood is less than 10 mg/l. If an ill person is given an initial dose of medicine to raise the concentration to 30 mg/l, for how long will the medicine be effective?
24. A particular controlled substance has a half-life of 12 hours, and it can be detected in concentrations as low as 0.002 mg/l in the blood.
    (a) If a person has an initial concentration of the substance of 15 mg/l in the blood, for how long can it be detected?
    (b) If the detection test is improved by a factor of 100, so it can detect a concentration of 0.00002 mg/l, for how long can an initial concentration of 15 mg/l be detected?
25. A doctor gave a patient 9 mg of a medicine that has half-life of 15 hours in the body. How much of the medicine does the patient need to take every 8 hours in order to maintain between 6 and 9 mg

of the medicine in the body all of the time? (See figure below.)




26. Each layer of a dark film transmits 40% of the light that strikes it.
    (a) How many layers are needed for an eye shield to transmit only 10% of the light?
    (b) How many layers are needed to transmit only 2% of the light?

27. A region has been contaminated with radioactive iodine-131 to a level five times the safe level. How long will it be until the area is safe?

28. A region has been contaminated with radioactive strontium-90 to a level five times the safe level. How long will it take until the area is safe?

29. The population of a country is 4 million and is growing at 5% per year. Currently the country has 10 million acres of forests that are being cut down (and not replanted) at a rate of 300,000 acres per year.
    (a) Find a formula for the number of acres of forest per person.
    (b) At what rate is the number of acres of forest per person changing?
    (c) If the population and harvest rates remain constant, in approximately how many years will there be one acre of forest per person?

### *6.4 Practice Answers*

1. (a) $f(5) = 2000e^{0.0488(5)} \approx 2552$

   (b) $f(T) = 5000 \Rightarrow 5000 = 2000e^{0.0488T} \Rightarrow 2.5 = e^{0.0488T} \Rightarrow \ln(2.5) = 0.0488T$
   $\Rightarrow T = \dfrac{\ln(2.5)}{0.0488} \approx 18.78$ hours; tripling time is $\dfrac{\ln(3)}{0.0488} \approx 22.51$ hours.

2. (a) $f(0) = 12000$ so $f(t) = 12000e^{kt}$ and $f(2) = 18000$ so:
   $$18000 = 12000e^{2k} \Rightarrow 1.5 = e^{2k} \Rightarrow \ln(1.5) = 2k \Rightarrow k = \frac{1}{2}\ln(1.5) \approx 0.2027$$
   and thus $f(t) \approx 12000e^{0.2027t}$.

   (b) Doubling time is $t_d = \dfrac{\ln(2)}{k} \approx \dfrac{\ln(2)}{0.2027} \approx 3.42$ seconds.

3. After 1 year, each \$1 invested will become $\$1 + (0.02)(\$1) = \$1.02$ so:
   $$f(1) = 1.02 = 1 \cdot e^{k \cdot 1} \Rightarrow \ln(1.02) = k \Rightarrow k \approx 0.0198$$
   The doubling time is therefore: $t_d = \dfrac{\ln(2)}{0.0198} \approx 35$ years.

4. $8 = t_h = -\dfrac{\ln(2)}{k} \Rightarrow k = -\dfrac{\ln(2)}{8} \approx -0.0866$ so $f(t) = 10e^{-0.0866t}$.

5. No. After 15.2 hours, the patient still has 0.3 mg/l of medicine in his blood, or $5(0.3) = 1.5$ mg. A dose of $20 - 1.5 = 18.5$ mg would return the medicine in his blood to the original level of 20 mg.

## 6.5 Heating, Cooling and Mixing

The previous section examined applications of exponential growth and decay. This section examines other applications involving separable IVPs that can be solved using the methods of Section 6.3.

### Newton's Law of Cooling

Some rates of change depend on how far the value of a variable quantity is from a fixed value. The rate at which a hot cup of soup cools (or a cool cup of water warms up) is proportional to the difference in temperature between the soup (or water) and the surrounding air. This principle, called Newton's Law of Cooling (or Warming), provides an approximate model for temperature change when the object in question is subject to forced convection (a steady breeze, for example).

It also applies to natural convection (no breeze or draft) if the temperature difference between the object and the surrounding air is not too great.

**Newton's Law of Cooling/Warming**

If $f(t)$ is the temperature at time $t$ of an object in an atmosphere with constant temperature $a$,

then the rate of change of $f$ is proportional to the difference between $f$ and $a$: $f'(t) = k\,[f(t) - a]$ and the solution to this ODE is:

$$f(t) = a + [f(0) - a] \cdot e^{kt}$$

The statement that the rate of change is proportional to the difference:

$$f'(t) = k\,[f(t) - a]$$

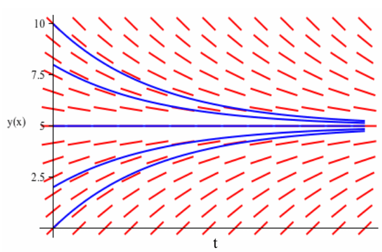
can be observed by experimentation.

This ODE is separable, so we can solve it using the methods of Section 6.3. Or we can let $g(t) = f(t) - a$ so that $g'(t) = f'(t)$ and $g(0) = f(0) - a$, transforming the ODE for $f(t)$ into the IVP:

$$g'(t) = kg(t), \qquad g(0) = f(0) - a$$

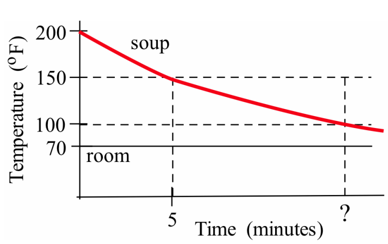
From Section 6.4 we know that $g(t) = g(0) \cdot e^{kt}$ so that:

$$f(t) - a = [f(0) - a]\, e^{kt} \quad \Rightarrow \quad f(t) = a + [f(0) - a]\, e^{kt}$$



Solutions of f '(t)= (-1)*( f(t) - 5 )

The margin figure shows some functions with different initial values that all satisfy the differential equation $f'(t) = f(t) - 5$.




**Example 1.** A cup of soup sits on a counter in a room with temperature 70° F. When first poured into the cup, the soup had a temperature of 200° F, and five minutes later its temperature was 150° F (see margin).

(a) Find a formula for the soup's temperature at any time $t > 0$.

(b) How long does it take for the soup to cool to 100° F?

(c) What will the temperature of the soup be after a "long" time?

**Solution.** (a) If $f(t)$ is the soup's temperature $t$ minutes after being poured and the room temperature is $a = 70°$ F, then we know that $f(0) = 200°$ F so $f'(t) = k\,[f(t) - 70]$ and hence:

$$f(t) = 70 + [200 - 70] \cdot e^{kt} = 70 + 130 \cdot e^{kt}$$

We can now use the information that $f(5) = 150°$ F to find the value of $k$ and an equation for $f(t)$:

$$150 = f(5) = 70 + 130.e^{k \cdot 5} \;\Rightarrow\; k = \frac{1}{5} \cdot \ln\left(\frac{80}{130}\right) \approx -0.0971$$

so $f(t) \approx 70 + 130 \cdot e^{-0.0971t}$.

(b) We now want to find the time $t$ so that $f(t) = 100$:

$$100 = f(t) \approx 70 + 130 \cdot e^{-0.0971t} \;\Rightarrow\; 30 = 130 \cdot e^{-0.0971t}$$

$$\Rightarrow\; t = \frac{\ln\left(\frac{30}{130}\right)}{-0.0971} \approx 15.1 \text{ minutes}$$

(c) After a "long time" means for very large values of $t$:

$$\lim_{t\to\infty} \left[70 + 130 \cdot e^{-0.0971t}\right] = 70$$

so (eventually) the temperature of the soup approaches 70° F, the temperature of the room. ◀

**Practice 1.** After eating some of the soup in the preceding Example, you refrigerate the leftover soup, cooling it to a temperature of 37° F. At noon the next day, you move the container of soup from the refrigerator back to the 70° F room. One hour later, the temperature of the soup is 48° F. At what time will the temperature of the soup be 60° F?

## *Mixing*

Many important applications involve one substance being mixed into another: chlorine being slowly introduced into a swimming pool, a toxic gas being mixed into the air in a closed room, or (in a classic example) salt (NaCl) being mixed into water ($H_2O$).

**Example 2.** A tank contains 350 grams of salt dissolved in 10 liters of water. Pure water begins flowing into the tank at 2 liters per hour, and the well-mixed solution flows out of the tank at the same rate.

(a) Find a formula for the amount of salt in the tank $t$ hours after this process begins.

(b) Find a formula for the concentration salt in the tank.

(c) What will the concentration be after a "long" time?

**Solution.** (a) Let $A(t)$ be the amount of salt (in grams) $t$ hours after this process begins. We know that $A(0) = 350$. Now consider $A'(t)$, the rate at which amount of salt in the tank is changing with respect to time. No salt is flowing in, but the rate at which salt is leaving the tank is:

$$\frac{A(t)\text{ g}}{10\text{ L}} \cdot 2\ \frac{\text{L}}{\text{hr}} = 0.2A(t)\ \frac{\text{g}}{\text{hr}}$$

which gives us the IVP:

$$\frac{dA}{dt} = -0.2A, \quad A(0) = 350$$

This is a separable IVP with solution $A(t) = 350e^{-0.2t}$.

(b) The concentration is:

$$C(t) = \frac{A(t)\text{ g}}{10\text{ L}} = \frac{350e^{-0.2t}\text{ g}}{10\text{ L}} = 35e^{-0.2t}\ \frac{\text{g}}{\text{L}}$$

(c) After a "long" time:

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} 35e^{-0.2t} = 0$$

This seems reasonable: after a very long time, most of the original salt in the tank will be flushed out, replaced by pure water. ◀

**Practice 2.** A tank contains 300 grams of salt dissolved in 10 liters of water. Water containing 15 grams of salt per liter begins flowing into the tank at a rate of 4 liters per hour, and the well-mixed solution flows out of the tank at the same rate.

(a) Find a formula for the amount of salt in the tank $t$ hours after this process begins.

(b) Find a formula for the concentration salt in the tank.

(c) What will the concentration be after a "long" time?

If the volume of liquid in the tank does not remain constant, the differential equation can become a bit more complicated.

**Example 3.** A 20-liter tank contains 250 grams of salt dissolved in 10 liters of water. Pure water begins flowing into the tank at a rate of 5 liters per hour, and the well-mixed solution flows out of the tank at a rate of 3 liters per hour.

(a) Find a formula for the amount of salt in the tank $t$ hours after this process begins.

(b) Find a formula for the concentration salt in the tank.

(c) What will the concentration be at the moment the tank overflows?

**Solution.** (a) Let $A(t)$ be the amount of salt (in grams) $t$ hours after this process begins. We know that $A(0) = 250$ and that the volume of water in the tank is $V(t) = 10 + 2t$ (because 5 liters of water flow in each hour, but only 3 liters flow out). No salt is flowing into the tank, but the rate at which salt is leaving the tank is:

$$\frac{A(t)\ \text{g}}{10+2t\ \text{L}} \cdot 3\ \frac{\text{L}}{\text{hr}} = \frac{3A(t)}{10+2t}\ \frac{\text{g}}{\text{hr}}$$

which gives us the IVP:

$$\frac{dA}{dt} = -\frac{3A}{10+2t}, \quad A(0) = 250$$

This is a separable IVP:

$$\frac{1}{A} \cdot \frac{dA}{dt} = -\frac{3}{10+2t} \Rightarrow \ln(A) = -\frac{3}{2}\ln(10+2t) + C$$

Using the initial condition:

$$\ln(250) = -\frac{3}{2}\ln(10) + C \Rightarrow C = \ln(250) + \frac{3}{2}\ln(10)$$

Solving for $A$:

$$\begin{aligned} &\ln(A) = \ln(10+2t)^{-\frac{3}{2}} + \ln(250) + \ln(10^{\frac{3}{2}}) \\ \Rightarrow \quad &A(t) = (10+2t)^{-\frac{3}{2}} \cdot 250 \cdot 10\sqrt{10} = \frac{2500\sqrt{10}}{(10+2t)^{\frac{3}{2}}} \end{aligned}$$

(b) The concentration is:

$$C(t) = \frac{A(t)\ \text{g}}{10+2t\ \text{L}} = \frac{2500\sqrt{10}}{(10+2t)^{\frac{5}{2}}}\ \frac{\text{g}}{\text{L}}$$

(c) The tank overflows when $20 = 10 + 2t \Rightarrow t = 5$, at which time the concentration is:

$$C(5) = \frac{2500\sqrt{10}}{(10+2\cdot 5)^{\frac{5}{2}}} = \frac{25}{4\sqrt{2}}\ \frac{\text{g}}{\text{L}}$$

or about 4.42 grams per liter. ◀

**Practice 3.** A 20-liter tank contains 250 grams of salt dissolved in 10 liters of water. Water containing 35 grams of salt per liter begins flowing into the tank at a rate of 5 liters per hour, and the well-mixed solution flows out of the tank at a rate of 3 liters per hour. Set up an initial value problem to model the amount of salt in the tank $t$ hours after this process begins. Is the ODE separable?

## 6.5 Problems

1. You remove a cheesecake from the oven with an ideal internal temperature of 165° F and place it into a 35° F refrigerator. After 10 minutes, the cheesecake has cooled to 150° F. If you must wait until the cheesecake has cooled to 70° before you eat it, how long will you have to wait?

2. When a pot of hot (200° F) water is removed from the stove in a 70° F kitchen, it takes 15 minutes to cool to a temperature of 150° F.
   (a) Find a formula for the temperature of the water $t$ minutes after it is removed from the stove.
   (b) When will the water be 100° F?
   (c) When will the water be 80° F?
   (d) When will the water be 60° F?

3. When you remove the pot of 200° F water from the previous problem from the stove and take it outside on a 40° F day, it only takes 4 minutes to cool to a temperature of 150° F.
   (a) Find a formula for the temperature of the water $t$ minutes after it is removed from the stove.
   (b) When will the water be 100° F?
   (c) When will the water be 80° F?
   (d) When will the water be 60° F?

4. When you remove a pitcher of orange juice from a 40° F refrigerator into a 70° F kitchen, it takes 25 minutes to warm up to a temperature of 60° F.
   (a) Estimate the temperature of the juice $t$ minutes after you removed it from the refrigerator.
   (b) When was the juice 50° F?
   (c) When will the juice be 65° F?

5. A tank initially holds 100 gallons of brine (water mixed with salt) containing 50 pounds of dissolved salt. Pure water flows into the tank at a rate of 3 gallons per minute and flows out at the same rate, with the brine concentration kept roughly uniform by constant stirring. How much salt remains in the tank after one hour?

6. A tank initially holds 1000 liters of brine containing 8 kg of dissolved salt. Pure water enters the tank at a rate of 10 liters per minute and brine (with concentration kept roughly uniform by constant stirring) leaves the tank at the same rate.
   (a) Find the concentration of salt in the tank after half an hour.
   (b) Determine the limiting concentration of salt in the tank after "a very long time."

7. A large tank initially contains 100 gallons of brine (water mixed with salt) containing 50 pounds of dissolved salt. Pure water flows into the tank at a rate of 3 gallons per minute and flows out at 2 gallons per minute, with the brine concentration kept roughly uniform by constant stirring. How much salt remains in the tank after one hour?

8. A tank initially holds 1000 liters of pure water. Brine containing 8 g of dissolved salt per liter enters the tank at a rate of 10 liters per minute and the well-mixed solution leaves the tank at the same rate.
   (a) Find the concentration of salt in the tank after half an hour.
   (b) Determine the limiting concentration of salt in the tank after "a very long time."

*6.5 Practice Answers*

1. The temperature of the soup $t$ minutes after being removed from the refrigerator is:

$$f(t) = 70 + [37 - 70] \cdot e^{kt} = 70 - 33e^{kt}$$

Using the fact that $f(60) = 48$:

$$48 = 70 - 33e^{kt} \ \Rightarrow\ \frac{22}{33} = e^{60k} \ \Rightarrow\ k = \frac{1}{60} \cdot \ln\left(\frac{22}{33}\right) \approx -0.006758$$

so that $f(t) \approx 70 - 33e^{-0.006758t}$. If $60 = 70 - 33e^{-0.006758T}$ then:

$$\frac{10}{33} = e^{-0.006758T} \ \Rightarrow\ T = -\frac{1}{0.006758} \cdot \ln\left(\frac{10}{33}\right) \approx 176.7$$

so the soup will warm up to 60° F after roughly three hours.

2. If $A(t)$ is the amount of salt (in grams) $t$ hours after this process begins, then:

$$\frac{dA}{dt} = 15\,\frac{\text{g}}{\text{L}} \cdot 4\,\frac{\text{L}}{\text{hr}} - \frac{A\ \text{g}}{10\ \text{L}} \cdot 4\,\frac{\text{L}}{\text{hr}}$$

so $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = 60 - \frac{2}{5}A, \quad A(0) = 300$$

(a) This IVP is separable, so we can solve it:

$$\int \frac{1}{60 - \frac{2}{5}A}\,dA = \int 1\,dt \ \Rightarrow\ -\frac{5}{2}\ln\left(\left|60 - \frac{2}{5}A\right|\right) = t + C$$
$$\Rightarrow\ \ln\left(\left|60 - \frac{2}{5}A\right|\right) = -\frac{2}{5}t + K$$

Using the initial condition:

$$\ln\left(\left|60 - \frac{2}{5} \cdot 300\right|\right) = -\frac{2}{5} \cdot 0 + K \ \Rightarrow\ K = \ln(60)$$

so that:

$$\ln\left(\left|60 - \frac{2}{5}A\right|\right) = -\frac{2}{5}t + \ln(60) \ \Rightarrow\ 60 - \frac{2}{5}A = -60e^{-\frac{2}{5}t}$$

where we need to use a $-$ sign when "undoing" the absolute value to ensure that the initial condition $A(0) = 300$ will still hold. Solving for $A$ yields $A(t) = 150\left[1 + e^{-\frac{2}{5}t}\right]$.

(b) Divide by the volume to get $C(t) = 15\left[1 + e^{-\frac{2}{5}t}\right]$.

(c) $\lim_{t\to\infty} 15\left[1 + e^{-\frac{2}{5}t}\right] = 15$, so the limiting concentration is 15 g/L (the concentration of solution flowing into the tank, as expected).

3. If $A(t)$ is the amount of salt (in grams) $t$ hours after this process begins, then:

$$\frac{dA}{dt} = 35\,\frac{\text{g}}{\text{L}} \cdot 5\,\frac{\text{L}}{\text{hr}} - \frac{A\text{ g}}{(10+2t)\text{ L}} \cdot 3\,\frac{\text{L}}{\text{hr}}$$

so $A(t)$ satisfies the IVP:

$$\frac{dA}{dt} = 175 - \frac{3A}{10+2t}, \quad A(0) = 250$$

This is not a separable equation. (You will learn how to solve equations of this type in a course about differential equations. For now, you could construct a direction field to analyze the behavior of solutions to the ODE.)