7
Transcendental Functions

An **algebraic number** is any number that can be a solution to a polynomial equation with integer coefficients. Any rational number is algebraic: for example, $\frac{5}{7}$ is a solution of $7x - 5 = 0$. Any square root or cube root is an algebraic number: $\sqrt{3}$ is a solution of $x^2 - 3 = 0$ and $-\sqrt[3]{5}$ is a solution of $x^3 + 5 = 0$. In fact, any number that can be expressed as the sum, difference, product, quotient or rational power of integers is an algebraic number, including something as horrible-looking as:

$$\sqrt[5]{\frac{\sqrt[4]{93} - \sqrt[3]{436 + \sqrt{2}}}{17 + \sqrt{37}}}$$

Any real number that is *not* an algebraic number is called a **transcendental number**. We have already met (and used extensively) two very important transcendental numbers: π and e.

It turns out that *proving* that these numbers are transcendental is rather difficult, although by the end of Chapter 8 we will have most of the tools necessary to prove that e is transcendental.

Functions defined using sums, differences, products, quotients or rational powers of rational coefficients and a real-valued variable x are called **algebraic functions**. Any non-algebraic functions of a real variable are called **transcendental functions**. Examples of transcendental functions with which you should be very familiar are $\sin(x)$, $\cos(x)$, $\tan(x)$ and the other trigonometric functions, as well as e^x.

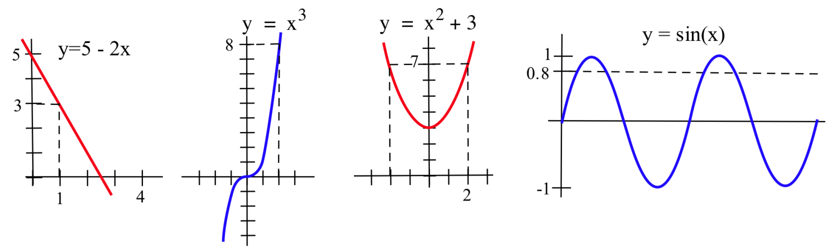
The geometric definitions of the trigonometric functions involve π, while the exponential function obviously involves e.

The inverse functions of these transcendental functions are also transcendental functions: $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$ and the other inverse trigonometric functions, as well as $\ln(x)$.

Because many important transcendental functions are defined as inverse functions, this chapter begins with a review of inverse functions and a discussion of finding derivatives of inverse functions. It continues with a review of inverse trigonometric functions and a discussion of their derivatives (and the usefulness of these derivative patterns in finding certain antiderivatives), then ties up some (very important) loose ends related to the definitions and properties of e^x and $\ln(x)$. The chapter concludes by introducing some new transcendental functions: the hyperbolic functions and their inverses, which play important roles in calculus and in applications.

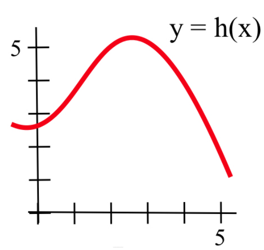
7.1 *One-to-One Functions*

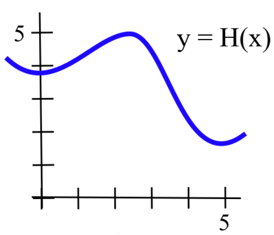
You've seen that some equations have only one solution (for example, $5 - 2x = 3$ and $x^3 = 8$), while some have two solutions ($x^2 + 3 = 7$) and some even have an infinite number of solutions ($\sin(x) = 0.8$). The graphs of $y = 5 - 2x$, $y = x^3$, $y = x^2 + 3$ and $y = \sin(x)$ and the solutions of the equations mentioned above appear below:



Functions f for which equations of the form $f(x) = k$ have at most one solution for each value of k (that is, each outcome k comes from only one input x) arise often in applications and possess a number of useful mathematical properties. This brief section focuses on those functions and examines some of their properties.

x	0	1	2	3	4	5
$g(x)$	5	7	3	5	0	7





Example 1. How many solutions does each equation have?

(a) $f(x) = 0$ for $f(x) = x(x - 4)$

(b) $g(x) = 3$ for g given in the margin table

(c) $h(x) = 4$ for h given by the graph in the margin

(d) $f(x) = k$ for $f(x) = e^x$.

Solution. (a) Two: $x(4 - x) = 0 \Rightarrow x = 0$ or $x = 4$. (b) One: $g(x) = 3$ only if $x = 2$. (c) Two: $h(x) = 4$ if $x \approx 1.2$ or if $x \approx 4$. (d) If $k > 0$, it has one solution: $x = \ln(k)$. If $k \le 0$, it has no solutions. ◀

Practice 1. How many solutions does each equation have?

(a) $f(x) = 4$ for $f(x) = x(4 - x)$

(b) $g(x) = 7$ for g given by the margin table

(c) $H(x) = 3$ for H given by the graph in margin

(d) $f(x) = 5$ for $f(x) = \ln(x)$

Horizontal Line Test

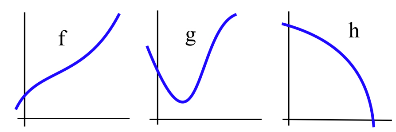
If not, review Section 0.3.

You should be familiar with the Vertical Line Test, a graphical tool you can use to help determine whether or not a curve in the xy-plane is the graph of a function. A similar geometrical test leads to the definition of a **one-to-one function** and provides a tool for helping to determine when a function is one-to-one.

> **Horizontal Line Test (Definition of One-to-One)**:
> A function is **one-to-one** if each horizontal line intersects the graph of the function at most once.

Equivalently, a function $y = f(x)$ is one-to-one if two *distinct* x-values always produce two *distinct* y-values: that is, $a \neq b \Rightarrow f(a) \neq f(b)$. This immediately tells us that every strictly increasing function is one-to-one, and that every strictly decreasing function is one-to-one. (Why?)

For any function, if we know an input value we can calculate the output, but an output may arise from any of several different inputs. With a one-to-one function, each output comes from only one input.



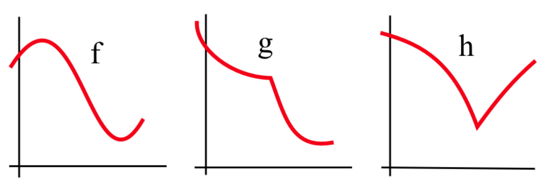
x	$f(x)$	$g(x)$	$h(x)$
0	5	7	2
1	2	3	-1
2	3	0	5
3	5	1	4
4	0	6	3
5	1	3	0

Example 2. (a) Which functions in the first margin figure are one–to-one? (b) Which functions in the first margin table are one-to-one?

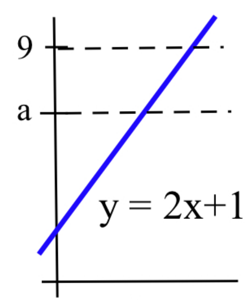
Solution. (a) In the figure, f and h are one-to-one; g fails the Horizontal Line Test, so g is not one-to-one. (b) In the table, h is one-to-one, while f and g are not one-to-one because $f(0) = f(3)$ and $g(1) = g(5)$. ◀

Practice 2. (a) Which functions graphed below are one-to-one? (b) Which functions in the second margin table are one-to-one?

x	$f(x)$	$g(x)$	$h(x)$
0	4	2	-2
1	2	3	5
2	-2	0	1
3	5	4	14
4	3	6	3
5	1	7	1



Example 3. Let $f(x) = 2x + 1$ (see margin). Find the values of x so that (a) $f(x) = 9$ and (b) $f(x) = a$ and then (c) solve $f(y) = x$ for y.



Solution. (a) $9 = f(x) = 2x + 1 \Rightarrow 8 = 2x \Rightarrow x = \frac{8}{2} = 4$ (b) $a = 2x + 1 \Rightarrow 2x = a - 1 \Rightarrow x = \frac{a-1}{2}$ (c) $x = f(y) = 2y + 1 \Rightarrow 2y = x - 1$ so $y = \dfrac{x-1}{2}$. Notice that this new function reverses the operations of $f(x)$, applied in reverse order: $f(x)$ multiplies x by 2, then adds 1; the new function subtracts 1, then divides by 2. ◀

Practice 3. Let $g(x) = 3x - 5$. Find the values of x so that (a) $g(x) = 7$ and (b) $g(x) = b$ and then (c) solve $g(y) = x$ for y.

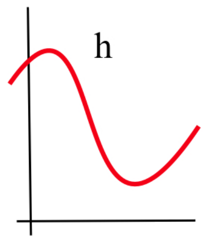
Practice 4. Show that exponential growth, for example $f(x) = e^{3x}$, and exponential decay, for example $g(x) = e^{-2x}$, are both one-to-one.

7.1 Problems

In Problems 1–4, explain why each given function is (or is not) one-to-one.

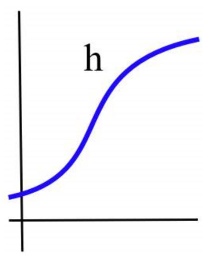
1. $f(x) = 3x - 5$, $y = 3 - x$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	3
1	4
2	5
3	2
4	4



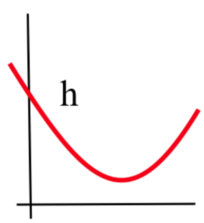
2. $f(x) = \frac{x}{4}$, $y = x^2 + 3$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	3
1	2
2	0
3	-2
4	1



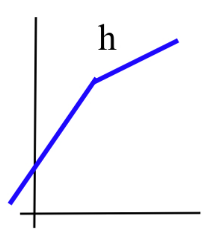
3. $f(x) = \sin(x)$, $y = e^x - 2$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	-1
1	5
2	3
3	1
4	0



4. $f(x) = 17$, $y = x^3 - 1$, $g(x)$ given by the table below, and $h(x)$ given by the graph below.

x	$g(x)$
0	2
1	5
2	4
3	1
4	2



5. Is the relation between people and Social Security numbers a function? A one-to-one function?

6. Is the relation between people and phone numbers a function? If so, is it one-to-one?

7. What would it mean if the scores on a calculus test were one-to-one?

8. The relation given below represents "y is married to x." (a) Is this relation a function? (b) Is it one-to-one? (c) Is P breaking the law? (d) Is A breaking the law?

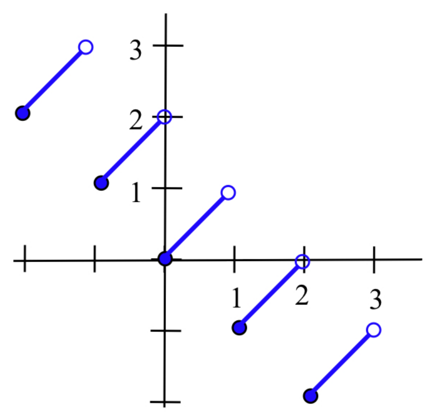
x	A	B	C	D
y	P	Q	P	R

9. In how many places can a one-to-one function touch the x-axis?

10. Can a continuous one-to-one function have the values given below? Explain.

x	1	3	5
$f(x)$	2	7	3

11. The graph of $f(x) = x - 2 \cdot \lfloor x \rfloor$ for $-2 \le x \le 3$ appears below.



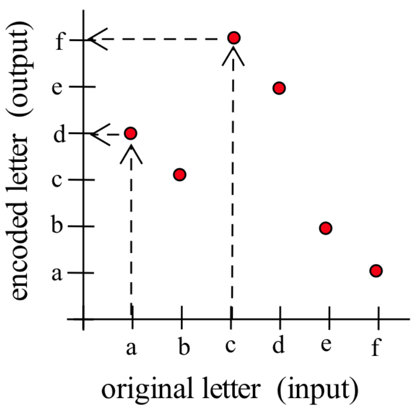
(a) Is f a one-to-one function?
(b) Is f an increasing function?
(c) Is f a decreasing function?

12. Is every linear function $L(x) = ax + b$ one-to-one? If not, which linear functions *are* one-to-one?

13. Show that $f(x) = \ln(x)$ is one-to-one for $x > 0$.
14. Show that $g(x) = e^x$ is one-to-one.
15. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
d	c	f	e	b	a

(a) Is the encoding rule a function?
(b) Is the encoding rule one-to-one?
(c) Encode the word "bad."
(d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).
(e) A graph of the encoding rule appears below. Create a graph of the decoding rule.



(f) Compare the encoding and decoding graphs.

16. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
b	d	b	b	a	c

(a) Is the encoding rule a function?
(b) Is the encoding rule one-to-one?
(c) Encode the word "bad."
(d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).
(e) Create a graph of the encoding rule.
(f) Create a graph of the decoding rule.
(g) Compare the encoding and decoding graphs.

17. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
d	f	e	a	c	b

(a) Is the encoding rule a function?
(b) Is the encoding rule one-to-one?
(c) Encode the word "bad."
(d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).
(e) Create a graph of the encoding rule.
(f) Create a graph of the decoding rule.
(g) Compare the encoding and decoding graphs.
(h) What happens if you encode a word, then encode the encoded word? For example, encode (encode ("bad")) = ?

18. The table below gives an encoding rule for a six-letter alphabet:

a	b	c	d	e	f
e	a	f	c	b	d

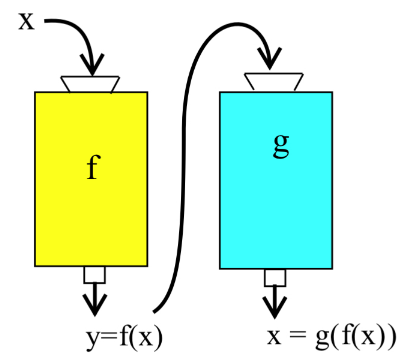
(a) Is the encoding rule a function?
(b) Is the encoding rule one-to-one?
(c) Encode the word "bad."
(d) Create a table for decoding the encoded letters and use it to decode your answer to part (c).
(e) Create a graph of the encoding rule.
(f) Create a graph of the decoding rule.
(g) Compare the encoding and decoding graphs.
(h) What happens if you apply this encoding rule three times in succession? For example, encode (encode (encode ("bad"))) = ?

7.1 Practice Answers

1. (a) One: solve $x(4-x) = 4$ to get $x = 2$.
   (b) Two: $x = 1$ and $x = 5$.
   (c) One: $x \approx 3.5$.
   (d) One: solve $5 = \ln(x)$ to get $x = e^5 \approx 148.4$.
2. (a) Only g is one-to-one; f and h fail the Horizontal Line Test.
   (b) Both f and g are one-to-one; h is not, because $h(2) = h(5)$.
3. (a) $3x - 5 = 7 \Rightarrow 3x = 12 \Rightarrow x = 4$
   (b) $3x - 5 = a \Rightarrow 3x = a + 5 \Rightarrow x = \dfrac{a+5}{3}$
   (c) $f(x) = 3x - 5 \Rightarrow f(y) = 3y - 5$ so $f(y) = x \Rightarrow 3y - 5 = x \Rightarrow 3y = x + 5 \Rightarrow y = \dfrac{x+5}{3}$
4. If $f(x) = e^{kx}$ where $k > 0$ then $f'(x) = k \cdot e^{kx} > 0$ so $f(x)$ is strictly increasing, hence one-to-one. If $g(x) = e^{rx}$ where $r < 0$ then $g'(x) = r \cdot e^{rx} < 0$ so $g(x)$ is strictly decreasing, hence one-to-one.

7.2 *Inverse Functions*

If f is any one-to-one function then equations of the form $f(x) = k$ have (at most) a single solution. Such functions can be uniquely "undone": if f is a one-to-one function, then there is another function g that "undoes" the effect of f, so that $g(f(x)) = x$. When g and f are composed in this manner, g retrieves the original input of f (see margin). We call this function g that "undoes" the effect of f the **inverse function** of f or simply **the inverse** of f.



If f is a function that encodes a message, then the inverse of f is the function that decodes an encoded message to retrieve the original message. The functions e^x and $\ln(x)$ "undo" the effects of each other:

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln(x)} = x \text{ (for } x > 0)$$

so the functions e^x and $\ln(x)$ are inverses of each other.

This section will examine some of the properties of inverse functions and explain how to find the inverse of a function given by a table of data, a graph or a formula.

> **Definition**: If f and g are functions that satisfy both $g(f(x)) = x$ and $f(g(y)) = y$ for all x in the domain of f and all y in the domain of g, then g is the **inverse** of f, f is the inverse of g, and f and g are a **pair of inverse functions**.

We often write the inverse function of f as f^{-1} (pronounced "eff inverse") but *you must be very careful*: f^{-1} does **not** mean $\dfrac{1}{f}$.

Example 1. The values of a function f appear in the margin table. Evaluate $f^{-1}(0)$ and $f^{-1}(1)$.

x	$f(x)$
0	3
1	2
2	0
3	1

Solution. For every x, $f^{-1}(f(x)) = x$ so the value of $f^{-1}(0)$ is the solution x of the equation $f(x) = 0$. The value we want is $x = 2$, and we can check that $f^{-1}(0) = f^{-1}(f(2)) = 2$.

The value of $f^{-1}(1)$ is the solution x of the equation $f(x) = 1$, which is $x = 3$, and we can check that $f^{-1}(1) = f^{-1}(f(3)) = 3$.

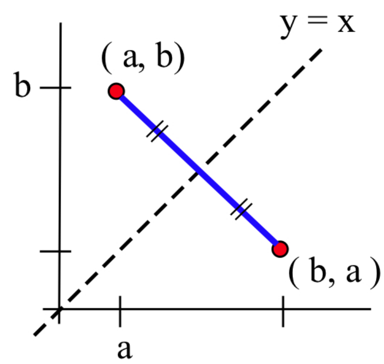
Similarly, $f^{-1}(2) = 1$ and $f^{-1}(3) = 0$. These results appear in the second margin table. You should notice that if the ordered pair (a, b) is in the table for f, then the reversed pair (b, a) is in the table for f^{-1}. ◄

x	$f^{-1}(x)$
0	2
1	3
2	1
3	0

Practice 1. The values of the function g appear in the margin table. Create a table of values for g^{-1}.

x	$g(x)$
0	2
1	1
2	3
3	4
4	0

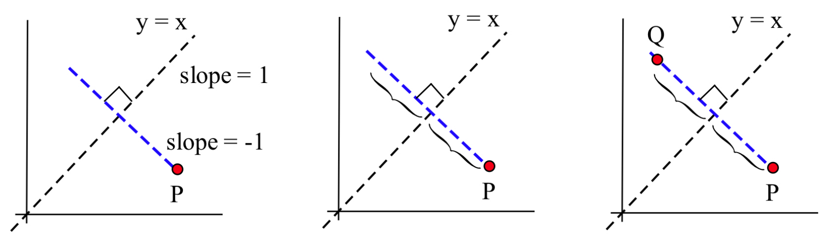
The method of interchanging the coordinates of a point on the graph (or in a table of values) of f to get a point on the graph (or in a table of values) of f^{-1} provides an efficient way to graph f^{-1}.



> **Theorem**: If the point (a,b) is on the graph of f, then the point (b,a) is on the graph of f^{-1}.

Proof. If (a,b) is on the graph of f, then $b = f(a)$, so $f^{-1}(b) = f^{-1}(f(a))$. By definition, $f^{-1}(f(a)) = a$, so $f^{-1}(b) = a$, which tells us that (b,a) is on the graph of f^{-1}. □

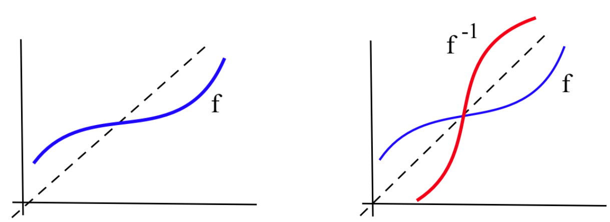
Graphically, when you interchange the coordinates of a point (a,b) to get a new point (b,a), the old point and the new point are symmetric about the line $y = x$. If you put a spot of wet ink at the point (a,b) on a piece of graph paper (see margin) and fold the paper along the line $y = x$, a new spot of ink will appear at the point (b,a). The figures below demonstrate another graphical method for finding the location of the point (b,a):



Draw the line $y = x$ along with a line of slope -1 that passes through the given point P (above left), then measure the distance from P to the line $y = x$ and move that same distance on the other side of $y = x$ (above center), and plot the new point Q at that location.

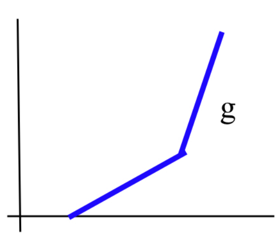
> **Corollary**: The graphs of f and f^{-1} are symmetric about the line $y = x$.

Example 2. A graph of f appears below left; sketch a graph of f^{-1}.



Solution. Imagine the graph of f is drawn with wet ink and fold the xy-plane along the line $y = x$. When you unfold the plane, the new graph is f^{-1} (see figure above right). ◀

You could also proceed point by point: Pick several points (a,b) on the graph of f and plot the symmetric points (b,a), then use the new (b,a) points as a guide for sketching the graph of f^{-1}.



Practice 2. A graph of g appears in the margin. Sketch a graph of g^{-1}. What happens to points on the graph of g that lie on the line $y = x$?

Finding a Formula for $f^{-1}(x)$

When you have a table of values for f, you can create a table of values for f^{-1} by interchanging the values of x and y in the table for f (as in Example 1). When you have a graph of f, you can sketch a graph of f^{-1} by reflecting the graph of f about the line $y = x$ (as in Example 2). When you have a formula for f, you can try to find a formula for f^{-1}.

Example 3. The steps for wrapping a gift are (1) put the gift in a box, (2) cover the box with paper and (3) attach a ribbon. What are the steps for opening the gift — the inverse of the wrapping operation?

Solution. (i) Remove the ribbon (undo step 3), (ii) remove the paper (undo step 2) and (iii) remove the gift from the box (undo step 1). ◀

Then show happiness and gratitude.

The reason for the preceding trivial example is to point out that the first unwrapping step undoes the last wrapping step, the second unwrapping step undoes the second-to-last wrapping step . . . and the last unwrapping step undoes the first wrapping step. This pattern holds for functions and their inverses too.

Example 4. The steps to evaluate $f(x) = 9x + 6$ are (1) multiply the input by 9 and (2) add 6 to the result. Write the steps, in words, for the inverse of this function, and then translate the verbal steps for the inverse into a formula for the inverse function.

Solution. (i) Subtract 6, undoing (2), and (ii) divide by 9, undoing (1):

$$x \quad \longrightarrow \quad x - 6 \quad \longrightarrow \quad \frac{x-6}{9}$$

so $f^{-1}(x) = \dfrac{x-6}{9}$ provides a formula for the inverse function. ◀

If (x, y) is a point on the graph of f, we know that (y, x) is on the graph of f^{-1}, so interchanging the roles of x and y in the formula for f should lead us to a formula for f^{-1}. Applying this idea to the function $f(x) = 9x + 6$ from the previous Example, we swap x and y in the formula $y = 9x + 6$ to get $x = 9y + 6 \Rightarrow x - 6 = 9y \Rightarrow y = \dfrac{x-6}{9}$, yielding the formula $f^{-1}(x) = \dfrac{x-6}{9}$ as in Example 4.

The following algorithm provides a general "recipe" for finding a formula for an inverse function:

- Start with a formula for f: $y = f(x)$.
- Interchange the roles of x and y: $x = f(y)$.
- Solve $x = f(y)$ for y.
- The resulting formula for y is the inverse of f: $y = f^{-1}(x)$.

The “interchange” and "solve" steps in the algorithm effectively undo the original operations in reverse order.

Example 5. Find formulas for the inverses $f^{-1}(x)$ and $g^{-1}(x)$ of $f(x) = \dfrac{7x-5}{4}$ and $g(x) = 2e^{5x}$.

Solution. Starting with $y = f(x) = \dfrac{7x-5}{4}$ and interchanging the roles of x and y yields:

$$x = \frac{7y-5}{4} \Rightarrow 4x = 7y-5 \Rightarrow 4x+5 = 7y \Rightarrow y = \frac{4x+5}{7}$$

so $f^{-1}(x) = \dfrac{4x+5}{7}$. Starting with $y = g(x) = 2e^{5x}$ and interchanging the roles of x and y yields:

$$x = 2e^{5y} \Rightarrow \frac{x}{2} = e^{5y} \Rightarrow \ln\left(\frac{x}{2}\right) = 5y \Rightarrow y = \frac{1}{5}\ln\left(\frac{x}{2}\right)$$

so $g^{-1}(x) = \dfrac{1}{5}\ln\left(\dfrac{x}{2}\right)$. ◀

Practice 3. Find formulas for the inverses $f^{-1}(x)$, $g^{-1}(x)$ and $h^{-1}(x)$ of $f(x) = 2x-5$, $g(x) = \dfrac{2x-1}{x+7}$ and $h(x) = 2+\ln(3x)$.

Sometimes it is easy to “solve $x = f(y)$ for y," but often it is not. When we try to find a formula for the inverse of $y = f(x) = x + e^x$, the first step is easy: interchanging the roles of x and y yields $x = y + e^y$. At this point, unfortunately, there is no way to algebraically solve the equation $x = y + e^y$ explicitly for y. The function $y = x + e^x$ has an inverse function, but we cannot find an explicit formula for that inverse.

Which Functions Have Inverse Functions?

We have seen how to find the inverse function for some functions given by tables of values, by graphs and by formulas, but there are functions that do not have inverse functions. The only way a graph and its reflection about the line $y = x$ can *both* be function graphs (so that f and f^{-1} are both functions) is if the graph of f passes both the Vertical Line Test (so that f is a function) and the Horizontal Line Test (so that the graph of f^{-1} passes the Vertical Line Test, verifying that f^{-1} is a function). This idea leads to the next theorem (stated without proof).

> **Theorem**: The function f has an inverse function if and only if f is one-to-one.

Two useful corollaries follow from this theorem.

> **Corollary 1**: If f is strictly increasing or strictly decreasing, then f has an inverse function.

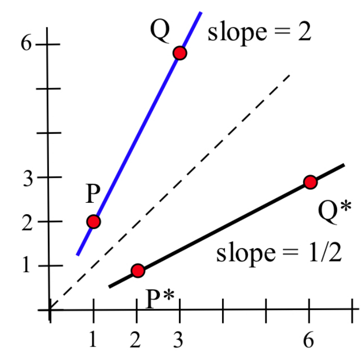
Corollary 2: If $f'(x) > 0$ for all x or $f'(x) < 0$ for all x, then f has an inverse function.

Applying this last Corollary to $f(x) = x + e^x$, we know that $f'(x) = 1 + e^x > 0$ for all x, so f must have an inverse (even though we can't find an explicit formula for f^{-1}).

Slopes of Inverse Functions

When a function f has an inverse, the symmetry of the graphs of f and f^{-1} also provides us with information about slopes and derivatives.

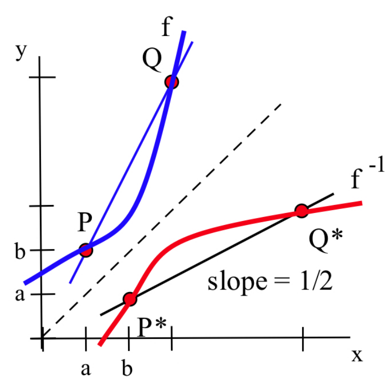
Example 6. Suppose the points $P = (1,2)$ and $Q = (3,6)$ are on the graph of a function f. (a) Sketch the line passing through P and Q. (b) Compute the slope of that line. (c) Graph the reflected points P^* and Q^* on the graph of f^{-1}. (d) Sketch the line passing through P^* and Q^*. Find the slope of the line through P^* and Q^*.



Solution. (a) See margin figure. (b) The slope through P and Q is $m = \frac{6-2}{3-1} = \frac{4}{2} = 2$. (c) The reflected points, obtained by interchanging the first and second coordinates of each point on the graph of f, are $P^* = (2,1)$ and $Q^* = (6,3)$. (d) See margin figure. (e) The slope of the line though P^* and Q^* is $\frac{3-1}{6-2} = \frac{1}{2}$ ◀

In the preceding Example, you may have noticed that:

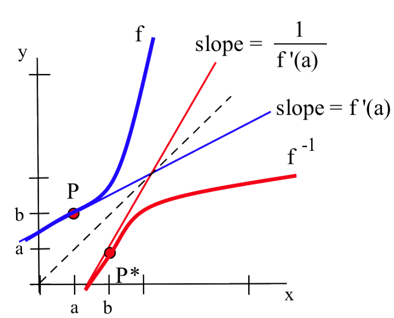
$$\text{slope of line through } P^* \text{ and } Q^* = \frac{1}{\text{slope of line through } P \text{ and } Q}$$



This was not a coincidence. In general, if $P = (a,b)$ and $Q = (x,y)$ are points on the graph of f (see margin), then the reflected points $P^* = (b,a)$ and $Q^* = (y,x)$ are on the graph of f^{-1} and:

$$\text{slope of segment } P^*Q^* = \frac{1}{\text{slope of segment } PQ}$$

Because the slope of a tangent line is the limit of slopes of secant lines, a similar relationship holds between the slope of the tangent line to f at the point (a,b) and slope of the tangent line to f^{-1} at the point (b,a). If we let the point Q^* approach the point P^* along the graph of f^{-1} (as in the margin figure) then:



$$\left(f^{-1}\right)'(b) = \lim_{Q^* \to P^*} [\text{slope of segment } P^*Q^*]$$
$$= \lim_{Q \to P} \frac{1}{\text{slope of segment } PQ} = \frac{1}{f'(a)}$$

This geometric idea leads to the following result:

Derivative of an Inverse Function

If $\quad b = f(a)$, f is differentiable at the point (a, b) and $f'(a) \neq 0$

then $\quad a = f^{-1}(b)$, f^{-1} is differentiable at the point (b, a) and:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$$

Example 7. The point $\left(e^2, 2\right)$ is on the graph of $f(x) = \ln(x)$ and $f'(x) = \dfrac{1}{x} \Rightarrow f'\left(e^2\right) = \dfrac{1}{e^2}$. Let g be the inverse function of f. Give one point on the graph of g and evaluate g' at that point.

Solution. The point $(2, e^2)$ is on the graph of g and:

$$g'(2) = \frac{1}{f'(e^2)} = \frac{1}{\frac{1}{e^2}} = e^2$$

In fact, the inverse of $f(x) = \ln(x)$ is the exponential function $g(x) = e^x$ and we can check that $g'(x) = e^x \Rightarrow g'(2) = e^2$. ◀

Example 8. In the table below, fill in the values of $f^{-1}(x)$ and $\left(f^{-1}\right)'(x)$ for $x = 0$ and $x = 1$.

Solution. $f(3) = 0$, so $f^{-1}(0) = 3$, while $\left(f^{-1}\right)'(0) = \dfrac{1}{f'(3)} = \dfrac{1}{2}$; $f(2) = 1$, so $f^{-1}(1) = 2$ while $\left(f^{-1}\right)'(1) = \dfrac{1}{f'(2)} = \dfrac{1}{-1} = -1$. ◀

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$\left(f^{-1}\right)'(x)$
0	2	3		
1	3	-2		
2	1	-1		
3	0	2		

Practice 4. Fill in the missing values for $x = 2$ and $x = 3$.

7.2 Problems

1. Given the values of f and f' in the table below, compute the specified values of f^{-1} and $\left(f^{-1}\right)'$.

x	$f(x)$	$f'(x)$	$f^{-1}(x)$	$\left(f^{-1}\right)'(x)$
1	3	-3		
2	1	2		
3	2	3		

2. Given the values of g and g' in the table below, compute the specified values of g^{-1} and $\left(g^{-1}\right)'$.

x	$g(x)$	$g'(x)$	$g^{-1}(x)$	$\left(g^{-1}\right)'(x)$
1	2	-2		
2	1	4		
3	3	2		

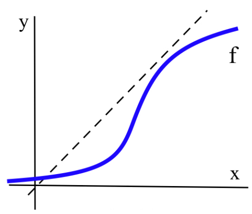
3. Given the values of h and h' in the table below, compute the specified values of h^{-1} and $\left(h^{-1}\right)'$.

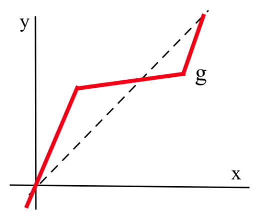
x	$h(x)$	$h'(x)$	$h^{-1}(x)$	$\left(h^{-1}\right)'(x)$
1	2	2		
2	3	-2		
3	1	0		

4. Given the values of w and w' in the table below, compute the specified values of w^{-1} and $\left(w^{-1}\right)'$.

x	$w(x)$	$w'(x)$	$w^{-1}(x)$	$\left(w^{-1}\right)'(x)$
1	1	2		
2	3	0		
3	2	5		

5. The figure below left shows a graph of f. Sketch a graph of f^{-1}.





6. The figure above right shows a graph of g. Sketch a graph of g^{-1}.

7. If the graphs of f and f^{-1} intersect at the point (a, b), how are a and b related?

8. If the graph of f intersects the line $y = x$ at $x = a$, does the graph of f^{-1} intersect the line $y = x$? If so, where?

9. The steps to evaluate the function $f(x) = \dfrac{7x-5}{4}$ are (1) multiply by 7, (2) subtract 5 and (3) divide by 4. Write the steps, in words, for the inverse of this function, and then translate these verbal steps into a formula for the inverse function.

10. Find a formula for the inverse function of $f(x) = 3x - 2$. Verify that $f^{-1}\left(f(5)\right) = 5$ and $f\left(f^{-1}(2)\right) = 2$.

11. Find a formula for the inverse function of $g(x) = 2x + 1$. Verify that $g^{-1}\left(g(1)\right) = 1$ and $g\left(g^{-1}(7)\right) = 7$.

12. Find a formula for the inverse of $h(x) = 2e^{3x}$. Verify that $h^{-1}\left(h(0)\right) = 0$.

13. Find a formula for the inverse of $w(x) = 5 + \ln(x)$. Verify that $w^{-1}\left(w(1)\right) = 1$.

14. If the graph of f goes through the point $(2, 5)$ and $f'(2) = 3$, then the graph of f^{-1} goes through the point (____ , ____) and $\left(f^{-1}\right)'(5) =$ ____.

15. If the graph of f goes through the point $(1, 3)$ and $f'(1) > 0$, then the graph of f^{-1} goes through the point (____ , ____). What can you say about the value of $\left(f^{-1}\right)'(3)$?

16. If $f(6) = 2$ and $f'(6) < 0$, then the graph of f^{-1} goes through the point (____ , ____). What can you say about $\left(f^{-1}\right)'(2)$?

17. If $f'(x) > 0$ for all values of x, what can you say about $\left(f^{-1}\right)'(x)$? What does this tell you about the graphs of f and f^{-1}?

18. If $f'(x) < 0$ for all values of x, what can you say about $\left(f^{-1}\right)'(x)$? What does this tell you about the graphs of f and f^{-1}?

19. Find a linear function $f(x) = ax + b$ so the graphs of f and f^{-1} are parallel and do not intersect.

20. Does $f(x) = 3 + \sin(x)$ have an inverse function? Justify your answer.

21. Does $f(x) = 3x + \sin(x)$ have an inverse function? Justify your answer.

22. For which positive integers n is $f(x) = x^n$ one-to-one? Justify your answer.

23. Some functions are their own inverses. For which four of these functions does $f^{-1}(x) = f(x)$?

(a) $f(x) = \dfrac{1}{x}$

(b) $f(x) = \dfrac{x+1}{x-1}$

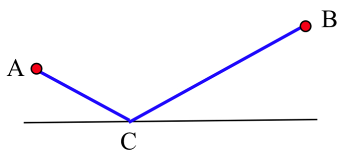
(c) $f(x) = \dfrac{3x-5}{7x-3}$

(d) $f(x) = \dfrac{ax+b}{cx-a}$

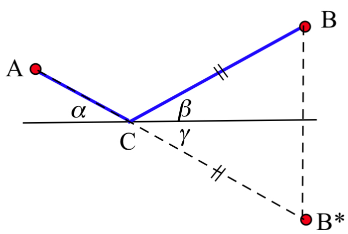
(e) $f(x) = x + a$ $(a \neq 0)$

Reflections on Folding

The symmetry of the graphs of a function and its inverse about the line $y = x$ make sketching a graph of the inverse function relatively easy if you have a graph of f: fold your graph paper along the line $y = x$ so the graph of f^{-1} is the "folded" image of f. This simple idea of obtaining a new image of something by folding along a line can enable us to quickly "see" solutions to some otherwise difficult problems.



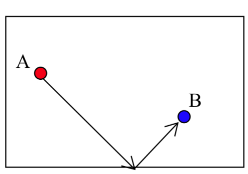
The minimization problem of finding the shortest path from town A to town B with an intermediate stop at a (straight) river, as depicted in the margin figure, is moderately difficult to solve using derivatives (see Problem 11 in Section 3.5). Geometry allows us to solve the problem much more easily (see second margin figure):

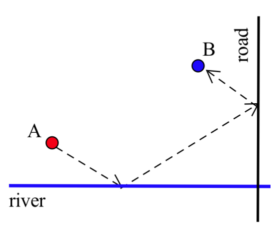


- obtain the point B^* by folding the image of B across the river line
- connect A and B^* with a line segment (the shortest path connecting A and B^*)
- fold the C-to-B^* segment back across the river to obtain the A-to-C-to-B solution.

As an almost-free bonus, we see that—for the minimum path—the angle of incidence at the river equals the angle of reflection.

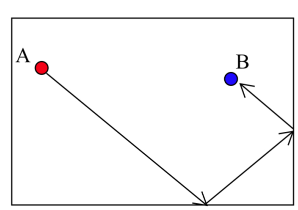
24. Devise an algorithm using "folding" to locate the point at the bottom edge of the billiards table you should aim ball A in the figure below left so that ball A will hit ball B after bouncing off the bottom edge of the table. (Assume that the angle of incidence equals the angle of reflection.)





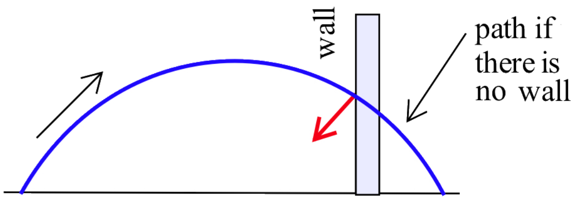
25. Devise an algorithm using "folding" to sketch the shortest path in the figure above right from town A to town B that includes a stop at the river *and* at the road. (One fold may not be enough.)

26. Devise an algorithm using "folding" to find where you should aim ball A at the bottom edge of the billiards table in the figure below so that ball A will hit ball B after bouncing off the bottom edge *and* the right edge of the table. (Assume that the angle of incidence equals the angle of reflection. Unfortunately, in a real-life game of billiards, the ball picks up a spin, called "English," when it bounces off the first bank, so at the second bounce the angle of incidence does not equal the angle of reflection.)



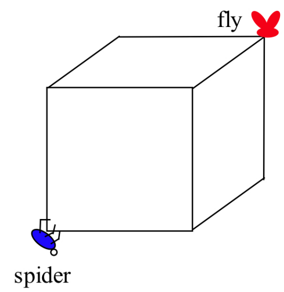
The "folding" idea can even be useful if the path is not a straight line.

27. The first figure below shows the parabolic path of a thrown ball. If the ball bounces off the tall vertical wall in the second margin figure, where will it hit the ground? (Assume that the angle of incidence equals the angle of reflection and that the ball does not lose energy during the bounce.)



Sometimes "unfolding" a problem is useful too.

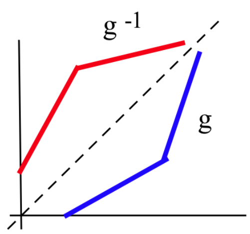
28. A spider and a fly are located at opposite corners of the cube as shown below. Sketch the shortest path the spider can travel along the surface of the cube to reach the fly.



7.2 Practice Answers

1. See table below left.

x	$g^{-1}(x)$
0	4
1	1
2	0
3	2
4	3



2. See figure above right. If a point lies on the line $y = x$, it must have the form (a, a) so that $g(a) = a \Rightarrow a = g^{-1}(a)$: the point (a, a) is also on the graph of g^{-1}.

3. (a) Swapping x and y in $y = 2x - 5$ yields $x = 2y - 5 \Rightarrow x + 5 = 2y \Rightarrow y = \dfrac{x+5}{2}$ so $f^{-1}(x) = \dfrac{x+5}{2}$.

   (b) Swapping x and y in $y = \dfrac{2x-1}{x+7}$: $x = \dfrac{2y-1}{y+7} \Rightarrow xy + 7x = 2y - 1 \Rightarrow 7x + 1 = 2y - xy \Rightarrow y = \dfrac{7x+1}{2-x}$ so $g^{-1}(x) = \dfrac{7x+1}{2-x}$.

   (c) Interchanging x and y in $y = 2 + \ln(3x)$ yields $x = 2 + \ln(3y) \Rightarrow x - 2 = \ln(3y) \Rightarrow e^{x-2} = 3y \Rightarrow y = \dfrac{1}{3}e^{x-2}$ so $h^{-1}(x) = \dfrac{1}{3}e^{x-2}$.

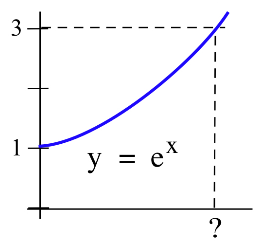
4. $f^{-1}(2) = 0$ and $f^{-1}(3) = 1$ while $\left(f^{-1}\right)'(2) = \dfrac{1}{f'(0)} = \dfrac{1}{3}$ and $\left(f^{-1}\right)'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{-2}$.

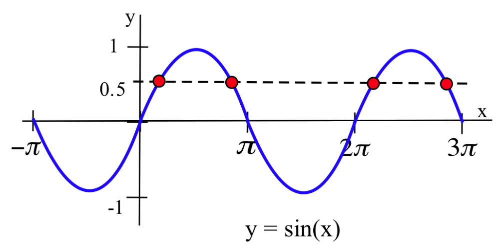
7.3 *Inverse Trigonometric Functions*

We now turn our attention to the inverse trigonometric functions, their properties and their graphs, focusing on properties and techniques needed to investigate derivatives and integrals of these functions. We will concentrate on the inverse sine and inverse tangent functions, the two inverse trigonometric functions that arise most often in calculus.

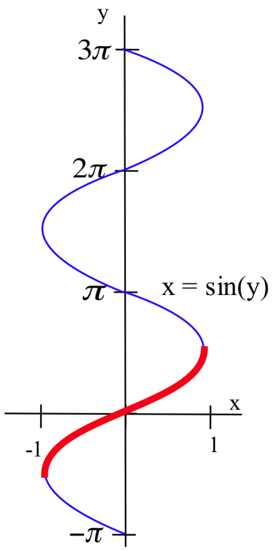
Inverse Sine: Solving $k = \sin(x)$ for x

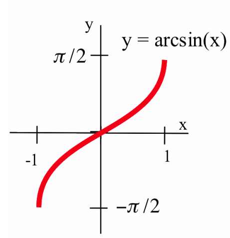
It is straightforward to solve the equation $3 = e^x$ (see margin figure below left): simply apply the natural logarithm function, the inverse of the exponential function e^x, to each side of the equation to get $\ln(3) = \ln(e^x) = x$. Because the function $f(x) = e^x$ is one-to-one, the equation $3 = e^x$ has only the one solution $x = \ln(3) \approx 1.1$.

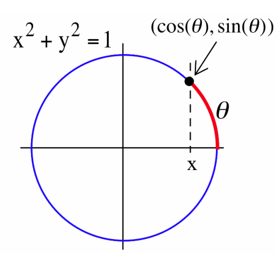


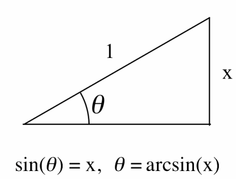


The solution of the equation $0.5 = \sin(x)$ presents more difficulties. As the figure above right illustrates, the function $f(x) = \sin(x)$ is *not* one-to-one: its graph reflected about the line $y = x$ (see margin figure) is *not* the graph of a function. Sometimes, however, it is necessary to "undo" the sine function, and we can do so by restricting its domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, the function $f(x) = \sin(x)$ is one-to-one and has an inverse function — and the graph of the inverse function (see below left) is the reflection about the line $y = x$ of the (restricted) graph of $y = \sin(x)$.









We call this inverse of the (restricted) sine function the **arcsine** and denote it $\arcsin(x)$. The name "arcsine" comes from the unit-circle definition of the sine function. On the unit circle (above center), if θ is the length of the arc whose sine is x, then $\sin(\theta) = x$ and $\theta = \arcsin(x)$. Using the right-triangle definition of sine (above right), θ represents an angle whose sine is x.

Many textbooks — and most calculators — use the notation $\sin^{-1}(x)$ for $\arcsin(x)$. You must be very careful to **never** interpret $\sin^{-1}(x)$ to mean:

$$(\sin(x))^{-1} = \frac{1}{\sin(x)} = \csc(x)$$

We avoid the $\sin^{-1}(x)$ notation for this reason and suggest that you do as well.

Definition of Inverse Sine
For $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$:

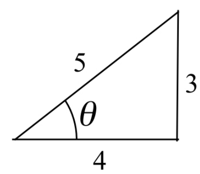
$$y = \arcsin(x) \iff x = \sin(y)$$

The domain of $\arcsin(x)$ is $[-1, 1]$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The (restricted) sine function and the arcsine are inverses of each other:

$$-1 \le x \le 1 \;\Rightarrow\; \sin(\arcsin(x)) = x$$
$$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \;\Rightarrow\; \arcsin(\sin(y)) = y$$

Right Triangles and Arcsine



$\sin(\theta) = \dfrac{3}{5}$

$\theta = \arcsin\left(\dfrac{3}{5}\right)$

$\theta \approx 0.6435$

For the right triangle shown in the margin, $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}$ so $\theta = \arcsin\left(\frac{3}{5}\right)$. It is possible to evaluate other trigonometric functions (such as cosine and tangent) of an angle expressed as an arcsine without explicitly solving for the value of the angle. For example:

$$\cos\left(\arcsin\left(\frac{3}{5}\right)\right) = \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5}$$
$$\tan\left(\arcsin\left(\frac{3}{5}\right)\right) = \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$$

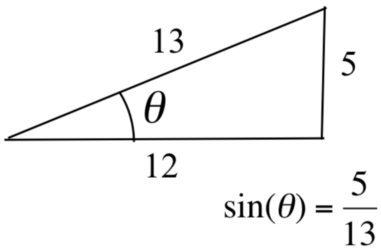
Once you know the sides of the right triangle, you can compute the values of the other trigonometric functions using their standard right-triangle definitions:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adjacent}}{\text{opposite}}$$

If you are given an angle θ as the arcsine of a number, but not given the sides of a right triangle, you can construct your own triangle with the given angle: select values for the opposite side and hypotenuse so the ratio $\dfrac{\text{opposite}}{\text{hypotenuse}}$ is the value whose arcsine we want: $\arcsin\left(\dfrac{\text{opposite}}{\text{hypotenuse}}\right)$. You can calculate the length of the third ("adjacent") side using the Pythagorean Theorem.

Example 1. Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin\left(\frac{5}{13}\right)$. Use the triangle to determine the values of $\tan\left(\arcsin\left(\frac{5}{13}\right)\right)$ and $\csc\left(\arcsin\left(\frac{5}{13}\right)\right)$.

Solution. We want the sine of θ, the ratio $\dfrac{\text{opposite}}{\text{hypotenuse}}$, to be $\frac{5}{13}$ so we can choose the opposite side to be 5 and the hypotenuse to be 13 (see



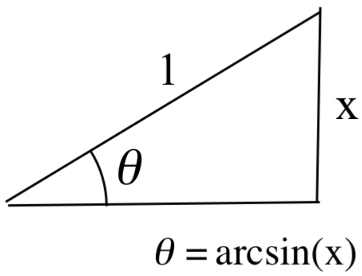
margin figure). Then $\sin(\theta) = \frac{5}{13}$, as desired. Using the Pythagorean Theorem, the length of the adjacent side is $\sqrt{13^2 - 5^2} = 12$. So:

$$\tan(\theta) = \tan\left(\arcsin\left(\frac{5}{13}\right)\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12}$$

$$\csc(\theta) = \csc\left(\arcsin\left(\frac{5}{13}\right)\right) = \frac{1}{\sin\left(\arcsin\left(\frac{5}{13}\right)\right)} = \frac{1}{\frac{5}{13}} = \frac{13}{5}$$

Any choice of values for the opposite side and the hypotenuse will work (for example opposite $= 500$ and hypotenuse $= 1300$), as long as the ratio of the opposite side to the hypotenuse is $\frac{5}{13}$. ◀

Practice 1. Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin\left(\frac{6}{11}\right)$. Use the triangle to determine the values of $\tan\left(\arcsin\left(\frac{6}{11}\right)\right)$, $\csc\left(\arcsin\left(\frac{6}{11}\right)\right)$ and $\cos\left(\arcsin\left(\frac{6}{11}\right)\right)$.



Example 2. Determine the lengths of the sides of a right triangle so one angle is $\theta = \arcsin(x)$. Use the triangle to determine the values of $\tan\left(\arcsin(x)\right)$ and $\cos\left(\arcsin(x)\right)$.

Solution. We want the sine of θ, the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$, to be x so we can choose the opposite side to be x and the hypotenuse to be 1 (see margin figure). Then $\sin(\theta) = \frac{x}{1} = x$ and, using the Pythagorean Theorem, the length of the adjacent side is $\sqrt{1-x^2}$ so that:

$$\tan\left(\arcsin(x)\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1-x^2}}$$

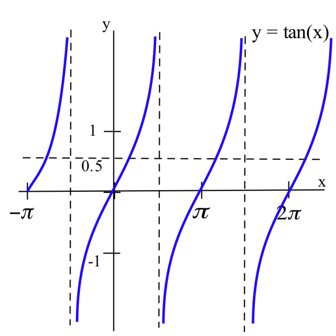
$$\cos\left(\arcsin(x)\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

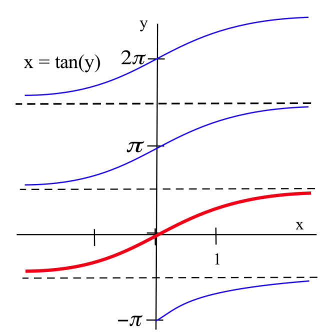
Other choices for the lengths of the opposite side and hypotenuse, such as $3x$ and 3, will work, but x and 1 are the simplest choices. ◀

Practice 2. Evaluate $\sec\left(\arcsin(x)\right)$ and $\csc\left(\arcsin(x)\right)$.

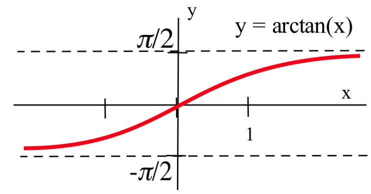
Inverse Tangent: Solving $k = \tan(x)$ for x

The equation $0.5 = \tan(x)$ (see below left) has many solutions: the function $f(x) = \tan(x)$ is not one-to-one, and its graph reflected across the line $y = x$ (below right) is not the graph of a function.





If, however, we restrict the domain of the tangent function to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the restricted $f(x) = \tan(x)$ *is* one-to-one and has an inverse function. The graph of this inverse tangent function (see margin figure) is the reflection about the line $y = x$ of the (restricted) graph of $y = \tan(x)$. We call this inverse of the (restricted) tangent function the **arctangent** and denote it $\arctan(x)$. For $x > 0$, the number $\arctan(x)$ is the length of the arc on the unit circle whose tangent is x, and $\arctan(x)$ is the angle whose tangent is x: $\tan(\arctan(x)) = x$.



Many textbooks — and most calculators — use the notation $\tan^{-1}(x)$ for $\arctan(x)$. You must be very careful to **never** interpret $\tan^{-1}(x)$ to mean:

$$(\tan(x))^{-1} = \frac{1}{\tan(x)} = \cot(x)$$

We avoid the $\tan^{-1}(x)$ notation for this reason and suggest that you do as well.

Definition of Inverse Tangent
For all x and $-\frac{\pi}{2} < y < \frac{\pi}{2}$:

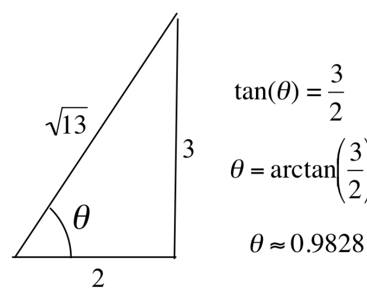
$$y = \arctan(x) \iff x = \tan(y)$$

The domain of $\arctan(x)$ is $(-\infty, \infty)$ and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The (restricted) tangent function and the arctangent are inverses:

$$-\infty < x < \infty \Rightarrow \tan(\arctan(x)) = x$$
$$-\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow \arctan(\tan(y)) = y$$

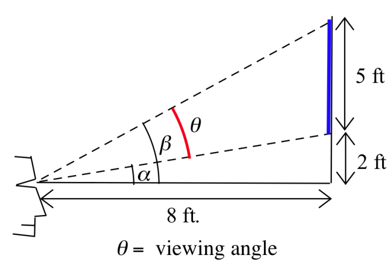
Right Triangles and Arctangent



For the right triangle in the margin, $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{2}$ so that $\theta = \arctan\left(\frac{3}{2}\right)$, hence:

$$\sin\left(\arctan\left(\frac{3}{2}\right)\right) = \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{\sqrt{13}} \approx 0.832$$
$$\cot\left(\arctan\left(\frac{3}{2}\right)\right) = \frac{1}{\tan\left(\arctan\left(\frac{3}{2}\right)\right)} = \frac{1}{\frac{3}{2}} = \frac{2}{3} \approx 0.667$$

Practice 3. Determine the lengths of the sides of a right triangle so that one angle is $\theta = \arctan\left(\frac{3}{4}\right)$, then use the triangle to determine the values of $\sin\left(\arctan\left(\frac{3}{4}\right)\right)$, $\cot\left(\arctan\left(\frac{3}{4}\right)\right)$ and $\cos\left(\arctan\left(\frac{3}{4}\right)\right)$.



Example 3. On a wall 8 feet in front of you, the lower edge of a 5-foot-tall painting rests 2 feet above your eye level (see margin). Represent your viewing angle θ using arctangents.

Solution. The viewing angle α to the bottom of the painting satisfies:

$$\tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{8} \Rightarrow \alpha = \arctan\left(\frac{1}{4}\right)$$

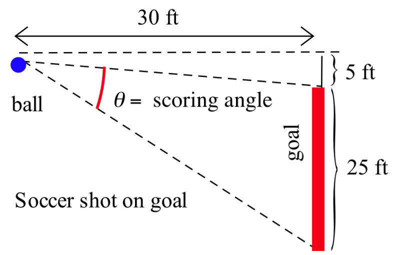
Similarly, the angle β to the top of the painting satisfies:

$$\tan(\beta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{7}{8} \Rightarrow \alpha = \arctan\left(\frac{7}{8}\right)$$

The viewing angle θ for the painting is therefore:

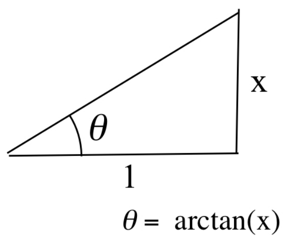
$$\theta = \beta - \alpha = \arctan\left(\frac{7}{8}\right) - \arctan\left(\frac{1}{4}\right) \approx 0.719 - 0.245 = 0.474$$

or about $27°$. ◀



Practice 4. Determine the scoring angle for the soccer player in the margin figure.

Example 4. Determine the lengths of the sides of a right triangle so that one angle is $\theta = \arctan(x)$, then use the triangle to determine the values of $\sin(\arctan(x))$ and $\cos(\arctan(x))$.



Solution. We want the tangent of θ (the ratio of opposite to adjacent) to be x, so we can choose the opposite side to be x and the adjacent side to be 1 (see margin). Then $\tan(\theta) = \frac{x}{1} = x$ and, using the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{1+x^2}$ so that:

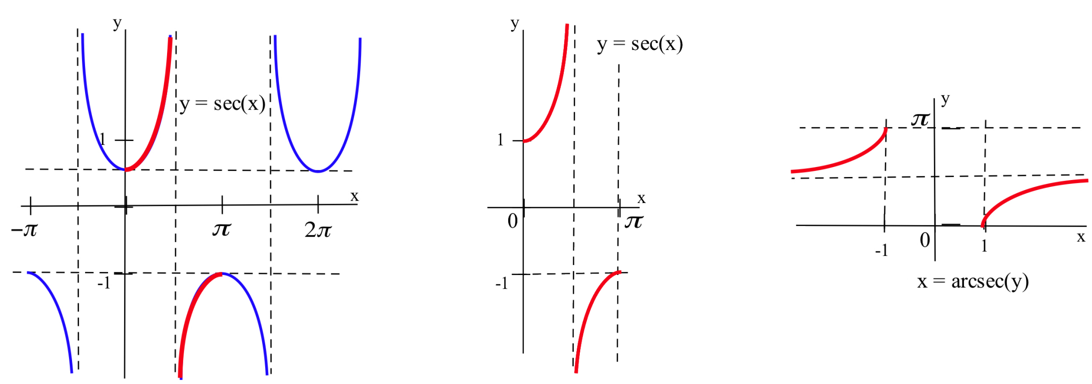
$$\sin(\arctan(x)) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{1+x^2}}$$
$$\cos(\arctan(x)) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}$$

We could have chosen other values for the opposite and adjacent sides (such as x^2 and x), but x and 1 provide the simplest option. ◀

Practice 5. Evaluate $\sec(\arctan(x))$ and $\cot(\arctan(x))$.

Inverse Secant: Solving $k = \sec(x)$ for x

The equation $2 = \sec(x)$ (see figure below left) has many solutions, but we can create an inverse function for secant—much the same way we did for sine and tangent—by suitably restricting the domain of the secant function so that it becomes a one-to-one function:



The figure above center shows the restriction $0 \le x \le \pi$ ($x \neq \frac{\pi}{2}$), which results in a one-to-one function that has an inverse. The graph of the inverse function (above right) is the reflection about the line $y = x$ of the (restricted) graph of $y = \sec(x)$. We call this inverse of the (restricted) secant function the **arcsecant** and denote it $\text{arcsec}(x)$.

There are alternate ways to restrict the secant function to get a one-to-one function, and they lead to slightly different definitions of the inverse secant. We chose to use this restriction because it seems more "natural" than the alternatives, it is easier to evaluate on a calculator, and it is the most commonly used.

Definition of Inverse Secant
If $|x| \geq 1$ and $0 \leq y \leq \pi$ with $y \neq \frac{\pi}{2}$:

$$y = \text{arcsec}(x) \iff x = \sec(y)$$

The domain of $\text{arcsec}(x)$ is $(-\infty, -1] \cup [1, \infty)$ and the range of $\text{arcsec}(x)$ is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Many textbooks use the notation $\sec^{-1}(x)$ for $\text{arcsec}(x)$. You must be very careful to **never** interpret $\sec^{-1}(x)$ to mean:

$$(\sec(x))^{-1} = \frac{1}{\sec(x)} = \cos(x)$$

We avoid the $\sec^{-1}(x)$ notation for this reason and suggest that you do as well.

The (restricted) secant function and the arcsecant are inverses:

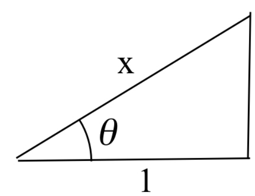
$$|x| \geq 1 \;\Rightarrow\; \sec(\text{arcsec}(x)) = x$$
$$0 \leq y \leq \pi \;\left(y \neq \frac{\pi}{2}\right) \;\Rightarrow\; \text{arcsec}(\sec(y)) = y$$

Example 5. Evaluate $\tan(\text{arcsec}(x))$.

Solution. We want the secant of θ (the ratio of hypotenuse to adjacent) to be x, so we can choose the hypotenuse to be x and the adjacent side to be 1 (see margin). Then $\sec(\theta) = \frac{x}{1} = x$ and, using the Pythagorean Theorem, the length of the opposite side is $\sqrt{x^2 - 1}$, so:

$$\tan(\text{arcsec}(x)) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$

As usual, x and 1 are the simplest—but not the only—choices. ◀



Practice 6. Evaluate $\sin(\text{arcsec}(x))$ and $\cot(\text{arcsec}(x))$.

The Other Inverse Trigonometric Functions

The inverse tangent and inverse sine functions are by far the most commonly used of the six inverse trigonometric functions in calculus. The inverse secant function turns up less often. The other three inverse trigonometric functions ($\arccos(x)$, $\text{arccot}(x)$ and $\text{arccsc}(x)$) can be defined as the inverses of restricted versions of $\cos(x)$, $\cot(x)$ and $\csc(x)$, respectively, but these functions are almost dispensable in calculus.

The reasons for this will become apparent in the next section.

Calculators and Inverse Trigonometric Functions

Most calculators only have keys for $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$, but the following identities allow you to compute values of the other inverse trigonometric functions.

If $x \neq 0$ and x is in the appropriate domain
then $\text{arccot}(x) = \arctan\left(\frac{1}{x}\right)$, $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ and $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$.

Proof. If $x \neq 0$, then:

$$\tan\left(\operatorname{arccot}(x)\right) = \frac{1}{\cot\left(\operatorname{arccot}(x)\right)} = \frac{1}{x}$$

Applying the arctangent function to each side of this equation:

$$\arctan\left(\tan\left(\operatorname{arccot}(x)\right)\right) = \arctan\left(\frac{1}{x}\right) \Rightarrow \operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

Proofs of the other two identities are left to you. □

If	$x \neq 0$ and x is in the appropriate domain
then	$\arcsin(x) + \arccos(x) = \frac{\pi}{2}$, $\arctan(x) + \operatorname{arccot}(x) = \frac{\pi}{2}$ and $\operatorname{arcsec}(x) + \operatorname{arccsc}(x) = \frac{\pi}{2}$.

Proof. If α and β are complementary angles in a right triangle, so that $\alpha + \beta = \frac{\pi}{2}$, then $\sin(\alpha) = \cos(\beta)$. Let $x = \sin(\alpha) = \cos(\beta)$ so that $\alpha = \arcsin(x)$ and $\beta = \arccos(x)$, hence:

$$\alpha + \beta = \arcsin(x) + \arccos(x) = \frac{\pi}{2}$$

This proves the first result for $0 < x < 1$. You can easily check that the result also holds for $x = 0$, $x = 1$ and $x = -1$. To check that it holds for $-1 < x < 0$, we need the the next set of identities listed below. Proofs of the other two identities above are left to you. □

If	x is in the appropriate domain
then	$\arcsin(-x) = -\arcsin(x)$, $\arccos(-x) = \pi - \arccos(x)$, $\arctan(x) = -\arctan(x)$, $\operatorname{arcsec}(x) = \pi - \operatorname{arcsec}(x)$, $\operatorname{arccsc}(-x) = -\operatorname{arccsc}(x)$ and $\operatorname{arccot}(-x) = -\operatorname{arccot}(x)$.

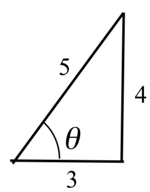
Proof. If $-1 \leq x \leq 1$, let $\theta = \arcsin(-x)$ so that $\sin(\theta) = -x \Rightarrow x = -\sin(\theta) = \sin(-\theta) \Rightarrow \arcsin(x) = -\theta \Rightarrow -\arcsin(x) = \theta = \arcsin(-x)$. This proves the first identity; the others are left to you. □

Some programming languages only include a *single* inverse trigonometric function, $\arctan(x)$, but it suffices to enable you to evaluate the other five inverse trigonometric functions:

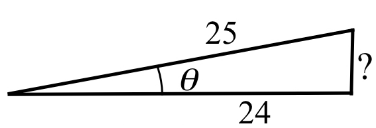
- $\arcsin(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$
- $\arccos(x) = \frac{\pi}{2} - \arcsin(x) = \frac{\pi}{2} - \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$
- $\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$
- $\operatorname{arcsec}(x) = \arctan\left(\sqrt{x^2-1}\right)$
- $\operatorname{arccsc}(x) = \frac{\pi}{2} - \operatorname{arcsec}(x) = \frac{\pi}{2} - \arctan\left(\sqrt{x^2-1}\right)$

7.3 Problems

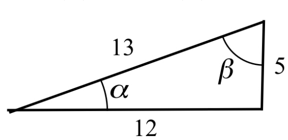
1. (a) List the three smallest positive angles θ that are solutions of the equation $\sin(\theta) = 1$.
   (b) Evaluate $\arcsin(1)$ and $\operatorname{arccsc}(1)$.
2. (a) List the three smallest positive angles θ that are solutions of the equation $\tan(\theta) = 1$.
   (b) Evaluate $\arctan(1)$ and $\operatorname{arccot}(1)$.
3. Find all x between 1 and 7 so that: (a) $\sin(x) = 0.3$ (b) $\sin(x) = -0.4$ (c) $\sin(x) = 0.5$
4. Find all values of x between 1 and 7 so that: (a) $\sin(x) = 0.3$ (b) $\sin(x) = -0.4$
5. Find all values of x between 2 and 7 so that: (a) $\tan(x) = 3.2$ (b) $\tan(x) = -0.2$
6. Find all values of x between 1 and 5 so that: (a) $\tan(x) = 8$ (b) $\tan(x) = -3$
7. In the figure below, angle θ is (a) the arcsine of what number? (b) the arctangent of what number? (c) the arcsecant of what number? (d) the arccosine of what number?



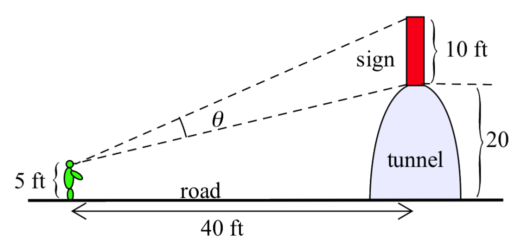
8. In the figure below, angle θ is (a) the arcsine of what number? (b) the arctangent of what number? (c) the arcsecant of what number? (d) the arccosine of what number?



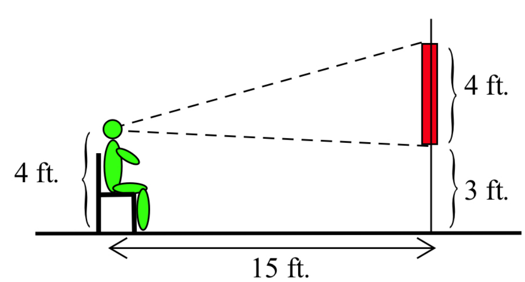
9. For the angle α in the triangle below, evaluate: (a) $\sin(\alpha)$ (b) $\tan(\alpha)$ (c) $\sec(\alpha)$ (d) $\cos(\alpha)$



10. For the angle β in the triangle above, evaluate: (a) $\sin(\beta)$ (b) $\tan(\beta)$ (c) $\sec(\beta)$ (d) $\cos(\beta)$
11. For $\theta = \arcsin\left(\frac{2}{7}\right)$, find the exact values of: (a) $\tan(\theta)$ (b) $\cos(\theta)$ (c) $\csc(\theta)$ (d) $\cot(\theta)$
12. For $\theta = \arctan\left(\frac{9}{2}\right)$, find the exact values of: (a) $\sin(\theta)$ (b) $\cos(\theta)$ (c) $\csc(\theta)$ (d) $\cot(\theta)$
13. For $\theta = \arccos\left(\frac{1}{5}\right)$, find the exact values of: (a) $\tan(\theta)$ (b) $\sin(\theta)$ (c) $\csc(\theta)$ (d) $\cot(\theta)$
14. For $\theta = \arcsin\left(\frac{a}{b}\right)$ with $0 < a < b$, find the exact values of: (a) $\tan(\theta)$ (b) $\cos(\theta)$ (c) $\csc(\theta)$ (d) $\cot(\theta)$
15. For $\theta = \arctan\left(\frac{a}{b}\right)$ with $0 < a < b$, find the exact values of: (a) $\tan(\theta)$ (b) $\sin(\theta)$ (c) $\cos(\theta)$ (d) $\cot(\theta)$
16. For $\theta = \arctan(x)$, find the exact values of: (a) $\sin(\theta)$ (b) $\cos(\theta)$ (c) $\sec(\theta)$ (d) $\cot(\theta)$
17. Find the exact values of (a) $\sin(\arccos(x))$ (b) $\cos(\arcsin(x))$ (c) $\sec(\arccos(x))$
18. Find the exact values of (a) $\tan(\arccos(x))$ (b) $\cos(\arctan(x))$ (c) $\sec(\arcsin(x))$
19. (a) Does $\arcsin(1) + \arcsin(1) = \arcsin(2)$?
    (b) Does $\arccos(1) + \arccos(1) = \arccos(2)$?
20. (a) What is the viewing angle for the tunnel sign in the figure below?
    (b) Use arctangents to describe the viewing angle when the observer is x feet from the entrance of the tunnel.



21. (a) What is the viewing angle for the whiteboard in the figure below?
    (b) Use arctangents to describe the viewing angle when the student is x feet from the wall.



22. Graph $y = \arcsin(2x)$ and $y = \arctan(2x)$.
23. Graph $y = \arcsin\left(\frac{x}{2}\right)$ and $y = \arctan\left(\frac{x}{2}\right)$.
24. Which curve is longer, $y = \sin(x)$ from $x = 0$ to $x = \pi$, or $y = \arcsin(x)$ from $x = -1$ to $x = 1$?

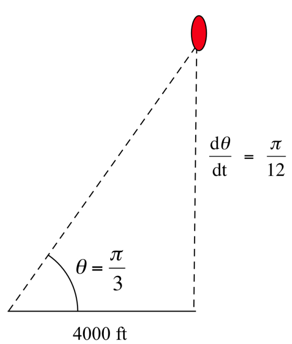
For Problems 25–28, $\left.\frac{d\theta}{dt}\right|_{\theta=1.3} = 12$, and θ and h are related by the given formula. Find $\left.\frac{dh}{dt}\right|_{\theta=1.3}$.

25. $\sin(\theta) = \frac{h}{20}$
26. $\tan(\theta) = \frac{h}{50}$
27. $\cos(\theta) = 3h + 20$
28. $3 + \tan(\theta) = 7h$

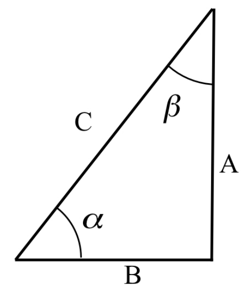
For Problems 29–32, $\left.\frac{dh}{dt}\right|_{\theta=1.3} = 4$, and θ and h are related by the given formula. Find $\left.\frac{d\theta}{dt}\right|_{\theta=1.3}$.

29. $\sin(\theta) = \frac{h}{38}$
30. $\tan(\theta) = \frac{h}{40}$
31. $\cos(\theta) = 7h - 23$
32. $\tan(\theta) = h^2$

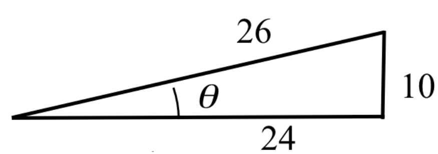
33. You are observing a rocket launch from a position located 4000 feet from the launch pad (see below). When your observation angle of the rocket is $\frac{\pi}{3}$, the angle is increasing at $\frac{\pi}{12}$ feet per second. How fast is the rocket traveling?



34. You are observing a rocket launch from a position 3000 feet from the launch pad. NASA's Twitter feed reports that when the rocket is 5000 feet high, its velocity is 100 feet per second.
    (a) What is the angle of elevation of the rocket when it is 5000 feet above the launch pad?
    (b) How fast is the angle of elevation increasing when the rocket is 5000 feet high?
35. Refer to the right triangle shown below.
    (a) Angle α is arcsine of what number?
    (b) Angle β is arccosine of what number?
    (c) For positive numbers A and C, evaluate $\arcsin\left(\frac{A}{C}\right) + \arccos\left(\frac{A}{C}\right)$.



36. Refer to the right triangle shown above.
    (a) Angle α is arctangent of what number?
    (b) Angle β is arccotangent of what number?
    (c) For positive numbers A and B, evaluate $\arctan\left(\frac{A}{B}\right) + \text{arccot}\left(\frac{A}{B}\right)$.
37. Refer to the triangle from Problems 35–36.
    (a) Angle α is arcsecant of what number?
    (b) Angle β is arccosecant of what number?
    (c) For positive numbers B and C, evaluate $\text{arcsec}\left(\frac{C}{B}\right) + \text{arccsc}\left(\frac{C}{B}\right)$.
38. Describe the pattern apparent in your results from the previous three problems.
39. Refer to the right triangle shown below.
    (a) Angle θ is arctangent of what number?
    (b) Angle θ is arccotangent of what number?



40. Refer to the right triangle shown above.
    (a) Angle θ is arcsine of what number?
    (b) Angle θ is arccosecant of what number?
41. Refer to the triangle from Problems 39–40.
    (a) Angle θ is arccosine of what number?
    (b) Angle θ is arcsecant of what number?
42. Describe the pattern apparent in your results from the previous three problems.

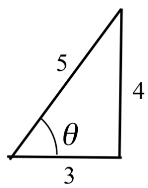
In 43–51, use a calculator (as necessary) and appropriate identities to compute the given values.

43. $\text{arcsec}(3)$ 44. $\text{arcsec}(-2)$ 45. $\text{arcsec}(-1)$

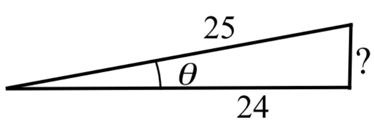
46. $\arccos(0.5)$ 47. $\arccos(-0.5)$ 48. $\arccos(1)$

49. $\text{arccot}(1)$ 50. $\text{arccot}(0.5)$ 51. $\text{arccot}(-3)$

52. For the triangle shown below:
   (a) $\theta = \arctan(\_\_\_\_)$
   (b) $\theta = \text{arccot}(\_\_\_\_)$
   (c) $\text{arccot}(\_\_\_\_) = \arctan(\_\_\_\_)$



53. For the triangle shown below:
   (a) $\theta = \arcsin(\_\_\_\_)$
   (b) $\theta = \arccos(\_\_\_\_)$
   (c) $\arccos(\_\_\_\_) = \arcsin(\_\_\_\_)$



54. Prove that $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$.

55. Prove that $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$.

Using a right triangle you can show that:

$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$$
$$\Rightarrow \arcsin(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$$

Imitate this reasoning in Problems 56–57.

56. Evaluate $\tan(\text{arccot}(x))$ and use the result to find a formula for $\text{arccot}(x)$ in terms of arctangent.

57. Evaluate $\tan(\text{arcsec}(x))$ and use the result to find a formula for $\text{arcsec}(x)$ in terms of arctangent.

58. Let $a = \arctan(x)$ and $b = \arctan(y)$. Use the identity:

$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\cdot\tan(b)}$$

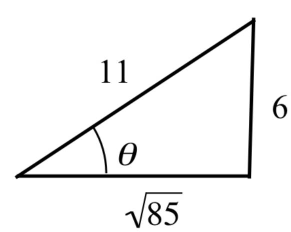
to show that:

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$$

7.3 Practice Answers

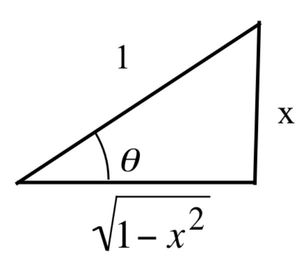
1. See the margin figure; with hypotenuse 11 and opposite side 6, the adjacent side must have length $\sqrt{11^2-6^2} = \sqrt{85}$, so:

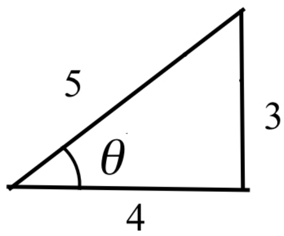
$$\tan\left(\arcsin\left(\frac{6}{11}\right)\right) = \frac{6}{\sqrt{85}}$$
$$\csc\left(\arcsin\left(\frac{6}{11}\right)\right) = \frac{11}{6}$$
$$\cos\left(\arcsin\left(\frac{6}{11}\right)\right) = \frac{\sqrt{85}}{11}$$



2. See the margin figure; with hypotenuse 1 and opposite side x, the adjacent side must have length $\sqrt{1^2-x^2} = \sqrt{1-x^2}$, so:

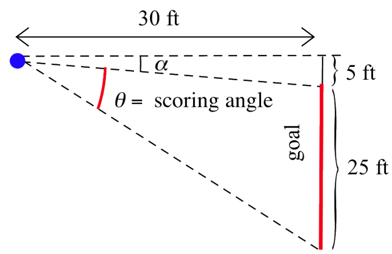
$$\sec(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$$
$$\csc(\arcsin(x)) = \frac{1}{x}$$





3. See the margin figure; with opposite side 3 and adjacent side 4, the hypotenuse must have length $\sqrt{3^2+4^2} = \sqrt{25} = 5$, so:

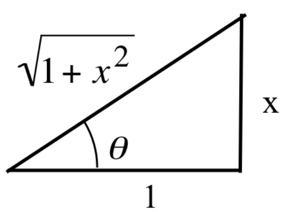
$$\sin\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{3}{5}$$
$$\cot\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{4}{3}$$
$$\cos\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{4}{5}$$



4. See margin figure; $\tan(\alpha) = \frac{5}{30} = \frac{1}{6}$ so $\alpha = \arctan\left(\frac{1}{6}\right) \approx 0.165$ (or about $9.46°$). Likewise, $\tan(\alpha+\theta) = \frac{30}{30} = 1$ so $\alpha + \theta = \arctan(1) = \frac{\pi}{4} \approx 0.785$ (or $45°$). Finally:

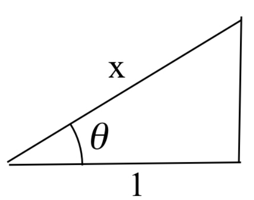
$$\theta = (\alpha+\theta) - \alpha \approx 0.785 - 0.165 = 0.62$$

or about $35.54°$.



5. See the margin figure; with opposite side x and adjacent side 1, the hypotenuse must have length $\sqrt{1^2+x^2} = \sqrt{1+x^2}$, so:

$$\sec(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$$
$$\cot(\arctan(x)) = \frac{1}{x}$$



$\theta =$ arcsec(x)

6. See the margin figure; with hypotenuse x and adjacent side 1, the opposite side must have length $\sqrt{x^2-1^2} = \sqrt{x^2-1}$, so:

$$\sin(\operatorname{arcsec}(x)) = \frac{\sqrt{x^2-1}}{x}$$
$$\cot(\operatorname{arcsec}(x)) = \frac{1}{\sqrt{x^2-1}}$$

7.4 *Derivatives of Inverse Trigonometric Functions*

The three previous sections introduced the ideas of one-to-one functions and inverse functions, then used those concepts to define arcsine, arctangent and the other inverse trigonometric functions. In this section, we obtain derivative formulas for the inverse trigonometric functions and consider an important application and some of its variations.

Derivative Formulas for Inverse Trigonometric Functions

$$\mathbf{D}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}} \quad (\text{for } |x| < 1) \qquad \mathbf{D}\left(\arccos(x)\right) = \frac{-1}{\sqrt{1-x^2}} \quad (\text{for } |x| < 1)$$

$$\mathbf{D}\left(\arctan(x)\right) = \frac{1}{1+x^2} \quad (\text{for all } x) \qquad \mathbf{D}\left(\text{arccot}(x)\right) = \frac{-1}{1+x^2} \quad (\text{for all } x)$$

$$\mathbf{D}\left(\text{arcsec}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x| > 1) \qquad \mathbf{D}\left(\text{arccsc}(x)\right) = \frac{-1}{|x|\sqrt{x^2-1}} \quad (\text{for } |x| > 1)$$

Proof. If $y = \arctan(x)$ then $\tan(y) = x$, so differentiating with respect to x and applying the Chain Rule yields:

$$\sec^2(y) \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

Recalling the Pythagorean identity $\sec^2(\theta) = 1 + \tan^2(\theta)$, we can write:

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

Notice that the derivative of $\arctan(x)$ is defined for all x, which corresponds to the domain of $\arctan(x)$.

proving the second formula listed above. To prove the first formula, put $y = \arcsin(x)$ so that $\sin(y) = x$. Implicitly differentiating with respect to x yields:

$$\cos(y) \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$$

Solving the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ for $\cos(\theta)$:

$$\cos^2(\theta) = 1 - \sin^2(\theta) \Rightarrow \cos(\theta) = \pm\sqrt{1-\sin^2(\theta)}$$

The range of $y = \arcsin(x)$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, so $\cos(y) \geq 0$ and:

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

Notice that the derivative of $\arcsin(x)$ is defined only when $1 - x^2 > 0$ or, equivalently, if $|x| < 1$, corresponding to the domain of $\arcsin(x)$ (omitting endpoints).

For the third formula, if $y = \text{arcsec}(x)$ then $\sec(y) = x$, so differentiation yields:

$$\sec(y) \cdot \tan(y) \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec(y) \cdot \tan(y)}$$

Solving the Pythagorean identity $\sec^2(\theta) = 1 + \tan^2(\theta)$ for $\tan(\theta)$:

$$\tan^2(\theta) = \sec^2(\theta) - 1 \;\Rightarrow\; \tan(\theta) = \pm\sqrt{\sec^2(\theta) - 1}$$

If $x \geq 1$ then $0 \leq y < \dfrac{\pi}{2}$ and on this interval $\tan(y) \geq 0$, so:

$$\frac{dy}{dx} = \frac{1}{\sec(y)\cdot\tan(y)} = \frac{1}{\sec(y)\cdot\sqrt{\sec^2(y)-1}} = \frac{1}{x\cdot\sqrt{x^2-1}}$$

If $x \leq -1$ then $\dfrac{\pi}{2} < y \leq \pi$ and on this interval $\tan(y) \leq 0$ so:

$$\frac{dy}{dx} = \frac{1}{\sec(y)\cdot\tan(y)} = \frac{1}{\sec(y)\cdot\left[-\sqrt{\sec^2(y)-1}\right]} = \frac{1}{-x\cdot\sqrt{x^2-1}}$$

Recognizing that $x = |x|$ when $x \geq 1$ and that $-x = |x|$ when $x \leq -1$, we can combine these two cases into a single compact form:

Notice that the derivative of $\operatorname{arcsec}(x)$ is defined only when $x^2 - 1 > 0$ or, equivalently, if $|x| > 1$, corresponding to the domain of $\operatorname{arcsec}(x)$ (omitting endpoints).

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$$

Proofs of the the remaining three formulas are nearly identical to those given above and are left for you as Problems 24–26. □

Example 1. Compute $\mathbf{D}\left(\arcsin\left(e^x\right)\right)$, $\mathbf{D}\left(\arctan\left(x-3\right)\right)$, $\mathbf{D}\left(\arctan^3\left(5x\right)\right)$ and $\mathbf{D}\left(\ln\left(\arcsin(x)\right)\right)$.

Solution. Each of these functions involves a composition of an inverse trigonometric function with other functions, so we need to use the Chain Rule:

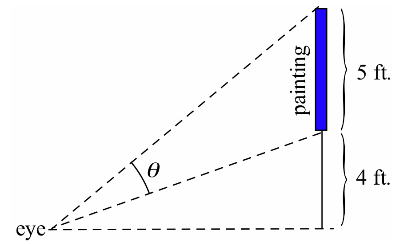
$$\begin{aligned}
\mathbf{D}\left(\arcsin\left(e^x\right)\right) &= \frac{1}{\sqrt{1-\left(e^x\right)^2}}\cdot\mathbf{D}\left(e^x\right) = \frac{e^x}{\sqrt{1-e^{2x}}} \\
\mathbf{D}\left(\arctan\left(x-3\right)\right) &= \frac{1}{1+\left(x-3\right)^2}\cdot\mathbf{D}\left(x-3\right) \\
&= \frac{1}{1+\left(x-3\right)^2} = \frac{1}{x^2-6x+10} \\
\mathbf{D}\left(\arctan^3\left(5x\right)\right) &= 3\arctan^2\left(5x\right)\cdot\mathbf{D}\left(\arctan(5x)\right) \\
&= 3\arctan^2\left(5x\right)\cdot\frac{1}{1+\left(5x\right)^2}\cdot 5 = \frac{15\arctan^2\left(5x\right)}{1+25x^2} \\
\mathbf{D}\left(\ln\left(\arcsin(x)\right)\right) &= \frac{1}{\arcsin(x)}\cdot\mathbf{D}\left(\arcsin(x)\right) = \frac{1}{\arcsin(x)}\cdot\frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

You have already practiced some of these differentiation patterns in Chapter 2 (see Problems 72–83 in Section 2.4). ◀

Practice 1. Compute $\mathbf{D}\left(\arcsin\left(5x\right)\right)$, $\mathbf{D}\left(\arctan\left(x+2\right)\right)$, $\mathbf{D}\left(\operatorname{arcsec}\left(7x\right)\right)$ and $\mathbf{D}\left(e^{\arctan(7x)}\right)$.

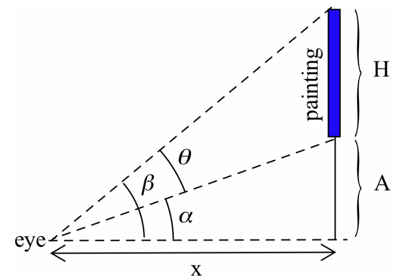
A Classic Application

Mathematics is the study of patterns, and one of the pleasures of mathematics is that patterns can crop up in some unexpected places. The mathematician Johannes Müller first posed the version of the classical Museum Problem below in 1471; it is one of the oldest known maximization problems.



Museum Problem: The lower edge of a 5-foot-tall painting is 4 feet above your eye level (see margin). At what distance should you stand from the wall so that your viewing angle of the painting is maximum?

On a typical sunny weekend, many people might rather be watching or playing a football or soccer game than visiting a museum or solving a calculus problem about a painting. But the pattern of the Museum Problem even appears in football and soccer. Because we also want to examine the Museum Problem in these other contexts, let's solve the general version.



Example 2. The lower edge of an H-foot-tall painting is A feet above your eye level (see margin). At what distance x should you stand from the painting so that your viewing angle is maximum?

Solution. Let $B = A + H$. Then $\tan(\alpha) = \dfrac{A}{x}$ and $\tan(\beta) = \dfrac{B}{x}$ so $\alpha = \arctan\left(\dfrac{A}{x}\right)$ and $\beta = \arctan\left(\dfrac{B}{x}\right)$. The viewing angle is thus:

$$\theta = \beta - \alpha = \arctan\left(\frac{B}{x}\right) - \arctan\left(\frac{A}{x}\right)$$

We can maximize θ by calculating the derivative $\dfrac{d\theta}{dx}$ and finding where that derivative is 0. Differentiation yields:

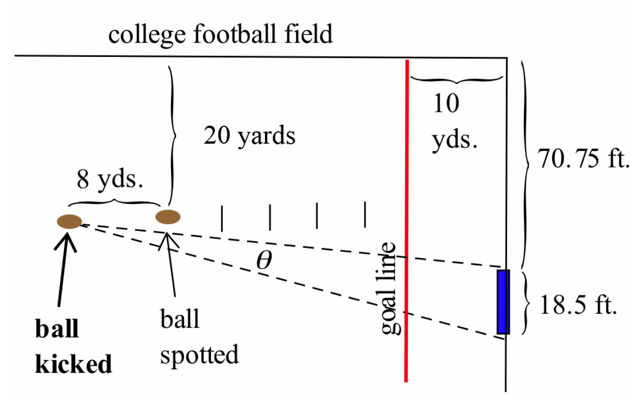
$$\begin{aligned}\frac{d\theta}{dx} &= \mathbf{D}\left[\arctan\left(\frac{B}{x}\right)\right] - \mathbf{D}\left[\arctan\left(\frac{A}{x}\right)\right] \\ &= \frac{1}{1+\left(\frac{B}{x}\right)^2}\cdot\left(\frac{-B}{x^2}\right) - \frac{1}{1+\left(\frac{A}{x}\right)^2}\cdot\left(\frac{-A}{x^2}\right) \\ &= \frac{-B}{x^2+B^2} + \frac{A}{x^2+A^2}\end{aligned}$$

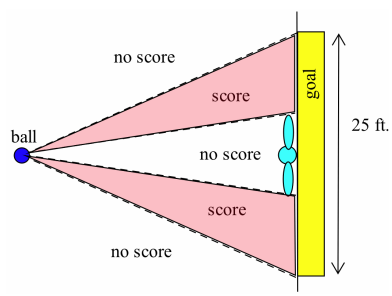
Setting $\dfrac{d\theta}{dx} = 0$ and solving for x yields $x = \sqrt{AB} = \sqrt{A(A+H)}$. ◄

We can disregard the endpoints, as you clearly do not have a maximum viewing angle with your nose pressed against the wall, or from an infinite distance far away from the wall.

Now the original Museum Problem above becomes straightforward (as do the football and soccer variations below). With $A = 4$ and $H = 5$, the maximum viewing angle occurs when $x = \sqrt{4(4+5)} = 6$ feet. The maximum angle is therefore $\theta = \arctan\left(\dfrac{9}{6}\right) - \arctan\left(\dfrac{4}{6}\right) \approx 0.983 - 0.588 \approx 0.395$, or about $22.6°$.

Practice 2. In a football game, a kicker is attempting a field goal by kicking the football between the goal posts (see figure below). At what distance from the goal line should the ball be "spotted" so the kicker has the largest angle for making the field goal? The goal posts are 18.5 feet wide and located 10 yards beyond the goal line. (Assume that the ball is placed on a "hash mark" that is 20 yards from the edge of the field and is actually kicked from a point about 8 yards farther from the goal line than where the ball is spotted.)





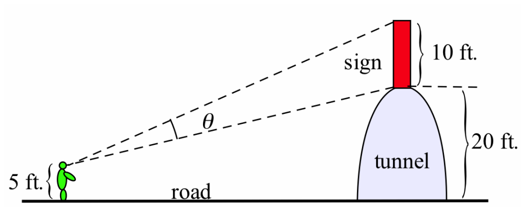
Practice 3. Kelcey is bringing a ball down the middle of a soccer field toward the 25-foot wide goal, which is defended by a goalie (see margin figure) positioned in the center of the goal. The goalie can stop a shot that is within four feet of the center of the goal. At what distance from the goal should Kelcey shoot so the scoring angle is maximum?

7.4 Problems

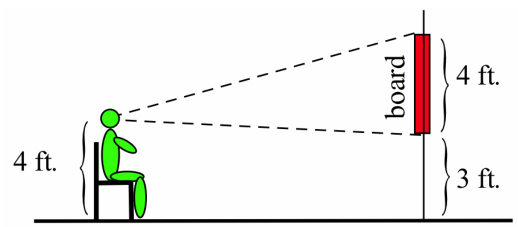
In Problems 1–21, compute the derivative.

1. $\mathbf{D}\left(\arcsin(3x)\right)$
2. $\mathbf{D}\left(\arctan(7x)\right)$
3. $\mathbf{D}\left(\arctan(x+5)\right)$
4. $\mathbf{D}\left(\arcsin\left(\frac{x}{2}\right)\right)$
5. $\mathbf{D}\left(\arctan\left(\sqrt{x}\right)\right)$
6. $\mathbf{D}\left(\operatorname{arcsec}\left(x^2\right)\right)$
7. $\mathbf{D}\left(\ln\left(\arctan(x)\right)\right)$
8. $\mathbf{D}\left(x\cdot\arctan(x)\right)$
9. $\mathbf{D}\left(\left(\operatorname{arcsec}(x)\right)^3\right)$
10. $\mathbf{D}\left(\arctan\left(\frac{5}{x}\right)\right)$
11. $\mathbf{D}\left(\arctan\left(\ln(x)\right)\right)$
12. $\mathbf{D}\left(\arcsin(x+2)\right)$
13. $\mathbf{D}\left(e^x\cdot\arctan(2x)\right)$
14. $\mathbf{D}\left(\tan(x)\cdot\arctan(x)\right)$
15. $\mathbf{D}\left(\arcsin(x)+\arccos(x)\right)$
16. $\mathbf{D}\left(\frac{1}{\arcsin(x)}\right)$
17. $\mathbf{D}\left(\sqrt{\arcsin(x)}\right)$
18. $\mathbf{D}\left(\left(1+\operatorname{arcsec}(x)\right)^3\right)$
19. $\mathbf{D}\left(\sin\left(3+\arctan(x)\right)\right)$
20. $\mathbf{D}\left(\frac{\arcsin(x)}{\arccos(x)}\right)$
21. $\mathbf{D}\left(x\cdot\arctan\left(\frac{1}{x}\right)\right)$

22. (a) Use arctangents to describe the viewing angle for the sign in the figure below when the observer is x feet from the tunnel entrance.
    (b) At what distance x is the angle maximized?



23. (a) Use arctangents to describe the viewing angle for the whiteboard in figure below when the student is x feet from the front wall.
    (b) At what distance x is the angle maximized?



24. Mimic the first part of the proof at the beginning of this section to show that (for all x):

$$\mathbf{D}\left(\operatorname{arccot}(x)\right) = \frac{-1}{1+x^2}$$

25. Mimic the second part of the proof at the beginning of this section to show that (for $|x| < 1$):

$$\mathbf{D}\left(\arccos(x)\right) = \frac{-1}{\sqrt{1-x^2}}$$

26. Mimic the third part of the proof at the beginning of this section to show that (for $|x| > 1$):

$$\mathbf{D}\left(\operatorname{arccsc}(x)\right) = \frac{-1}{|x| \cdot \sqrt{x^2-1}}$$

27. Revisit Problem 15 and draw a triangle to arrive at your answer without using any of the derivative patterns for inverse trignometric functions developed in this section.

7.4 Practice Answers

1. Applying the Chain Rule:

$$\mathbf{D}\left(\arcsin(5x)\right) = \frac{1}{\sqrt{1-(5x)^2}} \cdot 5 = \frac{5}{\sqrt{1-25x^2}}$$

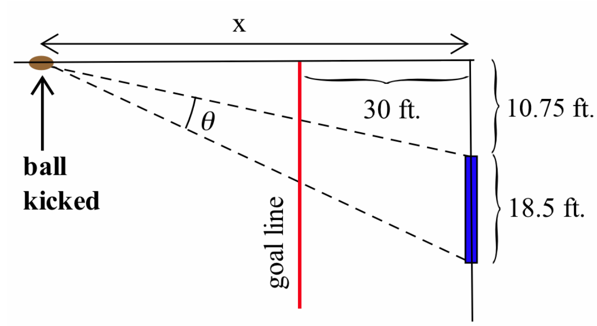
$$\mathbf{D}\left(\arctan(x+2)\right) = \frac{1}{1+(x+2)^2} = \frac{1}{x^2+4x+5}$$

$$\mathbf{D}\left(\operatorname{arcsec}(7x)\right) = \frac{1}{|7x| \cdot \sqrt{(7x)^2-1}} \cdot 7 = \frac{1}{|x| \cdot \sqrt{49x^2-1}}$$

$$\mathbf{D}\left(e^{\arctan(7x)}\right) = e^{\arctan(7x)} \cdot \mathbf{D}\left(\arctan(7x)\right)$$

$$= e^{\arctan(7x)} \cdot \frac{1}{1+(7x)^2} \cdot 7 = \frac{7e^{\arctan(7x)}}{1+49x^2}$$

2. $A = 10.75$ ft. and $H = 18.5$ ft. (see below) so $x = \sqrt{A(A+H)} = \sqrt{314.4375} \approx 17.73$ feet from the back edge of the endzone. Unfortunately, that is still more than 12 feet into the endzone. Our mathematical analysis shows that to maximize the angle for kicking a field goal, the ball should be kicked from a position 12 feet into the endzone, a touchdown!



If the ball is placed on a hash mark just outside the endzone, then the kicking distance is 54 feet (10 yards for the depth of the endzone plus 8 yards that the ball is hiked) and the scoring angle is:

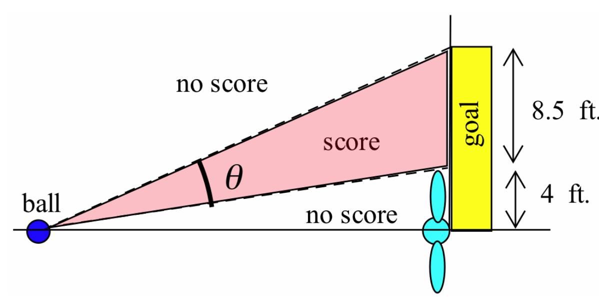
$$\theta = \arctan\left(\frac{29.25}{54}\right) - \arctan\left(\frac{10.75}{54}\right) \approx 17.2^\circ$$

It is somewhat interesting to see investigate how the scoring angle:

$$\theta = \arctan\left(\frac{29.25}{x}\right) - \arctan\left(\frac{10.75}{x}\right)$$

changes with the distance x (in feet), and also to compare the scoring angle for balls placed on the hash mark with those placed in the center of the field.

3. $A = 4$ ft. and $H = 8.5$ ft. so $x = \sqrt{A(A+H)} = \sqrt{50} \approx 7.1$ ft.:



x	θ
5	29.5°
7	31.0°
10	29.5°
15	24.9°
20	20.7°
30	15.0°
40	11.6°

From 7.1 feet, the scoring angle on one side of the goalie is:

$$\theta = \arctan\left(\frac{12.5}{7.1}\right) - \arctan\left(\frac{4}{7.1}\right) \approx 31^\circ$$

For comparison, the margin table gives values of the general scoring angle $\theta = \arctan\left(\frac{12.5}{x}\right) - \arctan\left(\frac{4}{x}\right)$ for some other distances x.

7.5 Integrals Involving Inverse Trig Functions

Aside from the Museum Problem and its sporting variations introduced in the previous section, the primary use of the inverse trigonometric functions in calculus involves their role as antiderivatives of rational and algebraic functions. Each of the six differentiation patterns from the previous section provides us with an integral formula, but they give rise to only three essentially different patterns:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$$

Valid for: $-1 < x < 1$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

Valid for: $-\infty < x < \infty$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(x) + C$$

Valid for: $|x| > 1$

Most of the related antiderivative patterns you will need in practice arise from variations of these basic ones. Typically, you need to transform an integrand so that it exactly matches one of the basic patterns.

Why are the derivative patterns for arccos, arccot and arccsc of little use to us when finding antiderivatives of algebraic functions?

Example 1. Evaluate $\displaystyle\int \frac{1}{16+x^2}\,dx$.

Solution. We can transform this integrand into the arctangent pattern by factoring 16 from the denominator

$$\int \frac{1}{16+x^2}\,dx = \int \frac{1}{16\left(1+\frac{x^2}{16}\right)}\,dx = \frac{1}{16}\int \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$$

and then using the substitution $u = \dfrac{x}{4} \Rightarrow du = \dfrac{1}{4}\,dx \Rightarrow 4\,du = dx$:

$$\frac{1}{16}\int \frac{1}{1+u^2}\cdot 4\,du = \frac{4}{16}\int \frac{1}{1+u^2}\,du = \frac{1}{4}\arctan(u) + C$$

Replacing u with $\dfrac{x}{4}$ we get $\dfrac{1}{4}\arctan\left(\dfrac{x}{4}\right) + C$ as our final answer. ◀

Practice 1. Evaluate $\displaystyle\int \frac{1}{1+9x^2}\,dx$ and $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx$.

The integrands that arise most often contain patterns with the forms $a^2 - x^2$, $a^2 + x^2$ and $x^2 - a^2$, where a is some positive constant, so it is worthwhile to develop general integral patterns for these forms, list them in Appendix I, and refer to them when necessary:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

Valid for: $-a < x < a$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Valid for: $-\infty < x < \infty$

$$\int \frac{1}{|x|\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$$

Valid for: $|x| > a$

You can arrive at each of these general formulas by factoring the a^2 out of the denominator and making a suitable change of variable (as in Example 1, with a in place of 4). You can then check the result by differentiating. The arctan pattern is, by far, the most common. The arcsin pattern appears occasionally, and the arcsec pattern only rarely.

Example 2. Develop the general formula for $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ from the known formula for $\int \frac{1}{\sqrt{1-x^2}}\,dx$. (Assume that $a>0$.)

Solution. Using Example 1 as a guide, factor a^2 out of the denominator:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}\,dx = \frac{1}{a}\int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx$$

Now substitute $u=\frac{x}{a} \Rightarrow du = \frac{1}{a}\,dx \Rightarrow a\,du = dx$ to get:

$$\frac{1}{a}\int \frac{1}{\sqrt{1-u^2}}\cdot a\,du = \frac{a}{a}\int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u)+C$$

and replace u with $\frac{x}{a}$ to get $\arcsin\left(\frac{x}{a}\right)+C$, the desired result. ◀

Practice 2. Verify that the derivative of $\frac{1}{a}\cdot\arctan\left(\frac{x}{a}\right)$ is $\frac{1}{a^2+x^2}$.

Example 3. Evaluate $\int \frac{1}{\sqrt{5-x^2}}\,dx$ and $\int_1^3 \frac{1}{5+x^2}\,dx$.

Solution. The constant a needn't be an integer, so take $a^2=5 \Rightarrow a=\sqrt{5}$:

$$\int \frac{1}{\sqrt{5-x^2}}\,dx = \arcsin\left(\frac{x}{\sqrt{5}}\right)+C$$

using the pattern from Example 2, while the general arctan pattern yields:

$$\begin{aligned}\int_1^3 \frac{1}{5+x^2}\,dx &= \left[\frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)\right]_1^3 \\ &= \frac{1}{\sqrt{5}}\left[\arctan\left(\frac{3}{\sqrt{5}}\right)-\arctan\left(\frac{1}{\sqrt{5}}\right)\right]\end{aligned}$$

or about 0.228. ◀

The easiest way to integrate certain rational functions is to split the original integrand into two pieces.

Example 4. Evaluate $\int \frac{6x+7}{25+x^2}\,dx$.

Solution. The integrand splits nicely into the sum of two other functions that you can integrate more easily:

$$\int \frac{6x+7}{25+x^2}\,dx = \int \frac{6x}{25+x^2}\,dx + \int \frac{7}{25+x^2}\,dx$$

In the first integral, use the subsitution $u = 25+x^2 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$:

$$\int \frac{6x}{25+x^2}\,dx = \frac{6}{2}\int \frac{1}{u}\,du = 3\ln\left(|u|\right) + C_1 = 3\ln\left(25+x^2\right) + C_1$$

Why can we remove the absolute value signs in the last step?

Meanwhile, the second integral matches the general arctangent pattern with $a = 5$:

$$\int \frac{7}{25+x^2}\,dx = 7\int \frac{1}{5^2+x^2}\,dx = 7\cdot\frac{1}{5}\arctan\left(\frac{x}{5}\right) + C_2$$

Combining these two results yields:

$$\int \frac{6x+7}{25+x^2}\,dx = \ln\left(25+x^2\right)^3 + \frac{7}{5}\arctan\left(\frac{x}{5}\right) + C$$

The two constants C_1 and C_2 add up to another arbitrary constant, which we can simply call C.

as our final answer. ◀

The antiderivative of a linear function divided by an irreducible quadratic polynomial will typically result in the sum of a logarithm and an arctangent.

Practice 3. Evaluate $\displaystyle\int \frac{4x+3}{x^2+7}\,dx$.

7.5 Problems

In Problems 1–24, evaluate the integral.

1. $\displaystyle\int \frac{7}{\sqrt{9-x^2}}\,dx$
2. $\displaystyle\int \frac{9}{\sqrt{7-y^2}}\,dy$
3. $\displaystyle\int_0^1 \frac{3}{x^2+25}\,dx$
4. $\displaystyle\int_5^7 \frac{5}{x\sqrt{x^2-16}}\,dx$
5. $\displaystyle\int \frac{9}{\sqrt{49-x^2}}\,dx$
6. $\displaystyle\int_1^4 \frac{2}{7+x^2}\,dx$
7. $\displaystyle\int_6^{10} \frac{3}{x\sqrt{x^2-25}}\,dx$
8. $\displaystyle\int \frac{7}{(x-5)^2+9}\,dx$
9. $\displaystyle\int \frac{1}{(x-1)^2+1}\,dx$
10. $\displaystyle\int \frac{1}{x^2-2x+2}\,dx$
11. $\displaystyle\int_{-1}^1 \frac{e^x}{1+e^{2x}}\,dx$
12. $\displaystyle\int_1^e \frac{1}{x}\cdot\frac{3}{1+[\ln(x)]^2}\,dx$
13. $\displaystyle\int \frac{\cos(\theta)}{\sqrt{9-\sin^2(\theta)}}\,d\theta$
14. $\displaystyle\int \frac{8x}{16+x^2}\,dx$
15. $\displaystyle\int \frac{3x}{\sqrt{9+x^2}}\,dx$
16. $\displaystyle\int \frac{3x}{\sqrt{9-x^2}}\,dx$
17. $\displaystyle\int \frac{6x}{9+x^4}\,dx$
18. $\displaystyle\int \frac{6x}{\sqrt{9-x^4}}\,dx$
19. $\displaystyle\int \frac{1}{1+4x^2}\,dx$
20. $\displaystyle\int \frac{1}{\sqrt{1-9x^2}}\,dx$
21. $\displaystyle\int_0^\infty \frac{1}{3+x^2}\,dx$
22. $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$
23. $\displaystyle\int_0^{\sqrt{7}} \frac{1}{\sqrt{7-x^2}}\,dx$
24. $\displaystyle\int_0^\infty \frac{x}{1+x^4}\,dx$

In Problems 25–28, solve the initial value problem.

25. $\dfrac{dy}{dx} = \dfrac{y}{\sqrt{1-x^2}}$, $y(0) = e$

26. $\dfrac{dy}{dx} = \dfrac{1}{y(1+x^2)}$, $y(0) = 4$

27. $\dfrac{dy}{dx} = \dfrac{y^2}{9+x^2}$, $y(1) = 2$

28. $\dfrac{dy}{dx} \cdot \sqrt{16-x^2} = y$, $y(4) = 1$

In Problems 29–32, evaluate the integral by splitting the integrand into two simpler functions.

29. $\displaystyle\int \frac{8x+5}{x^2+9}\,dx$

30. $\displaystyle\int \frac{1-4x}{x^2+1}\,dx$

31. $\displaystyle\int \frac{7x+3}{x^2+10}\,dx$

32. $\displaystyle\int \frac{x+5}{x^2+16}\,dx$

Problems 33–40 illustrate how we can sometimes decompose a difficult integral into simpler ones. (Hints: For 33, complete the square in the denominator; for 34, let u = denominator; for 35, write $4x+20 = (4x+12)+8$.)

33. $\displaystyle\int \frac{8}{x^2+6x+10}\,dx$

34. $\displaystyle\int \frac{4x+12}{x^2+6x+10}\,dx$

35. $\displaystyle\int \frac{4x+20}{x^2+6x+10}\,dx$

36. $\displaystyle\int \frac{7}{x^2+4x+5}\,dx$

37. $\displaystyle\int \frac{12x+24}{x^2+4x+5}\,dx$

38. $\displaystyle\int \frac{12x+31}{x^2+4x+5}\,dx$

39. $\displaystyle\int \frac{6x+15}{x^2+4x+20}\,dx$

40. $\displaystyle\int \frac{2x+5}{x^2-4x+13}\,dx$

7.5 Practice Answers

1. For the first integral, write:

$$\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx$$

and substitute $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$ to get:

$$\int \frac{1}{1+u^2} \cdot \frac{1}{3}\,du = \frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan(u) + C = \frac{1}{3}\arctan(3x) + C$$

For the second integral, factor out 25 to get:

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \int \frac{1}{5\sqrt{1-\frac{x^2}{25}}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}}\,dx$$

and then substitute $u = \dfrac{x}{5} \Rightarrow du = \dfrac{1}{5}\,dx \Rightarrow 5\,du = dx$:

$$\frac{1}{5}\int \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}} \cdot 5\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + K$$

Replacing u with $\dfrac{x}{5}$ yields:

$$\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\left(\frac{x}{5}\right) + K$$

2. Using the Chain Rule:

$$\mathbf{D}\left[\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right] = \frac{1}{a} \cdot \frac{1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1}{a} = \frac{1}{a^2\left[1+\left(\frac{x}{a}\right)^2\right]} = \frac{1}{a^2+x^2}$$

3. Split the integrand into two pieces:

$$\int \frac{4x+3}{x^2+7}\,dx = \int \frac{4x}{x^2+7}\,dx + \int \frac{3}{x^2+7}\,dx$$

For the first integral, let $u = x^2+7 \;\Rightarrow\; du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$:

$$\int \frac{4x}{x^2+7}\,dx = \int \frac{2}{u}\,du = 2\ln\left(|u|\right) + C_1 = 2\ln\left(x^2+7\right) + C_1$$

The second integral matches the arctangent pattern with $a^2 = 7 \Rightarrow a = \sqrt{7}$:

$$\int \frac{3}{x^2+7}\,dx = 3\int \frac{1}{\left(\sqrt{7}\right)^2 + x^2}\,dx = 3 \cdot \frac{1}{\sqrt{7}} \arctan\left(\frac{x}{\sqrt{7}}\right) + C_2$$

Combining these results yields:

$$\int \frac{4x+3}{x^2+7}\,dx = \ln\left(\left[x^2+7\right]^2\right) + \frac{3}{\sqrt{7}} \arctan\left(\frac{x}{\sqrt{7}}\right) + C$$

7.6 Calculus Done Right

This section, currently in preparation, will be included in the official first printing of this edition. It will begin with the function $L(x) = \int_1^x \frac{1}{t}\,dt$ and show that this function possesses all of the properties of the natural logarithm. It will then show $L(x)$ has an inverse function $E(x)$, argue that there is a number e so that $L(e) = 1$ and E(1)= e, show that $E(r) = e^r$ for any rational number r, and define $e^x = E(x)$ for all other values of x.

See `http://contemporarycalculus.com` for a digital version of this section as soon as it becomes available.

7.7 Hyperbolic Functions

This (optional) section, currently in preparation, will be included in the official first printing of this edition. It will present the hyperbolic functions, compute their derivatives and compute antiderivatives of related functions.

See `http://contemporarycalculus.com` for a digital version of this section as soon as it becomes available.

7.8 Inverse Hyperbolic Functions

This (optional) section, currently in preparation, will be included in the official first printing of this edition. It will present the inverse hyperbolic functions, compute their derivatives and compute antiderivatives of related functions.

See `http://contemporarycalculus.com` for a digital version of this section as soon as it becomes available.