8
Integration Techniques

In our journey through integral calculus, we have: developed the concept of a Riemann sum that converges to a definite integral; learned how to use the Fundamental Theorem of Calculus to evaluate a definite integral — as long as we can find an antiderivative for the integrand; examined numerical methods to approximate values of definite integrals; applied the concept of a Riemann sum to a variety of geometric and physical situations to compute lengths, areas, volumes, work and more; and employed integration to solve differential equations.

Finding an approximate value for a definite integral is often "good enough," but exact values are sometimes necessary — and this requires us to find antiderivatives of integrand functions. We have already learned how to find antiderivatives of many basic functions, and repeatedly employed substitution to turn complicated integrands into ones that are easier to integrate. This chapter begins with a review of these integration techniques you already know, then develops several new techniques that will allow you to integrate even more functions. It concludes by presenting a way to find "approximate antiderivatives" that will allow you to compute approximate numerical values for certain definite integrals much more efficiently than the techniques introduced in Section 4.9.

8.1 Finding Antiderivatives: A Review

Success at integration is primarily a matter of recognizing standard patterns and being able to manipulate functions into a form that corresponds to one of these patterns. Integral tables — such as the brief one on the next page and the longer one in Appendix I — list antiderivatives for many basic patterns of functions. Often a change of variable (employing the u-substitution method introduced in Section 4.6) will allow you to see a pattern more easily. For most people, developing the skill of recognizing these patterns comes with practice, and this section provides a variety of problems to review and hone your skills.

This is an extended version of the table listed in Section 4.6.

Constant Functions: $\int k\,du = ku + C$

Powers of u: $\int u^p\,du = \frac{u^{p+1}}{p+1} + C$ if $p \neq -1$, $\int \frac{1}{u}\,du = \ln|u| + C$

Exponential Functions: $\int e^u\,du = e^u + C$, $\int a^u\,du = \frac{a^u}{\ln(a)} + C$

Trig Functions: $\int \cos(u)\,du = \sin(u) + C$, $\int \sin(u)\,du = -\cos(u) + C$

$\int \tan(u)\,du = \ln(|\sec(u)|) + C$, $\int \cot(u)\,du = \ln(|\sin(u)|) + C$

These two patterns may be new to you. See the discussion below.

$\int \sec(u)\,du = \ln(|\sec(u) + \tan(u)|) + C$

$\int \csc(u)\,du = -\ln(|\csc(u) + \cot(u)|) + C$

$\int \sec^2(u)\,du = \tan(u) + C$, $\int \csc^2(u)\,du = -\cot(u) + C$

$\int \sec(u)\cdot\tan(u)\,du = \sec(u) + C$, $\int \csc(u)\cdot\cot(u)\,du = -\csc(u) + C$

Inverse-Trig–Related Functions: $\int \frac{1}{1+u^2}\,du = \arctan(u) + C$

$\int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u) + C$, $\int \frac{1}{|u|\cdot\sqrt{u^2-1}}\,du = \text{arcsec}(u) + C$

The most generally useful and powerful integration technique remains Changing the Variable. The first Problems in this section provide additional practice changing variables to calculate integrals. As we develop more complicated and more specialized techniques for finding antiderivatives, your first thought should still be whether the integral can be simplified by changing the variable. Sometimes the appropriate change of variable is not obvious, and we may need to manipulate the integrand using algebra, trigonometric identities or some clever "tricks" before employing a u-substitution.

Antiderivatives of $\sec(\theta)$ *and* $\csc(\theta)$

In the list of basic antiderivatives at the top of this page, you may have noticed two unfamiliar patterns: those for $\int \sec(\theta)\,d\theta$ and $\int \csc(\theta)\,d\theta$. Antiderivatves for $\cos(\theta)$ and $\sin(\theta)$ essentially came "free" from the derivative patterns we discovered in Chapter 2. Antiderivatives for $\tan(\theta)$ and $\cot(\theta)$ were among the first applications of u-substitution: for example, we can write $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and put $u = \cos(\theta)$ so that $du = -\sin(\theta)\,d\theta$ and the integral in question becomes:

$$\int \tan(\theta)\,d\theta = \int \frac{\sin(\theta)}{\cos(\theta)}\,d\theta = \int \frac{-1}{u}\,du = -\ln(|u|) + C$$
$$= -\ln(|\cos(\theta)|) + C = \ln(|\sec(\theta)|) + C$$

Finding an antiderivative of $\sec(\theta)$, however, requires a special "trick."

Before attempting a substitution, write:

$$\sec(\theta) = \frac{\sec(\theta)}{1} \cdot \frac{(\sec(\theta) + \tan(\theta))}{(\sec(\theta) + \tan(\theta))} = \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}$$

This is an example of "multiplying by 1," a tactic often employed in mathematics where we multiply the top and bottom of a fraction by the same expression.

Why would we want to take the nice, simple function $\sec(\theta)$ and rewrite it as this monstrosity? Look at the denominator and notice that the derivative of $\sec(\theta)$ is $\sec(\theta) \cdot \tan(\theta)$, while the derivative of $\tan(\theta)$ is $\sec^2(\theta)$. Both of these derivatives appear in the numerator. So if we use the substitution:

$$u = \sec(\theta) + \tan(\theta) \;\Rightarrow\; du = \left[\sec(\theta)\tan(\theta) + \sec^2(\theta)\right] d\theta$$

the integral of $\sec(\theta)$ becomes:

$$\begin{aligned}\int \sec(\theta)\, d\theta &= \int \frac{\sec^2(\theta) + \sec(\theta)\tan(\theta)}{\sec(\theta) + \tan(\theta)}\, d\theta \\ &= \int \frac{1}{u}\, du = \ln(|u|) + C = \ln\left(|\sec(\theta) + \tan(\theta)|\right) + C\end{aligned}$$

proving the result listed on the previous page.

Practice 1. Prove that $\int \csc(\theta)\, d\theta = -\ln\left(|\csc(\theta) + \cot(\theta)|\right) + C$.

The trick used to integrate $\sec(\theta)$ and $\csc(\theta)$ only applies in these special situations, so rather than remembering the trick, you might want to simply memorize the result if you find yourself needing to integrate $\sec(\theta)$ on a regular basis.

See Problems 41–42 in Section 8.3 (and the related discussion on page 575) for a more intuitive—albeit more tedious—method to obtain antiderivatives for $\sec(\theta)$ and $\csc(\theta)$.

An Irreducible Quadratic Denominator

The following examples review and extend techniques (introduced in Section 7.5) involving variations on the arctangent derivative pattern.

Example 1. Evaluate $\int \frac{18}{1 + (x-3)^2}\, dx$ and $\int \frac{18}{x^2 - 6x + 10}\, dx$.

Solution. The form of the first integrand reminds us of the derivative of the arctangent function:

$$\mathbf{D}\left(\arctan(u)\right) = \frac{1}{1 + u^2}$$

If we make the substitution $u = x - 3 \Rightarrow du = dx$ the integral becomes:

$$\int \frac{18}{1 + (x-3)^2}\, dx = 18 \int \frac{1}{1 + u^2}\, du = 18\arctan(u) + C$$

Replacing u with $x - 3$, we get $18\arctan(x-3) + C$. The second integrand appears much more complicated, until you notice that:

$$1 + (x-3)^2 = 1 + x^2 - 6x + 9 = x^2 - 6x + 10$$

These integrands are in fact equal, so the second integral also equals $18\arctan(x-3) + C$. ◀

In the preceding Example, we showed that the two integrands were equal by expanding the denominator of the first integrand to get the denominator of the second integrand. If we had started with the second integral in Example 1, we could have rewritten the second denominator employing the method of **completing the square**:

Recall from algebra that to complete the square with a polynomial, you need to take half of the x coefficient:

$$\frac{-6}{2} = -3 \quad \text{or} \quad \frac{1}{2} \cdot \frac{b}{a}$$

and then square it:

$$(-3)^2 = 9 \quad \text{or} \quad \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

to get the number that must be added and subtracted to create a perfect square (plus a leftover constant term). A quadratic polynomial is **irreducible** if it can't be factored into two linear terms using real coefficients; for $ax^2 + bx + c$ this occurs when $b^2 - 4ac < 0$.

$$x^2 - 6x + 10 = \left(x^2 - 6x + 9\right) + (10 - 9) = (x-3)^2 + 1$$

so that we would make the substitution $u = x - 3$. For any irreducible quadratic polynomial of the form $ax^2 + bx + c$, we can write:

$$\begin{aligned} ax^2 + bx + c &= a\left[x^2 + \frac{b}{a}\right] + c = a\left[x^2 + \frac{b}{a} + \frac{b^2}{4a^2}\right] + c - a \cdot \frac{b^2}{4a^2} \\ &= a\left(x - \frac{b}{2a}\right)^2 + \left[c - \frac{b^2}{4a}\right] \end{aligned}$$

Rather than memorizing this formula, you should remember the *process* of completing the square.

Practice 2. Evaluate $\displaystyle\int \frac{5}{x^2 + 8x + 25}\,dx$ by completing the square and making a substitution.

Example 2. Evaluate $\displaystyle\int \frac{2x}{x^2 - 6x + 10}\,dx$.

Solution. This would be an easy problem if the numerator were $2x - 6$: the numerator would then be the derivative of the denominator and the pattern of the integral would be $\int \frac{1}{u}\,du$ with $u = x^2 - 6x + 10$. Using a bit of cleverness, we can rewrite the numerator as $2x - 6 + 6$. Then the integral becomes:

This is an example of "adding 0," another tactic often employed in mathematics in which we add and subtract the same quantity from an expression.

$$\int \frac{2x - 6 + 6}{x^2 - 6x + 10}\,dx = \int \frac{2x-6}{x^2 - 6x + 10}\,dx + \int \frac{6}{x^2 - 6x + 10}\,dx$$

For the first integral, substitute $u = x^2 - 6x + 10 \Rightarrow du = (2x - 6)\,dx$:

Why can we omit the absolute values in the last step?

$$\int \frac{2x-6}{x^2 - 6x + 10}\,dx = \int \frac{1}{u}\,du = \ln(|u|) + C_1 = \ln\left(x^2 - 6x + 10\right) + C_1$$

For the second integral, complete the square in the denominator:

$$x^2 - 6x + 10 = x^2 - 6x + 9 + 1 = (x-3)^2 + 1$$

and use the substitution $w = x - 3 \Rightarrow dw = dx$ to get:

$$\begin{aligned} \int \frac{6}{x^2 - 6x + 10}\,dx &= \int \frac{6}{(x-3)^2 + 1}\,dx = 6\int \frac{1}{w^2+1}\,dw \\ &= 6\arctan(w) + C_2 = 6\arctan(x-3) + C_2 \end{aligned}$$

so that the final answer is $\ln\left(x^2 - 6x + 10\right) + 6\arctan(x-3) + C$. ◀

This "logarithm plus an arctangent" pattern that arose in the previous Example turns up quite often with integrals of linear functions divided by irreducible quadratic polynomials. If the quadratic denominator can be factored into a product of two linear factors, we will instead use a technique discussed in Section 8.3 (Partial Fraction Decomposition).

Practice 3. Evaluate $\int \frac{4x+21}{x^2+8x+25}\,dx$.

8.1 Problems

In Problems 1–54, evaluate the integral. A well-chosen substitution will often turn a complicated-looking integral into a much simpler one.

1. $\int 6x\left(x^2+7\right)^2\,dx$
2. $\int 6x\left(x^2-1\right)^3\,dx$
3. $\int_2^4 \frac{6t}{\sqrt{t^2-3}}\,dt$
4. $\int_0^{\pi} 12\cos(\theta)\,[2+\sin(\theta)]^2\,d\theta$
5. $\int \frac{12x}{x^2+3}\,dx$
6. $\int \frac{\cos(\varphi)}{2+\sin(\varphi)}\,d\varphi$
7. $\int \sin(3y+2)\,dy$
8. $\int \cos\left(\frac{x}{5}\right)\,dx$
9. $\int_{-1}^{0} e^x\cdot\sec^2\left(e^x+3\right)\,dx$
10. $\int_0^{\frac{\pi}{2}} \cos(\theta)\sqrt{1+\sin(\theta)}\,d\theta$
11. $\int \frac{\ln(x)}{x}\,dx$
12. $\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$
13. $\int \cos(\theta)\cdot e^{\sin(\theta)}\,d\theta$
14. $\int e^z\sin(e^z)\,dz$
15. $\int_1^3 \frac{5}{1+9x^2}\,dx$
16. $\int_0^1 \frac{7}{1+(x+5)^2}\,dx$
17. $\int_1^2 \frac{1}{x^2}\cdot\cos\left(\frac{1}{x}\right)\,dx$
18. $\int_1^e \frac{\sec\left(2+\ln(x)\right)}{x}\,dx$
19. $\int \frac{6\sin(\theta)\cos(\theta)}{5+\sin^2(\theta)}\,d\theta$
20. $\int \frac{6\cos(\alpha)}{5+\sin^2(\alpha)}\,d\alpha$
21. $\int \frac{10}{2x+5}\,dx$
22. $\int \frac{3}{8y+1}\,dy$
23. $\int_1^3 \frac{20x}{5x^2+3}\,dx$
24. $\int_1^5 \frac{4x}{x^2+9}\,dx$
25. $\int_0^1 \frac{7}{(x+3)^2+4}\,dx$
26. $\int_{-2.3}^{-2} \frac{1}{\sqrt{1-(x+2)^2}}\,dx$
27. $\int \frac{e^t}{1+e^{2t}}\,dt$
28. $\int \frac{4x+10}{x^2+5x+9}\,dx$
29. $\int_1^3 \frac{3}{x\left[1+\ln(x)\right]}\,dx$
30. $\int_0^1 \frac{e^t}{1+e^t}\,dt$
31. $\int_0^1 2x\sqrt{1-x^2}\,dx$
32. $\int_0^3 \frac{2x}{\sqrt{5+x^2}}\,dx$
33. $\int \cos(\theta)\,[1+\sin(\theta)]^3\,d\theta$

34. $\int \cos(\varphi) \sin^4(\varphi)\, d\varphi$

35. $\int_1^e \frac{\sqrt{\ln(x)}}{x}\, dx$

36. $\int_1^2 e^x \sqrt{2 + e^x}\, dx$

37. $\int \frac{\sec^2(\theta)}{5 + \tan(\theta)}\, d\theta$

38. $\int \frac{6x}{(x^2 - 1)^3}\, dx$

39. $\int \tan(y - 5)\, dy$

40. $\int \left(x^3 + 3\right)^2\, dx$

41. $\int_0^1 e^{5u}\, du$

42. $\int \sec(2 + 3t)\, dt$

43. $\int t \cdot \sec(2 + 3t^2)\, dt$

44. $\int_1^1 \arctan\left(\sqrt{5 - x^3}\right)\, dx$

45. $\int_e^e \ln\left(\sqrt{5 + x^3}\right)\, dx$

46. $\int_1^\infty \frac{1}{1 + 9x^2}\, dx$

47. $\int_1^\infty \frac{x}{1 + 9x^4}\, dx$

48. $\int_1^\infty \frac{e^{-x}}{1 + e^{-2x}}\, dx$

In 49–54, complete the square in the denominator, make an appropriate substitution, then integrate.

49. $\int \frac{7}{x^2 + 4x + 5}\, dx$

50. $\int \frac{3}{x^2 + 4x + 29}\, dx$

51. $\int \frac{2}{x^2 - 6x + 58}\, dx$

52. $\int \frac{11}{x^2 - 2x + 10}\, dx$

53. $\int \frac{3}{x^2 + 10x + 29}\, dx$

54. $\int \frac{5}{x^2 + 2x + 5}\, dx$

In 55–60, first split the integral into two integrals. (Hint: In Problem 55, $2x + 11 = (2x + 4) + 7$.)

55. $\int \frac{2x + 11}{x^2 + 4x + 5}\, dx$

56. $\int \frac{4x + 11}{x^2 + 4x + 5}\, dx$

57. $\int \frac{4x + 7}{x^2 - 6x + 10}\, dx$

58. $\int \frac{6x + 28}{x^2 + 10x + 34}\, dx$

59. $\int \frac{6x + 5}{x^2 - 4x + 13}\, dx$

60. $\int \frac{4x + 9}{x^2 + 6x + 13}\, dx$

In 61–66, remember that completing the square only helps with irreducible quadratic denominators.

61. $\int \frac{1}{x^2 + 4x + 4}\, dx$

62. $\int \frac{x + 2}{x^2 + 4x + 4}\, dx$

63. $\int \frac{x + 3}{x^2 - 6x + 9}\, dx$

64. $\int_4^\infty \frac{1}{x^2 - 6x + 9}\, dx$

65. $\int_3^\infty \frac{1}{x^2 - 6x + 9}\, dx$

66. $\int_4^\infty \frac{8x - 24}{x^2 - 6x + 9}\, dx$

8.1 Practice Answers

1. First use the "multiply by 1" trick to write:

$$\csc(\theta) = \frac{\csc(\theta)}{1} \cdot \frac{\csc(\theta) + \cot(\theta)}{\csc(\theta) + \cot(\theta)} = \frac{\csc^2(\theta) + \csc(\theta)\cot(\theta)}{\csc(\theta) + \cot(\theta)}$$

and then make the substitution $u = \csc(\theta) + \cot(\theta)$ so that $du = \left(-\csc(\theta)\cot(\theta) - \csc^2(\theta)\right)\, d\theta$ and:

$$\begin{aligned}\int \csc(\theta)\, d\theta &= \int \frac{\csc^2(\theta) + \csc(\theta)\cot(\theta)}{\csc(\theta) + \cot(\theta)}\, d\theta = \int \frac{-1}{u}\, du \\ &= -\ln(|u|) + C = -\ln(|\csc(\theta) + \cot(\theta)|) + C\end{aligned}$$

2. First complete the square in the denominator:

$$x^2 + 8x + 25 = \left(x^2 + 8x + 16\right) + (25 - 16) = (x+4)^2 + 9$$

so the integral becomes:

$$\int \frac{5}{x^2 + 8x + 25}\, dx = \int \frac{5}{(x+4)^2 + 9}\, dx$$

Now make the substitution $u = x + 4 \Rightarrow du = dx$ to get:

$$5 \int \frac{1}{u^2 + 3^2}\, du = 5 \cdot \frac{1}{3} \arctan\left(\frac{u}{3}\right) + C = \frac{5}{3} \arctan\left(\frac{x+4}{3}\right) + C$$

3. If we substitute $u = x^2 + 8x + 25$ then $du = (2x + 8)\, dx$. We would be in good shape if the numerator of the integrand were $4x + 16 = 2(2x + 8)$, so split the numerator into $4x + 21 = (4x + 16) + 5$ to get:

$$\int \frac{4x + 21}{x^2 + 8x + 25}\, dx = \int \frac{(4x+16) + 5}{x^2 + 8x + 25}\, dx = \int \frac{4x + 16}{x^2 + 8x + 25}\, dx + \int \frac{5}{x^2 + 8x + 25}\, dx$$

The first integral (with $u = x^2 + 8x + 25$) now becomes:

$$\int \frac{4x + 16}{x^2 + 8x + 25}\, dx = \int \frac{2}{u}\, du = 2\ln(|u|) + C_1 = 2\ln\left(x^2 + 8x + 25\right) + C_1$$

The second integral is just the integral from Practice 2:

$$\int \frac{5}{x^2 + 8x + 25}\, dx = \frac{5}{3} \arctan\left(\frac{x+4}{3}\right) + C_2$$

Combining these results yields:

$$\int \frac{4x + 21}{x^2 + 8x + 25}\, dx = \ln\left(x^2 + 8x + 25\right)^2 + \frac{5}{3} \arctan\left(\frac{x+4}{3}\right) + C$$

8.2 *Integration by Parts*

Integration by parts is an integration method that enables us to find antiderivatives of certain functions for which our previous antidifferentiation methods fail, such as $\ln(x)$ and $\arctan(x)$, as well as antiderivatives of certain products of functions, such as $x^2 \ln(x)$ and $e^x \sin(x)$. It leads to many of the general integral formulas in Appendix I and (next to u-substitution) it is one most powerful and frequently used of antidifferentiation techniques.

The Integration by Parts formula for integrals arises from the Product Rule for derivatives. Recall that for functions $u = u(x)$ and $v = v(x)$, the Product Rule says:

$$\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

If we integrate both sides of this equation with respect to x we get:

$$u \cdot v = \int u \cdot \frac{dv}{dx}\, dx + \int v \cdot \frac{du}{dx}\, dx$$

Solving for the first of the two integrals on the right side, yields:

$$\int u \cdot \frac{dv}{dx}\, dx = u \cdot v - \int v \cdot \frac{du}{dx}\, dx$$

At first, this formula may not appear very promising, as it merely exchanges one integration problem for another. But in certain situations, if we choose $u(x)$ and $v(x)$ carefully, this formula exchanges a difficult integral for an easier one. We can restate the formula in slightly more compact form using differentials.

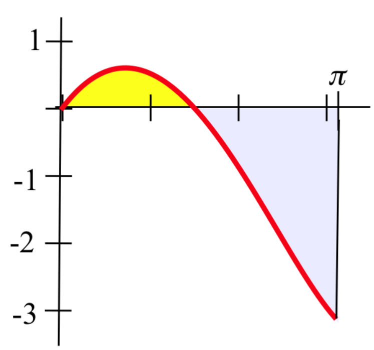
Integration by Parts Formula

If u, v, u' and v' are continuous functions,
then $\displaystyle\int u\, dv = u \cdot v - \int v\, du$

For definite integrals, the Integration By Parts Formula says:

Integration by Parts Formula (Definite Integrals)

If u, v, u' and v' are continuous functions,
then $\displaystyle\int_a^b u\, dv = \Big[u \cdot v\Big]_a^b - \int_a^b v\, du$



Example 1. Use integration by parts to evaluate $\int x \cos(x)\, dx$ and $\int_0^{\pi} x \cos(x)\, dx$ (see margin for a graphical interpretation).

Solution. Our first step is to write this integral in the form required by the Integration by Parts formula, $\int u\,dv$. If we let $u = x$, then we must have $dv = \cos(x)\,dx$ so that $u\,dv$ completely represents the integrand $x\cos(x)$. We also need to calculate du and v:

$$u = x \;\Rightarrow\; du = dx \qquad \text{and} \qquad dv = \cos(x)\,dx \;\Rightarrow\; v = \sin(x)$$

Putting these pieces into the Integration by Parts formula, we have:

$$\int x \cdot \cos(x)\,dx \;=\; x \cdot \sin(x) - \int \sin(x)\,dx \;=\; x \cdot \sin(x) + \cos(x) + C$$

To check this result, differentiate the answer to verify that:

$$[x\sin(x) + \cos(x)]' = x\cos(x)$$

Now use this result to evaluate the definite integral:

$$\int_0^{\pi} x \cdot \cos(x)\,dx = \Big[x \cdot \sin(x) + \cos(x)\Big]_0^{\pi}$$
$$= [\pi\sin(\pi) + \cos(\pi)] - [0 \cdot \sin(0) + \cos(0)] = -1 - 1$$

or -2, which appears reasonable based on the area interpretation of this integral in the margin graph on the previous page. ◀

Integration by parts allowed us to exchange the problem of evaluating $\displaystyle\int x\cos(x)\,dx$ for the much easier problem of evaluating $\displaystyle\int \sin(x)\,dx$.

Practice 1. Use the Integration by Parts formula on $\displaystyle\int x\cos(x)\,dx$ with $u = \cos(x)$ and $dv = x\,dx$. Why does this lead to a poor exchange?

Example 2. Evaluate $\displaystyle\int xe^{3x}\,dx$ and $\displaystyle\int_0^1 xe^{3x}\,dx$.

Solution. The integrand is a product of two functions, so it is reasonable to use integration by parts to search for an antiderivative. If $u = x \;\Rightarrow\; du = dx$, then $dv = e^{3x}\,dx \;\Rightarrow\; v = \frac{1}{3}e^{3x}$. Inserting these expressions into the Integration by Parts formula, we get:

$$\int xe^{3x}\,dx = x \cdot \frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}\,dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$$
$$\Rightarrow \int_0^1 xe^{3x}\,dx = \left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_0^1 = \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}\right] = \frac{2}{9}e^3 + \frac{1}{9}$$

or about 4.57. ◀

In practice, when you need to use integration by parts to evaluate a definite integral, it is often safest to first evaluate the corresponding indefinite integral and then use that antiderivative pattern to evaluate the definite integral, as we have done here.

In the previous Example another valid choice would have been $u = e^{3x}$ and $dv = x\,dx$, but that choice results in an integral that is more difficult than the original one: $du = 3e^{3x}\,dx$ and $v = \frac{1}{2}x^2$, so the Integration by Parts formula yields:

$$\int xe^{3x}\,dx = e^{3x} \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot 3e^{3x}\,dx$$

which exchanges $\int xe^{3x}\,dx$ for the more difficult integral $\int \frac{3}{2}x^2e^{3x}\,dx$.

Practice 2. Evaluate $\int x\sin(x)\,dx$ and $\int xe^{5x}\,dx$.

Once you have chosen u and dv to represent the integrand as $u\,dv$, you need to calculate du and v. The du calculation is usually easy, but finding v from dv can be difficult for some choices of dv. In practice, you need to select u so that the remaining dv is simple enough that you can find v, the antiderivative of dv.

Example 3. Evaluate $\int 2x\ln(x)\,dx$.

Solution. The choice $u = 2x$ seems fine until we get a little further into the process. If $u = 2x$, then $dv = \ln(x)\,dx$. We now need to find du and v. Computing $du = 2\,dx$ is simple, but then we face the difficult problem of finding an antiderivative v for our choice $dv = \ln(x)\,dx$.

If you cannot find a v for your original choice of dv, try a different u and dv.

The choice $u = \ln(x)$ results in easier calculations. Let $u = \ln(x)$. Then $dv = 2x\,dx$, so $du = \frac{1}{x}\,dx$ and $v = x^2$. Then the Integration by Parts formula gives:

$$\int 2x\ln(x)\,dx = \ln(x)\cdot x^2 - \int x^2 \cdot \frac{1}{x}\,dx = x^2\ln(x) - \int x\,dx$$

so the final result is $x^2\ln(x) - \dfrac{1}{2}x^2 + C$. ◀

Antiderivatives of Inverse Functions

So far we have applied the Integration by Parts method to products of simple functions, but it also enables us—perhaps surprisingly—to find antiderivatives of the inverse trigonometric functions and of the logarithm (the inverse exponential function).

Example 4. Evaluate $\int \arctan(x)\,dx$.

Solution. Let $u = \arctan(x)$. Then $dv = dx$, so $du = \dfrac{1}{1+x^2}\,dx$ and $v = x$. Putting these expressions into the Integration by Parts formula:

$$\int \arctan(x)\,dx = x\arctan(x) - \int x \cdot \frac{1}{1+x^2}\,dx$$

We can evaluate the new integral using the substitution $w = 1 + x^2 \Rightarrow dw = 2x\,dx \Rightarrow \frac{1}{2}\,dw = x\,dx$:

Why are absolute value signs not needed in the last term?

$$\int x \cdot \frac{1}{1+x^2}\,dx = \int \frac{1}{2}\cdot\frac{1}{w}\,dw = \frac{1}{2}\ln(|w|) + K \;=\; \frac{1}{2}\ln\left(1+x^2\right) + K$$

Combining these results:

We can now include the antiderivative of $\arctan(x)$ in our list of antiderivatives in Appendix I.

$$\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln\left(1+x^2\right) + C$$

or $x\arctan(x) - \ln\left(\sqrt{1+x^2}\right) + C$. ◀

Practice 3. Evaluate $\int \ln(x)\,dx$ and $\int_1^e \ln(x)\,dx$.

Note the following about the Integration by Parts formula:

- Once you choose u, then dv is completely determined.

dv = rest of the integrand

- Because you need to find an antiderivative of dv to get v, pick u and dv with this in mind.
- Integration by parts allows you to trade one integral for another. If the new integral is more difficult than the original integral, then you have made a poor choice of u and dv. Try a different choice for u and dv (or try a different technique).
- To evaluate the new integral $\int v\,du$ you may need to use substitution, integration by parts again, or some other technique (such as the ones discussed later in this chapter).

General Patterns

Sometimes a single application of integration by parts yields a formula that allows us to integrate an entire family of functions.

Example 5. Evaluate $\int x^p \ln(x)\,dx$ for any number $p \neq -1$.

Solution. Set $u = \ln(x)$ so $dv = x^p\,dx$, $du = \frac{1}{x}\,dx$ and $v = \frac{1}{p+1}x^{p+1}$. Putting all of this into the Integration by Parts formula:

$$\int x^p \ln(x)\,dx = \frac{1}{p+1}x^{p+1} \cdot \ln(x) - \int \frac{1}{p+1}x^{p+1} \cdot \frac{1}{x}\,dx$$

This new integral becomes:

$$\frac{1}{p+1}\int x^p\,dx = \frac{1}{p+1} \cdot \frac{1}{p+1}x^{p+1} + K = \frac{x^{p+1}}{(p+1)^2} + K$$

Combining these results yields:

$$\frac{x^{p+1}}{p+1}\ln(x) - \frac{x^{p+1}}{(p+1)^2} + C = \frac{x^{p+1}}{p+1}\left[\ln(x) - \frac{1}{p+1}\right] + C$$

for any number p as long as $p \neq -1$. ◀

What happens when $p = -1$? Can you evaluate that integral?

Practice 4. Use Example 5 to evaluate $\int x^2 \ln(x)\,dx$ and $\int \ln(x)\,dx$.

Reduction Formulas

Sometimes the result of an integration-by-parts procedure still contains an integral, but a simpler one with a smaller exponent. In these situations we can reuse the resulting **reduction formula** until the remaining integral is simple enough to integrate completely.

Example 6. Evaluate $\int x^n e^x\,dx$ and use the result to evaluate $\int x^2 e^x\,dx$.

Solution. Set $u = x^n$ so $dv = e^x\,dx$, $du = nx^{n-1}\,dx$ and $v = e^x$. The Integration by Parts formula gives

$$\int x^n e^x\,dx \;=\; x^n e^x - n\int x^{n-1}e^x\,dx$$

which is a reduction formula because we have reduced the power of x by 1, trading $\int x^n e^x\,dx$ for the "reduced" integral $\int x^{n-1}e^x\,dx$.

Because $\int x^2 e^x\,dx$ matches the general pattern of $\int x^n e^x\,dx$ with $n = 2$, we know that:

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x^1 e^x\,dx$$

The new integral also matches the pattern in the reduction formula (with $n = 1$ this time) so we know that:

$$\int x^1 e^x\,dx = x^1 e^x - 1\cdot\int x^0 e^x\,dx$$

This last integral is just $\int e^x\,dx = e^x + K$, so combining our results:

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\int x^1 e^x\,dx = x^2 e^x - 2\left[xe^x - \int e^x\,dx\right]\\ &= x^2 e^x - 2xe^x + 2e^x + C\end{aligned}$$

We can also write the answer as $e^x\left[x^2 - 2x + 2\right] + C$. ◀

Practice 5. Develop the reduction formula:

$$\int x^n \sin(x)\,dx \;=\; -x^n\cos(x) + n\int x^{n-1}\cos(x)\,dx$$

using integration by parts.

The Reappearing Integral

Sometimes the integral we are trying to evaluate shows up on both sides of the equation during our calculations in such a way that we can solve for the desired integral algebraically.

Example 7. Evaluate $\int e^x\cos(x)\,dx$.

Solution. Let $u = e^x$, so $dv = \cos(x)\,dx$, $du = e^x\,dx$ and $v = \sin(x)$:

$$\int e^x\cos(x)\,dx \;=\; e^x\sin(x) - \int e^x\sin(x)\,dx$$

The new integral does not look any easier than the original one, but it doesn't look any worse, so let's try to evaluate the new integral using integration by parts again. To evaluate $\int e^x \sin(x)\,dx$, let $u = e^x$ and $dv = \sin(x)\,dx$ so that $du = e^x\,dx$ and $v = -\cos(x)$, giving us:

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx$$

Putting this result back into the original problem, we get:

$$\begin{aligned}\int e^x \cos(x)\,dx &= e^x \sin(x) - \int e^x \sin(x)\,dx \\ &= e^x \sin(x) - \left[-e^x \cos(x) + \int e^x \cos(x)\,dx\right] \\ &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx\end{aligned}$$

Note that $\int e^x \cos(x)\,dx$ appears on both sides of this last equation, so we can solve for that expression algebraically:

$$2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) + K$$

We need an arbitrary constant on the right side of this equation because the left side is an indefinite integral.

and, finally, divide both sides by 2 to get:

$$\int e^x \cos(x)\,dx = \frac{1}{2}\left[e^x \sin(x) + e^x \cos(x)\right] + C$$

$C = \frac{K}{2}$

which we can also write as $\frac{1}{2}e^x\left[\sin(x) + \cos(x)\right] + C$. ◀

Practice 6. Evaluate $\int e^x \sin(x)\,dx$.

A Useful Shortcut

Repeated application of integration by parts, as in Example 6, can quickly become tedious, even with the aid of a reduction formula. For integrals of the form $\int x^n \cdot f(x)\,dx$ where n is an integer, it is often possible to arrange the integration-by-parts ingredients in a table to allow much speedier computation.

Example 8. Evaluate $\int x^2 e^{3x}\,dx$.

Solution. Make a table (see margin) with two columns. In the second entry of the left column, start with $u = x^2$ and below it list the successive derivatives of x^2: $2x$, 2 and 0 (you can stop when you get to 0). In the right column start with $v' = e^{3x}$ and below it list successive antiderivatives of e^{3x}: $\frac{1}{3}e^{3x}$, $\frac{1}{9}e^{3x}$ and $\frac{1}{27}e^{3x}$. Now multiply the functions in each row and add the results, alternating signs:

		e^{3x}	$= v'$
$u =$	x^2	$\frac{1}{3}e^{3x}$	$(+)$
	$2x$	$\frac{1}{9}e^{3x}$	$(-)$
	2	$\frac{1}{27}e^{3x}$	$(+)$
	0	$\frac{1}{81}e^{3x}$	$(-)$

$$x^2 \cdot \frac{1}{3}e^{3x} - 2x \cdot \frac{1}{9}e^{3x} + 2 \cdot \frac{1}{27}e^{3x} - 0 \cdot \frac{1}{81}e^{3x}$$

We can stop here, because all of the remaining terms will be 0. Now add a constant and you have the result: $\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$. ◀

Apply this new technique to the second integral in Example 6 to verify that you get the same result with the new method. Which method is faster?

This shortcut method only works when the chosen u in the initial integration-by-parts setup is x^n (so that taking several derivatives results in 0) and when the chosen dv is a function like e^{ax} or $\sin(bx)$ so that repeated antidifferentiation is fairly easy. But when this shortcut does apply, it can save you a great deal of time.

Practice 7. Evaluate $\int x^5 \cos(x)\,dx$ and $\int x^3 e^{-2x}\,dx$.

8.2 Problems

Problems 1–6 list one part (u or dv) needed for integration by parts. Find the other part (dv or u), calculate du and v, and apply the Integration by Parts formula to evaluate the integral.

1. $\int 12x \ln(x)\,dx, \quad u = \ln(x)$
2. $\int xe^{-x}\,dx, \quad u = x$
3. $\int x^4 \ln(x)\,dx, \quad dv = x^4\,dx$
4. $\int x \sec^2(3x)\,dx, \quad u = x$
5. $\int x \arctan(x)\,dx, \quad dv = x\,dx$
6. $\int x(5x+1)^{19}\,dx, \quad u = x$

In Problems 7–24, evaluate the integral.

7. $\int_0^1 \frac{x}{e^{3x}}\,dx$
8. $\int_0^1 10xe^{3x}\,dx$
9. $\int x\sec(x)\tan(x)\,dx$
10. $\int_0^{\pi} 5x\sin(2x)\,dx$
11. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 7x\cos(3x)\,dx$
12. $\int 6x\sin(x^2+1)\,dx$
13. $\int 12x\cos(3x^2)\,dx$
14. $\int x^2\cos(x)\,dx$
15. $\int_1^3 \ln(2x+5)\,dx$
16. $\int x^3\ln(5x)\,dx$
17. $\int_1^e (\ln(x))^2\,dx$
18. $\int_1^e \sqrt{x}\ln(x)\,dx$
19. $\int \arcsin(x)\,dx$
20. $\int x^2e^{5x}\,dx$
21. $\int x\arctan(3x)\,dx$
22. $\int x\ln(x+1)\,dx$
23. $\int_1^2 \frac{\ln(x)}{x}\,dx$
24. $\int_1^2 \frac{\ln(x)}{x^2}\,dx$

25. Write $\sin^n(x) = \sin^{n-1}(x)\cdot\sin(x)$ and use integration by parts to obtain the following reduction formula for $\int \sin^n(x)\,dx$:
$$\frac{1}{n}\left[-\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx\right]$$
26. Write $\cos^n(x) = \cos^{n-1}(x)\cdot\cos(x)$ and use integration by parts to obtain the following reduction formula for $\int \cos^n(x)\,dx$:
$$\frac{1}{n}\left[\cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx\right]$$
27. Use integration by parts to obtain a reduction formula for $\int \sec^n(x)\,dx$.
28. Use integration by parts to obtain a reduction formula for $\int \tan^n(x)\,dx$.

In Problems 29–40, use a result from Problems 25–28 to evaluate the integral.

29. $\int \sin^3(x)\,dx$
30. $\int \sin^4(x)\,dx$
31. $\int \sin^5(x)\,dx$
32. $\int \cos^3(x)\,dx$
33. $\int \cos^4(x)\,dx$
34. $\int \cos^5(x)\,dx$
35. $\int \sec^3(x)\,dx$
36. $\int \sec^4(x)\,dx$
37. $\int \sec^5(x)\,dx$
38. $\int \sin^3(5x-2)\,dx$
39. $\int \cos^3(2x+3)\,dx$
40. $\int \sec^3(7x-1)\,dx$

In Problems 41–44, obtain a reduction formula using integration by parts.

41. $\int x^n e^{ax}\,dx$

42. $\int x^n \sin(ax)\,dx$

43. $\int x\,(\ln(x))^n\,dx$

44. $\int x^n \cos(ax)\,dx$

45. The integral $\int x(2x+5)^{19}\,dx$ can be evaluated with integration by parts or by substitution.
  (a) Evaluate the integral using integration by parts with $u = x$ and $dv = (2x+5)^{19}\,dx$.
  (b) Evaluate the integral using a change of variable with $w = 2x+5$.
  (c) Which method is easier?

46. The integral $\int \frac{x}{\sqrt{1+x}}\,dx$ can be evaluated with integration by parts or by substitution.
  (a) Evaluate the integral using integration by parts with $u = x$ and $dv = \frac{1}{\sqrt{1+x}}\,dx$.
  (b) Evaluate the integral using a change of variable with $w = 1+x$.
  (c) Which method is easier?

In Problems 47–68, evaluate the integral using any appropriate method.

47. $\int x\,(\ln(x))^2\,dx$

48. $\int x^2 \arctan(x)\,dx$

49. $\int_0^1 e^{-x}\sin(x)\,dx$

50. $\int_0^1 \frac{\cos(x)}{e^x}\,dx$

51. $\int \sin(\ln(x))\,dx$

52. $\int \cos(\ln(x))\,dx$

53. $\int \cos\left(\sqrt{x}\right)\,dx$

54. $\int \sin\left(\sqrt{x}\right)\,dx$

55. $\int e^{3x}\sin(x)\,dx$

56. $\int e^x\cos(3x)\,dx$

57. $\int_0^\infty xe^{-x}\,dx$

58. $\int_0^\infty x^2e^{-3x}\,dx$

59. $\int_0^\infty e^{-x}\sin(x)\,dx$

60. $\int_0^\infty e^{-2x}\cos(3x)\,dx$

61. $\int x\sqrt{x+1}\,dx$

62. $\int x\sqrt{x^2+1}\,dx$

63. $\int x\cos(x^2)\,dx$

64. $\int x^2\sqrt{x^3+1}\,dx$

65. $\int x^2\cos(x)\,dx$

66. $\int x^3\sqrt{x^2+1}\,dx$

67. $\int x^3\sqrt[3]{x^2+1}\,dx$

68. $\int x^2\sin(x)\,dx$

In Problems 69–72, solve the initial value problem.

69. $y' = x\sin(x),\ y(0) = 0$

70. $y' = xe^{7x},\ y(0) = 1$

71. $y' = \frac{x}{e^{x+y}},\ y(0) = 1$

72. $y' = x\sin(x)\cos^2(y),\ y(0) = \frac{\pi}{4}$

73. Consider $\int_0^1 x\sin(x)\,dx$ and $\int_0^1 \sin(x)\,dx$.
  (a) Before evaluating the integrals, which do you think is larger? Why?
  (b) Evaluate both integrals. Was your prediction in part (a) correct?

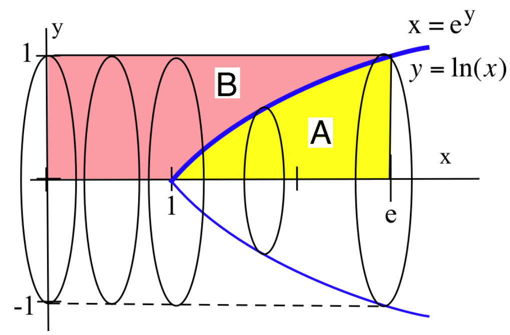
74. Consider $\int_0^\pi x\sin(x)\,dx$ and $\int_0^\pi \sin(x)\,dx$.
  (a) Before evaluating the integrals, which do you think is larger? Why?
  (b) Evaluate both integrals. Was your prediction in part (a) correct?

75. The figure below shows two regions, A and B. The volume swept out when region A is revolved about the x-axis is (using the disk method):

$$\int_{x=1}^{x=e} \pi\,(\ln(x))^2\,dx$$

and the volume swept out when region B is revolved about the x-xis is (using the tube method):

$$\int_{y=0}^{y=1} 2\pi y e^y\,dy$$



  (a) Before evaluating the integrals, which volume do you think is larger? Why?
  (b) Evaluate the integrals. Was your prediction in part (a) correct?

76. Refer to regions A and B from Problem 75.
    (a) Compute the volume of the solid generated when A is revolved around the y-axis.
    (b) Compute the volume of the solid generated when B is revolved around the y-axis.

77. Calculate the volume swept out when the region between the x-axis and the graph of $y = \sin(x)$ for $0 \le x \le \pi$ is rotated about the y-axis.

78. Calculate the volume swept out when the region between the x-axis and the graph of $y = \cos(x)$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the y-axis.

79. Calculate the volume swept out when the region between the x-axis and the graph of $y = x\sin(x)$ for $0 \le x \le \pi$ is rotated about the y-axis.

80. Calculate the volume swept out when the region between the x-axis and the graph of $y = x\cos(x)$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the y-axis.

81. Determine if the area of the region between the graph of $y = xe^{-x}$ and the positive x-axis is finite. (If so, compute the area.)

82. Determine if the area of the region between the graph of $y = x^2e^{-x}$ and the positive x-axis is finite. (If so, compute the area.)

83. Determine if the volume of the solid obtained by revolving the region between the graph of $y = xe^{-x}$ and the positive x-axis about the x-axis is finite. (If so, compute the volume.)

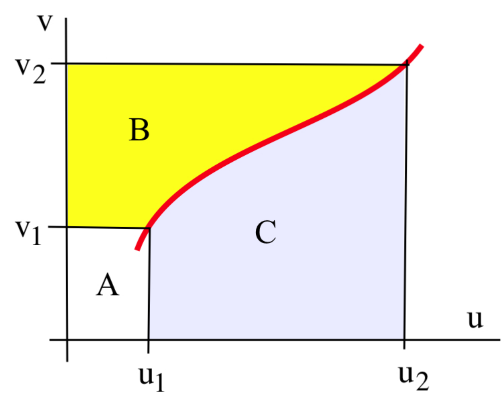
84. Determine if the volume of the solid obtained by revolving the region between the graph of $y = x^2e^{-x}$ and the positive x-axis about the x-axis is finite. (If so, compute the volume.)

85. Determine if the volume of the solid obtained by revolving the region between the graph of $y = xe^{-x}$ and the positive x-axis about the y-axis is finite. (If so, compute the volume.)

86. Determine if the volume of the solid obtained by revolving the region between the graph of $y = x^2e^{-x}$ and the positive x-axis about the y-axis is finite. (If so, compute the volume.)

87. We obtained the Integration by Parts formula analytically, starting with the Product Rule, but the formula also has a geometric interpretation. In the figure below, let D be the large rectangle formed by the regions A, B and C so that:

$$(\text{area of } C) = (\text{area of } D) - (\text{area of } A) - (\text{area of } B)$$



    (a) Represent the area of the large rectangle D as a function of u_2 and v_2.
    (b) Represent the area of the small rectangle A as a function of u_1 and v_1.
    (c) Represent the area of region C as an integral with respect to the variable u.
    (d) Represent the area of region B as an integral with respect to the variable v.
    (e) Rewrite the area equation using the results of parts (a)–(d). This should look familiar.

88. Suppose f and f' are continuous and bounded on the interval $[0, 2\pi]$, meaning that $|f(x)| < M$ and $|f'(x)| < M$ when $0 \le x \le 2\pi$. The n-th Fourier Sine Coefficient of f is defined as:

$$S_n = \int_0^{2\pi} f(x)\sin(nx)\,dx$$

    (a) Use the Integration By Parts Formula with $u = f(x)$ and $dv = \sin(nx)\,dx$ to represent the formula for S_n in a different way.
    (b) Use the new representation of S_n from part (a) to determine what happens to the values of S_n when n is very large ($n \to \infty$). (Hint: $|f'(x)\cos(nx)| = |f'(x)| \cdot |\cos(nx)| < M \cdot 1$.)
    (c) What happens to the n-th Fourier Cosine Coefficients $C_n = \displaystyle\int_0^{2\pi} f(x)\cos(nx)\,dx$ as $n \to \infty$?

8.2 Practice Answers

1. $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$ and $dv = x\,dx \Rightarrow v = \frac{1}{2}x^2$, so:

$$\begin{aligned}\int x\cos(x)\,dx = \int \cos(x)\cdot x\,dx = \int u\,dv = u\cdot v - \int v\,du \\ = \cos(x)\cdot\frac{1}{2}x^2 - \int \frac{1}{2}x^2\left[-\sin(x)\right]\,dx \\ = \frac{1}{2}x^2\cos(x) + \int \frac{1}{2}x^2\sin(x)\,dx\end{aligned}$$

resulting in a new integral worse than the original integral.

2. (a) With $u = x$, $dv = \sin(x)\,dx$ so $du = dx$ and $v = -\cos(x)$:

$$\begin{aligned}\int x\sin(x)\,dx = \int u\,dv = u\cdot v - \int v\,du \\ = x\cdot(-\cos(x)) - \int\left[-\cos(x)\right]\,dx \\ = -x\cos(x) + \int \cos(x)\,dx = -x\cos(x) + \sin(x) + C\end{aligned}$$

(b) With $u = x$, $dv = e^{5x}\,dx$ so $du = dx$ and $v = \frac{1}{5}e^{5x}$:

$$\begin{aligned}\int xe^{5x}\,dx = \int u\,dv = u\cdot v - \int v\,du \\ = x\cdot\frac{1}{5}e^{5x} - \int\frac{1}{5}e^{5x}\,dx = \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C\end{aligned}$$

3. Let $u = \ln(x)$ and $dv = dx$ so that $du = \frac{1}{x}\,dx$ and $v = x$:

$$\begin{aligned}\int \ln x\,dx = \int u\,dv = u\cdot v - \int v\,du \\ = \ln(x)\cdot x - \int x\cdot\frac{1}{x}\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C\end{aligned}$$

Using this result:

$$\int_1^e \ln x\,dx = \left[x\ln(x) - x\right]_1^e = \left[e\ln(e) - e\right] - \left[1\ln(1) - 1\right] = 1$$

4. With $n = 2$: $\displaystyle\int x^2\ln(x)\,dx = \frac{x^3}{3}\left[\ln(x) - \frac{1}{3}\right] + C$

With $n = 0$: $\displaystyle\int \ln(x)\,dx = \frac{x^1}{1}\left[\ln(x) - \frac{1}{1}\right] + C = x\ln(x) - x + C$

5. Set $u = x^n$ and $dv = \sin(x)\,dx$ so $du = nx^{n-1}\,dx$ and $v = -\cos(x)$:

$$\begin{aligned}\int x^n\sin(x)\,dx = \int u\,dv = uv - \int v\,du \\ = -x^n\cos(x) - \int\left[-\cos(x)\right]\cdot nx^{n-1}\,dx \\ = -x^n\cos(x) + n\int x^{n-1}\cos(x)\,dx\end{aligned}$$

6. Proceeding as in Example 7, set $u = e^x$ so that $dv = \sin(x)\,dx$. Then $du = e^x\,dx$ and $v = -\cos(x)\,dx$, yielding:

$$\begin{aligned}\int e^x \sin(x)\,dx &= \int u\,dv = uv - \int v\,du \\ &= e^x \cdot (-\cos(x)) - \int (-\cos(x)) \cdot e^x\,dx \\ &= -e^x \cos(x) + \int e^x \cos(x)\,dx\end{aligned}$$

For this new integral, set $u = e^x$ so that $dv = \cos(x)\,dx$. Then $du = e^x\,dx$ and $v = \sin(x)\,dx$, yielding:

$$\int e^x \cos(x)\,dx = \int u\,dv = uv - \int v\,du = e^x \sin(x) - \int e^x \sin(x)\,dx$$

Combining these results we get:

$$\begin{aligned}\int e^x \sin(x)\,dx &= -e^x \cos(x) + \int e^x \cos(x)\,dx \\ &= -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,dx\end{aligned}$$

and solving for the integral we started with yields:

$$2\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x) + K$$

so that:

$$\int e^x \sin(x)\,dx = \frac{1}{2}\left[-e^x \cos(x) + e^x \sin(x)\right] + C$$

7. For the first integral, let $u = x^5$ and take derivatives until you get 0:

$$x^5 \longrightarrow 5x^4 \longrightarrow 20x^3 \longrightarrow 60x^2 \longrightarrow 120x \longrightarrow 120 \longrightarrow 0$$

		$\cos(x)$	$= v'$
$u =$	x^5	$\sin(x)$	$(+)$
	$5x^4$	$-\cos(x)$	$(-)$
	$20x^3$	$-\sin(x)$	$(+)$
	$60x^2$	$\cos(x)$	$(-)$
	$120x$	$\sin(x)$	$(+)$
	120	$-\cos(x)$	$(-)$
	0	$-\sin(x)$	$(+)$

Put these in the left column of the margin table. Then set $v' = \cos(x)$ and start taking antiderivatives:

$$\cos(x) \longrightarrow \sin(x) \longrightarrow -\cos(x) \longrightarrow -\sin(x) \longrightarrow \cos(x)$$

and put these in the right column of the margin table. Now multiply the entries in each and add, using alternating signs:

$$\begin{aligned}\int x^5 \cos(x)\,dx =& x^5 \sin(x) - 5x^4\left[-\cos(x)\right] + 20x^3\left[-\sin(x)\right] \\ & - 60x^2 \cos(x) + 120x \sin(x) - 120\left[-\cos(x)\right] + C\end{aligned}$$

which we can rewrite as:

$$\left[x^5 - 20x^3 + 120x\right]\sin(x) + \left[5x^4 - 60x^2 + 120\right]\cos(x) + C$$

Using the same method with the second integral yields:

$$x^3\left[-\frac{1}{2}e^{-2x}\right] - 3x^2\left[\frac{1}{4}e^{-2x}\right] + 6x\left[-\frac{1}{8}e^{-2x}\right] - 6\left[\frac{1}{16}e^{-2x}\right] + C$$

so $\int x^3 e^{-2x}\,dx = -e^{-2x}\left[\frac{1}{2}x^3 + \frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{8}\right] + C.$

8.3 *Partial Fraction Decomposition*

Rational functions (polynomials divided by polynomials) and their integrals play important roles in mathematics and applications, but if you look through an integral table (such as Appendix I) you will find very few formulas for antiderivatives of rational functions. Partly this is because the general formulas are rather complicated and have many special cases, and partly it is because they can all be reduced to just a few cases using the algebraic technique discussed in this section: Partial Fraction Decomposition.

In algebra you learned to add rational functions to get a single rational function (just as in arithmetic you learned how to add many fractions to get a single fraction). Partial Fraction Decomposition is a technique for reversing that procedure to "decompose" a single rational function into a sum of simpler rational functions. This allows us to turn the integral of a single rational function into the sum of integrals of simpler functions.

Example 1. Verify that the algebraic decomposition:

$$\frac{17x-35}{2x^2-5x} = \frac{7}{x} + \frac{3}{2x-5}$$

is true and use this fact to evaluate $\displaystyle\int \frac{17x-35}{2x^2-5x}\,dx$.

Solution. Working from the right side of the above equality, combine the two rational expressions by converting to a common denominator:

$$\frac{7}{x} + \frac{3}{2x-5} = \frac{7}{x}\cdot\frac{(2x-5)}{(2x-5)} + \frac{3}{(2x-5)}\cdot\frac{x}{x} = \frac{14x-35+3x}{2x^2-5x} = \frac{17x-35}{2x^2-5x}$$

which is the expression on the left side. This decomposition allows us to exchange the original integral for two much easier ones:

$$\int \frac{17x-35}{2x^2-5x}\,dx = \int \frac{7}{x}\,dx + \int \frac{3}{2x-5}\,dx = 7\ln(|x|) + \frac{3}{2}\ln(|2x-5|) + C$$

which can also be written as $\ln\left(|x|^7\cdot|2x-5|^{\frac{3}{2}}\right)+C$. ◀

Practice 1. Verify that the algebraic decomposition:

$$\frac{7x-11}{3x^2-8x-3} = \frac{4}{3x+1} + \frac{1}{x-3}$$

is true and use this fact to evaluate $\displaystyle\int \frac{7x-11}{3x^2-8x-3}\,dx$.

Example 1 illustrates how to use a "decomposed" fraction to find an antiderivative of a rational function, but it does not show how to achieve this decomposition. The algebraic basis for the Partial Fraction Decomposition technique relies on the Fundamental Theorem of

Algebra, which guarantees that every polynomial with integer coefficients can be factored into a product of linear factors of the form $ax + b$ and irreducible quadratic factors of the form $ax^2 + bx + c$ (with $b^2 - 4ac < 0$). Unfortunately, the Fundamental Theorem of Algebra does not tell us *how* to find these factors, which typically will be more complicated than the examples in this section, but every polynomial has such factors. Before we apply the Partial Fraction Decomposition technique, the fraction must have the following form:

You might think that the Fundamental of Algebra would be easier to prove than the Fundamental Theorem of Calculus, because you studied algebra before calculus. Although we have already proved the Fundamental Theorem of Calculus in Chapter 4, a proof of the Fundamental Theorem of Algebra requires some higher-powered math.

- (degree of the numerator) < (degree of the denominator)
- The denominator has been factored into a product of linear factors and irreducible quadratic factors.

If the first assumption is not true, we can use polynomial division until we get a remainder with a smaller degree than the denominator. If the second assumption is not true, we simply cannot use the Partial Fraction Decomposition technique.

Now might be a good time to review polynomial division if you have not used it recently.

Example 2. Put each fraction into a form ready for Partial Fraction Decomposition:

(a) $\dfrac{2x^2 + 4x - 6}{x^2 - 2x}$ (b) $\dfrac{3x^3 - 3x^2 - 9x + 8}{x^2 - x - 6}$ (c) $\dfrac{7x^2 + 12x - 12}{x^3 - 4x}$

Solution. (a) The degree of the numerator is not lower than the degree of the denominator, so we need to use polynomial division to rewrite the rational function:

$$\frac{2x^2 + 4x - 6}{x^2 - 2x} = 2 + \frac{8x - 6}{x^2 - 2x} = 2 + \frac{8x - 6}{x(x - 2)}$$

and conclude by factoring the denominator.

(b) The degree of the numerator is bigger than the degree of the denominator, so we need to use polynomial division to rewrite the rational function:

$$\frac{3x^3 - 3x^2 - 9x + 8}{x^2 - x - 6} = 3x + \frac{9x + 8}{x^2 - x - 6} = 3x + \frac{9x + 8}{(x + 2)(x - 3)}$$

and conclude by factoring the denominator.

(c) Here the degree of the numerator is smaller than the degree of the denominator so we need only factor the denominator:

$$\frac{7x^2 + 12x - 12}{x^3 - 4x} = \frac{7x^2 + 12x - 12}{x(x + 2)(x - 2)}$$

and no polynomial division is required. ◀

Distinct Linear Factors

If you can factor the denominator into a product of distinct linear factors, then it turns out you can write the original fraction as the sum of fractions of the form $\frac{\text{number}}{\text{linear factor}}$. Your job is to find the numbers in the numerators, and that requires solving a system of equations.

We won't prove you can always do this, but you should be able to understand *why* this technique works by examining the solutions of Examples 3 and 4.

Example 3. Find values for A and B so that:

$$\frac{17x-35}{x(2x-5)} = \frac{A}{x} + \frac{B}{2x-5}$$

Solution. Multiply both sides of this equation by the common denominator $x(2x-5)$ to get:

$$17x - 35 = A(2x-5) + Bx = 2Ax - 5A + Bx = (2A+B)x - 5A$$

The leftmost and rightmost expressions in this equality are both linear functions, and we want them to be equal for all values of x, so the coefficients of x must be the same: $17 = 2A + B$. Similarly, the constant terms must be the same: $-5A = -35 \Rightarrow A = 7$. Putting this into the previous equation:

$$17 = 2A + B \Rightarrow 17 = 2(7) + B \Rightarrow B = 3$$

so we now know that:

$$\frac{17x-35}{x(2x-5)} = \frac{7}{x} + \frac{3}{2x-5}$$

which agrees with what we saw in Example 1. ◀

Practice 2. Find values of A and B so that:

$$\frac{6x-7}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

In general, there will be one unknown coefficient for each distinct linear factor of the denominator. If the number of distinct linear factors is large, we would need to solve a large system of equations for the unknowns. For any situation involving only distinct linear factors, however, a useful shortcut exists.

Example 4. Find values for A, B and C so that:

$$\frac{2x^2+7x+9}{x(x+1)(x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+3}$$

Solution. Multiplying both sides of the above equation by the common denominator $x(x+1)(x+3)$ yields:

$$2x^2 + 7x + 9 = A(x+1)(x+3) + Bx(x+3) + Cx(x+1)$$

At this point, you could combine the expressions on the right side of this new equation into one big quadratic polynomial and compare its coefficients to those of the quadratic polynomial on the left side, resulting in three equations in three unknowns. Or you could plug in some well-chosen values of x to avoid much of that algebra. If $x = 0$:

$$2 \cdot 0^2 + 7 \cdot 0 + 9 = A(0+1)(0+3) + B \cdot 0 \cdot (0+3) + C \cdot 0 \cdot (0+1)$$

so $9 = 3A \Rightarrow A = 3$. Similarly, if $x = -1$:

$$2(-1)^2 + 7(-1) + 9 = A(0)(2) + B(-1)(2) + C(-1)(0) \Rightarrow 4 = -2B$$

so $B = -2$. And if $x = -3$:

$$2(-3)^2 + 7(-3) + 9 = A(-2)(0) + B(-3)(0) + C(-3)(-2) \Rightarrow 6 = 6C$$

so $C = 1$. We now know that:

$$\frac{2x^2 + 7x + 9}{x(x+1)(x+3)} = \frac{3}{x} - \frac{2}{x+1} + \frac{1}{x+3}$$

and can easily integrate each of these three simpler fractions. ◀

Practice 3. Evaluate the integral $\displaystyle\int \frac{2x^2 + 7x + 9}{x(x+1)(x+3)}\, dx$.

Practice 4. Apply the method of Example 4 to the Partial Fraction Decomposition in Example 3.

For fractions whose denominators contain irreducible quadratic factors or repeated factors, the form of the decomposition becomes more complicated—and with it, the algebra required to find the values of the constants. We will not discuss why the following suggestions work to decompose more general rational functions, except to note that they provide enough (but not too many) unknown coefficients.

Distinct Irreducible Quadratic Factors

If the factored denominator includes a distinct irreducible quadratic factor, then the Partial Fraction Decomposition sum contains a fraction of the form:

$$\frac{\text{linear polynomial}}{\text{irreducible quadratic factor}} \quad \text{or} \quad \frac{Ax + B}{x^2 + px + q}$$

where we typically need to solve a system of equations to find the values of the unknown coefficients A and B, given p and q with $p^2 - 4q < 0$.

Example 5. Find values for A, B and C so that:

$$\frac{x^2 + 3x - 15}{(x^2 + 2x + 5)x} = \frac{Ax + B}{x^2 + 2x + 5} + \frac{C}{x}$$

Solution. Multiply the equation on both sides by the common denominator $(x^2+2x+5)x$ to get:

$$x^2+3x-15=(Ax+B)x+C(x^2+2x+5)$$

Put $x=0$ into the equation to get:

$$-15=(B)(0)+C(5)\Rightarrow C=-3$$

Unfortunately, there no other numbers we can use for x to make terms on the right side vanish, but we can plug in any other "nice" number we want. The "nicest" numbers after 0 are 1 and -1:

$$\begin{aligned} x=1: &\quad -11=(A+B)-3(8) &&\Rightarrow\ A+B=13 \\ x=-1: &\quad -17=(-A+B)(-1)-3(4) &&\Rightarrow\ A-B=-5 \end{aligned}$$

We now have two equations in two unknowns. Adding these equations yields $2A=8\Rightarrow A=4$ and subtracting the second equation from the first yields $2B=18\Rightarrow B=9$. This tells us that:

$$\frac{x^2+3x-15}{(x^2+2x+5)x}=\frac{4x+9}{x^2+2x+5}-\frac{3}{x}$$

The original function is now in a form that is ready for integration. ◀

Practice 5. Evaluate the integral $\int \frac{x^2+3x-15}{(x^2+2x+5)x}\,dx$.

In general, there are two unknown coefficients for each distinct irreducible quadratic factor in the denominator.

Example 6. Decompose the rational function $\frac{6x^3+36x^2+50x+53}{(x^2+4)(x^2+4x+5)}$.

Solution. The degree of the denominator (4) is bigger than the degree of the numerator (3) and fortunately the denominator has already been factored into two irreducible quadratics. We can write:

$$\frac{6x^3+36x^2+50x+53}{(x^2+4)(x^2+4x+5)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+4x+5}$$

The denominator in unfactored form is:

$$x^4+4x^3+9x^2+16x+20$$

Would you be able to factor this into two irreducible quadratic factors if it had not already been factored for you?

Multiplying by the common denominator $(x^2+4)(x^2+4x+5)$ yields:

$$\begin{aligned} 6x^3+36x^2+50x+53 &= (Ax+B)(x^2+4x+5)+(Cx+D)(x^2+4) \\ &= Ax^3+4Ax^2+5Ax+Bx^2+4Bx+5B+Cx^3+4Cx+Dx^2+4D \\ &= (A+C)x^3+(4A+B+D)x^2+(5A+4B+4C)x+(5B+4D) \end{aligned}$$

To simplify the algebra, note that:

$$C=6-A \text{ and } D=\frac{1}{4}(53-5B)$$

then substitute these expressions into the second and third equations to reduce a system of four equations in four unknowns to a system of two equations in two unknowns.

Compare coefficients between the first and last polynomials to see that $A+C=6$, $4A+B+D=36$, $5A+4B+4C=50$ and $5B+4D=53$. After much algebra (see margin note), this system of four equations in four unknowns reduces to $A=6$, $B=5$, $C=0$ and $D=7$. ◀

Repeated Factors

If the factored denominator contains a linear factor raised to a power (greater than one), the decomposition requires one unknown coefficient for each power of the linear factor. For example:

$$\frac{\text{something}}{(x+1)(x-2)^3} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

Similarly, if the factored denominator contains an irreducible quadratic factor raised to a power (greater than one), then the decomposition requires one linear term (with two unknown coefficients) for each power of the irreducible quadratic. For example:

$$\frac{\text{something}}{x^2(x^2+9)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+9} + \frac{Ex+F}{(x^2+9)^2} + \frac{Gx+H}{(x^2+9)^3}$$

This leads to a system of 8 equations with 8 unknowns.

Example 7. Evaluate $\displaystyle\int \frac{-4x^2+5x+3}{x^3-2x^2+x}\,dx$.

Solution. Before we can integrate, we need to decompose the integrand into simpler rational functions. The degree of the denominator (3) is bigger than the degree of the numerator (2) but we do need to factor:

$$x^3 - 2x^2 + x = x(x^2-2x+1) = x(x-1)^2$$

so that we can write:

$$\frac{-4x^2+5x+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiply this equation by the common denominator $x(x-1)^2$ to get:

$$-4x^2+5x+3 = A(x-1)^2 + Bx(x-1) + Cx$$

Inserting the values $x=0$ or $x=1$ yields:

$$\begin{aligned} x=0: &\quad 3 = (A)(-1)^2 + B(0)(-1) + C(0) &\Rightarrow\ A=3 \\ x=1: &\quad 4 = A(0) + B(1)(0) + C(1) &\Rightarrow\ C=4 \end{aligned}$$

Using $A=3$ and $C=4$ with $x=-1$ results in:

$$-6 = 3(-2)^2 + B(-1)(-2) - 4 \ \Rightarrow\ 2B = -14 \ \Rightarrow\ B = -7$$

We can now integrate:

$$\begin{aligned} \int \frac{-4x^2+5x+3}{x^3-2x^2+x}\,dx &= \int \left[\frac{3}{x} - \frac{7}{x-1} + \frac{4}{(x-1)^2}\right] dx \\ &= 3\ln(|x|) - 7\ln(|x-1|) - \frac{4}{x-1} + C \end{aligned}$$

which can also be written $\ln\left(\dfrac{|x|^3}{|x-1|^7}\right) - \dfrac{4}{x-1} + C$. ◀

Practice 6. Evaluate $\int \frac{2x^2 + 27x + 85}{(x+5)^2}\, dx$.

Be careful: Partial Fraction Decomposition only works with *rational* functions, although you may be able to turn another type of integrand into a rational function using substitution. For example, we can write:

$$\int \sec(\theta)\, d\theta = \int \frac{1}{\cos(\theta)}\, d\theta = \int \frac{\cos(\theta)}{\cos^2(\theta)}\, d\theta = \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\, d\theta$$

and substitute $u = \sin(\theta) \Rightarrow du = \cos(\theta)\, d\theta$ to convert this to:

$$\int \sec(\theta)\, d\theta = \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\, d\theta = \int \frac{1}{1-u^2}\, du$$

Problem 41 asks you to complete this integration and resubstitute to obtain the antiderivative pattern for $\sec(\theta)$.

which is now a job for partial fractions.

Additional Applications

The primary use of the partial fraction technique in this course is to convert rational functions into a form that makes them easier to integrate, but this algebraic technique can also be used to simplify the differentiation of certain rational functions. (In a later math course, you will also use partial fractions when computing certain inverse Laplace transforms.) The next Example illustrates the use of partial fractions to make a differentiation problem easier.

Example 8. For $f(x) = \frac{2x+13}{x^2+x-2}$, calculate $f'(x)$, $f''(x)$ and $f'''(x)$.

Solution. You already know how to calculate these derivatives using the Quotient Rule, but that process becomes rather tedious for the second and third derivatives of this function. Instead, we can use the partial fraction technique to rewrite f as:

$$f(x) = \frac{5}{x-1} - \frac{3}{x+2} = 5(x-1)^{-1} - 3(x+2)^{-1}$$

Computing the derivatives is now quite straightforward:

$$\begin{aligned} f'(x) &= -5(x-1)^{-2} + 3(x+2)^{-2} \\ f''(x) &= 10(x-1)^{-3} - 6(x+2)^{-3} \\ f'''(x) &= -30(x-1)^{-4} + 18(x+2)^{-4} \end{aligned}$$

You will appreciate the value of this shortcut if you attempt to compute $f'''(x)$ using the Quotient Rule. ◀

Practice 7. Use a partial fraction decomposition of $g(x) = \frac{9x+1}{x^2-2x-3}$ to calculate $g'(x)$, $g''(x)$ and $g^{(4)}(x)$.

The rational functions in the Examples and Problems in this section have been carefully constructed to allow you to easily factor denominators and to decompose fractions with a minimum of effort. In practice, factoring higher-order polynomials can require numerical methods (like Newton's Method) to approximate roots and decomposing rational functions whose denominators include repeated linear irreducible quadratic factors can lead to systems of many equations in many unknowns that benefit from the tools available in linear algebra. As a result, computers are best suited to handle the partial fraction decomposition of more complicated rational functions.

8.3 Problems

In Problems 1–32, decompose the integrand and then evaluate the integral.

1. $\displaystyle\int \frac{7x+2}{x(x+1)}\,dx$
2. $\displaystyle\int \frac{7x+9}{(x+3)(x-1)}\,dx$
3. $\displaystyle\int \frac{11x+25}{x^2+9x+8}\,dx$
4. $\displaystyle\int \frac{3x+7}{x^2-1}\,dx$
5. $\displaystyle\int \frac{2x^2+15x+25}{x^2+5x}\,dx$
6. $\displaystyle\int \frac{3x^3+3x^2}{x^2+x-2}\,dx$
7. $\displaystyle\int \frac{6x^2+9x-15}{x(x+5)(x-1)}\,dx$
8. $\displaystyle\int \frac{6x^2-x-1}{x^3-x}\,dx$
9. $\displaystyle\int \frac{8x^2-x+3}{x^3+x}\,dx$
10. $\displaystyle\int \frac{9x^2+13x+15}{x^3+2x^2-3x}\,dx$
11. $\displaystyle\int \frac{11x^2+23x+6}{x^2(x+2)}\,dx$
12. $\displaystyle\int \frac{6x^2+14x-9}{x(x+3)^2}\,dx$
13. $\displaystyle\int \frac{3x+13}{(x+2)(x-5)}\,dx$
14. $\displaystyle\int \frac{2x+11}{(x-7)(x-2)}\,dx$
15. $\displaystyle\int_2^5 \frac{2}{x^2-1}\,dx$
16. $\displaystyle\int_1^3 \frac{5x^2+5x+3}{x^3+x}\,dx$
17. $\displaystyle\int \frac{2x^2+5x-3}{x^2-1}\,dx$
18. $\displaystyle\int \frac{2x^2+19x+22}{x^2+x-12}\,dx$
19. $\displaystyle\int \frac{3x^2+19x+24}{x^2+6x+5}\,dx$
20. $\displaystyle\int \frac{7x^2+8x-2}{x^2+2x}\,dx$
21. $\displaystyle\int \frac{3x^2-1}{x^3+x}\,dx$
22. $\displaystyle\int \frac{x^4+5x^3+x-15}{x^2+5x}\,dx$
23. $\displaystyle\int \frac{x^3+3x^2-4x+30}{x^2+3x-10}\,dx$
24. $\displaystyle\int \frac{7x^3+x^2+7x+10}{x^4+2x^3}\,dx$
25. $\displaystyle\int \frac{12x^2+19x-6}{x^3+3x^2}\,dx$
26. $\displaystyle\int \frac{2x+5}{(x+1)^2}\,dx$
27. $\displaystyle\int \frac{7x^2+3x+7}{x^3+x}\,dx$
28. $\displaystyle\int \frac{7x^2-4x+4}{x^3+1}\,dx$
29. $\displaystyle\int_2^\infty \frac{2}{x^2-1}\,dx$
30. $\displaystyle\int_2^\infty \frac{7x+2}{6x^2-13x+16}\,dx$
31. $\displaystyle\int \frac{6x^2+5x+61}{(x-1)(x^2+4x+13)}\,dx$
32. $\displaystyle\int \frac{x-84}{(x+5)(x^2-6x+34)}\,dx$

Integrals can be very sensitive to small changes in the integrand. In 33–34, notice how similar functions require vastly different integration methods.

33. (a) $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$
    (b) $\displaystyle\int \frac{1}{x^2+2x+1}\,dx$
    (c) $\displaystyle\int \frac{1}{x^2+2x+0}\,dx$
34. (a) $\displaystyle\int \frac{1}{x^2-6x+8}\,dx$
    (b) $\displaystyle\int \frac{1}{x^2-6x+9}\,dx$
    (c) $\displaystyle\int \frac{1}{x^2-6x+10}\,dx$

In Problems 35–40, use a partial fraction decomposition to compute the first and second derivatives of the given function.

35. $\displaystyle f(x) = \frac{7x+2}{x(x+1)}$
36. $\displaystyle F(x) = \frac{7x+9}{(x+3)(x-1)}$
37. $\displaystyle g(x) = \frac{11x+25}{x^2+9x+8}$
38. $\displaystyle G(x) = \frac{3x+7}{x^2-1}$
39. $\displaystyle h(x) = \frac{2x^2+15x+25}{x^2+5x}$
40. $\displaystyle H(x) = \frac{9x^2+13x+15}{x^3+2x^2-3x}$

41. Obtain a formula for $\int \sec(\theta)\, d\theta$ by writing:

$$\int \frac{1}{\cos(\theta)}\, d\theta = \int \frac{\cos(\theta)}{\cos^2(\theta)}\, d\theta = \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\, d\theta$$

and using the substitution $u = \sin(\theta)$ to turn this integrand into a rational function.

42. Obtain a formula for $\int \csc(\theta)\, d\theta$ by writing:

$$\int \frac{1}{\sin(\theta)}\, d\theta = \int \frac{\sin(\theta)}{\sin^2(\theta)}\, d\theta = \int \frac{\sin(\theta)}{1-\cos^2(\theta)}\, d\theta$$

and using the substitution $u = \cos(\theta)$ to turn this integrand into a rational function.

Logistic Growth

The following two applications involve a type of differential equation that can be solved by separating variables and then using a partial fraction decomposition to calculate the antiderivatives. The same type of differential equation is also used to model the spread of rumors and diseases, as well as the growth of some populations and chemical reactions.

The growth rate of many different types of populations depends not only on the number of individuals currently in the population (leading to exponential growth) but also on a "carrying capacity" of the environment. (As a population grows, it might deplete a food source, slowing the growth of the population.) If x is the size of a population at time t and the growth rate of x is proportional only to the size of the population, we get the differential equation:

$$\frac{dx}{dt} = kx$$

which we investigated in Section 6.4. If we want to create a model in which the growth slows as the size of the population nears the carrying capacity M, we can multiply this model for $\frac{dx}{dt}$ by factor $\left(1 - \frac{x}{M}\right)$:

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{M}\right)$$

When x is small (relative to M), this new factor is close to 1 so that:

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{M}\right) \approx kx$$

leading to growth that is "almost exponential." When x gets close to M, this new factor is close to 0:

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{M}\right) \approx kx\left(1 - \frac{M}{M}\right) = 0$$

so that the growth of x slows down and x is (roughly) constant. We call this differential equation the **logistic equation** and its solution a **logistic function**.

43. Let $k = 1$ and $M = 100$, and assume the initial population is $x(0) = 5$.
    (a) Create a direction field for the ODE.
    (b) Solve the corresponding logistic IVP.
    (c) Graph the population $x(t)$ for $0 \le t \le 20$.
    (d) When will the population be 20? 50? 90? 100?
    (e) What is the population after a "long" time? (Find the limit of x, as $t \to \infty$.)
    (f) Explain the shape of the graph in (c) in the context of a population of bacteria.
    (g) When is the growth rate largest?
    (h) What is the population when the growth rate is largest?
    (i) What would happen if $x(0) > 100$?

44. Let $k = 1$ and $M = 100$, and assume the initial population is $x(0) = 150$.
    (a) Solve the corresponding logistic IVP.
    (b) Graph the population $x(t)$ for $0 \le t \le 20$.
    (c) When will the population be 120? 110? 100?
    (d) What is the population after a "long" time?
    (e) Explain the shape of the graph in (b).

45. Let k and M be positive constants, and assume the initial population is $x(0) = x_0$.
    (a) Solve the corresponding logistic IVP.
    (b) What is the population after a "long" time?
    (c) When is the growth rate largest?
    (d) What is the population at that time?

Chemical Reactions

In certain chemical reactions, a new material X is formed from materials A and B, and the rate at which X forms is proportional to the product of the amount of A and the amount of B remaining. Let x represent the amount of material X present at time t, and assume that the reaction begins with a grams of A, b grams of B and no material X (so that $x(0) = 0$). Then the rate of formation of material X can be described by the differential equation:

$$\frac{dx}{dt} = k(a - x)(b - x)$$

46. Solve the IVP given above for x if $k = 1$ and the reaction begins with
    (a) 7 grams of A and 5 grams of B.
    (b) 6 grams of A and 6 grams of B.

47. Solve the IVP given above for x if $k = 1$ and the reaction begins with
    (a) a grams of A and b grams of B with $a \neq b$.
    (b) c grams of A and c grams of B (for $c > 0$).

8.3 Practice Answers

1. To verify the identity:

$$\frac{4}{(3x+1)} \cdot \frac{(x-3)}{(x-3)} + \frac{1}{(x-3)} \cdot \frac{(3x+1)}{(3x+1)} = \frac{(4x-12)+(3x+1)}{(3x+1)(x-3)} = \frac{7x-11}{3x^2-8x-3}$$

To evaluate the integral:

$$\int \frac{7x-11}{3x^2-8x-3}\,dx = \int \left[\frac{4}{(3x+1)} + \frac{1}{(x-3)}\right] dx = \frac{4}{3}\ln(|3x+1|) + \ln(|x-3|) + C$$

2. Multiply both sides of the equality by the common denominator $(x+3)(x-2)$ to get:

$$6x-7 = A(x-2)+B(x+3) = Ax-2A+Bx+3B = (A+B)x+(-2A+3B)$$

so $A+B=6$ and $-2A+3B=-7$. Solving this system of equations yields $A=5$ and $B=1$.

3. Using the result of Example 4:

$$\int \left[\frac{3}{x} - \frac{2}{x+1} + \frac{1}{x+3}\right] dx = 3\ln(|x|) - 2\ln(|x+1|) + \ln(|x+3|) + C$$

4. Multiply each side of the equation by the common denominator $x(2x-5)$ to get:

$$\frac{17x-35}{x(2x-5)} = \frac{A}{x} + \frac{B}{2x-5} \;\Rightarrow\; 17x-35 = A(2x-5)+Bx$$

Now put $x=0$ and $x=\frac{5}{2}$ into this new equation to get:

$$\begin{aligned} x=0: &\quad -35 = -5A+0 \;\Rightarrow\; A=7 \\ x=\frac{5}{2}: &\quad \frac{15}{2} = A(0) + B\left(\frac{5}{2}\right) \;\Rightarrow\; B=3 \end{aligned}$$

5. Using the result of Example 5:

$$\begin{aligned} \int \left[\frac{4x+9}{x^2+2x+5} - \frac{3}{x}\right] dx &= \int \left[\frac{4x+8+1}{(x+1)^2+1} - \frac{3}{x}\right] dx \\ &= \int \left[\frac{2(2x+2)}{x^2+2x+5} + \frac{5}{(x+1)^2+4} - \frac{3}{x}\right] dx \\ &= 2\ln\left(x^2+2x+5\right) + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) - 3\ln(|x|) + C \end{aligned}$$

6. Because the degree of the numerator equals the degree of the denominator, we must use polynomial division to rewrite the integrand as:

$$\frac{2x^2+27x+85}{x^2+10x+25} = 2 + \frac{7x+35}{x^2+10x+25} = 2 + \frac{7(x+5)}{(x+5)^2} = 2 + \frac{7}{x+5}$$

This integrand does not require partial fraction decomposition:

$$\int \frac{2x^2+27x+85}{(x+5)^2}\,dx = \int \left[2 + \frac{7}{x+5}\right] dx = 2x + 5\ln(|x+5|) + C$$

7. First write:

$$g(x) = \frac{9x+1}{x^2-2x-3} = \frac{9x+1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

and then multiply through the last equality by the common denominator $(x-3)(x+1)$ to get:

$$9x+1 = A(x+1) + B(x-3)$$

Now put $x = 3$ and $x = -1$ into this new equation to get:

$$\begin{aligned} x = 3: &\quad 28 = 4A + B(0) \;\Rightarrow\; A = 7 \\ x = -1: &\quad -8 = A(0) + B(-4) \;\Rightarrow\; B = 2 \end{aligned}$$

We can now rewrite $g(x)$ as $g(x) = 7(x-3)^{-1} + 2(x+1)^{-1}$ so that:

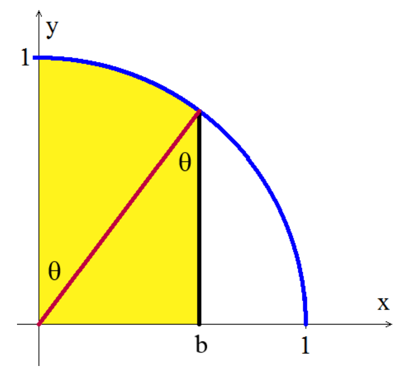
$$\begin{aligned} g'(x) &= -7(x-3)^{-2} - 2(x+1)^{-2} \\ g''(x) &= 14(x-3)^{-3} + 4(x+1)^{-3} \\ g'''(x) &= -42(x-3)^{-4} - 12(x+1)^{-4} \\ g^{(4)}(x) &= 168(x-3)^{-5} + 48(x+1)^{-5} \end{aligned}$$

8.4 Trigonometric Substitution

In the previous section, you learned how to decompose rational expressions into simpler forms that were easier to integrate. This section investigates a method for transforming integrands involving radical expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$ using a specialized change of variable.

While we know that $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x)+C$, and while we can compute $\int \frac{x}{\sqrt{1-x^2}}\,dx$ and $\int x\sqrt{1-x^2}\,dx$ without much difficulty using the substitution $u = 1-x^2$, the simpler-looking integral $\int \sqrt{1-x^2}\,dx$ poses a greater challenge. It does, however, have a geometric interpretation that allows us to find an antiderivative.

Consider $\int_0^b \sqrt{1-x^2}\,dx$ for any b with $0 < b < 1$. This definite integral represents the area of the region under the curve $y=\sqrt{1-x^2}$ between the y-axis (where $x=0$) and the vertical line $x=b$ (see margin). We can split this region into two pieces (as in the margin figure): a sector of a circle and a triangle.



The triangle has base b and height $\sqrt{1-b^2}$, so its area is $\frac{1}{2}b\cdot\sqrt{1-b^2}$. The area of a sector of a circle with radius r and central angle θ is $\frac{\theta}{2\pi}\cdot\pi r^2 = \frac{1}{2}r^2\theta$. But $r=1$ and θ is also the angle of the upper vertex of the triangle, so:

$$\sin(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{1} = b \;\Rightarrow\; \theta = \arcsin(b)$$

which tells us:

$$\int_0^b \sqrt{1-x^2}\,dx = \frac{1}{2}b\cdot\sqrt{1-b^2}+\frac{1}{2}\arcsin(b)$$

and we can easily deduce that:

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\cdot\sqrt{1-x^2}+\frac{1}{2}\arcsin(x)+C$$

If $b=1$, the integral should give the area of a quarter-circle of radius 1, and we can then verify that:

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{2}\cdot 1\cdot 0+\frac{1}{2}\arcsin(1) = 0+\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

The appearance of $\arcsin(x)$ in the antiderivative of $\sqrt{1-x^2}$ suggests a somewhat unusual substitution. Because $\arcsin(x)$ represents an angle, we can call it θ and write:

$$\arcsin(x) = \theta \;\Rightarrow\; x = \sin(\theta) \;\Rightarrow\; dx = \cos(\theta)\,d\theta$$

If we make the substitution $x = \sin(\theta)$, the integral becomes:

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2(\theta)}\cdot\cos(\theta)\,d\theta = \int \sqrt{\cos^2(\theta)}\cdot\cos(\theta)\,d\theta$$
$$= \int \cos(\theta)\cdot\cos(\theta)\,d\theta = \int \cos^2(\theta)\,d\theta$$

Ordinarily we would write:

$$\sqrt{\cos^2(\theta)} = |\cos(\theta)|$$

but $\theta = \arcsin(x)$ and the range of arcsine is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so $\cos(\theta)\geq 0$.

This is an integral we already know how to work out:

$$\int \cos^2(\theta)\,d\theta = \int\left[\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right]d\theta = \frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)+C$$
$$= \frac{1}{2}\theta+\frac{1}{4}\cdot 2\sin(\theta)\cos(\theta)+C = \frac{1}{2}\theta+\frac{1}{2}\sin(\theta)\cos(\theta)+C$$

Now we must convert the variable θ back to x. We started out with $\theta = \arcsin(x) \Rightarrow x = \sin(\theta)$ so we just need to rewrite $\cos(\theta)$ in terms of x. The Pythagorean identity tells us that:

$$\cos^2(\theta) + \sin^2(\theta) = 1 \ \Rightarrow \cos^2(\theta) = 1 - \sin^2(\theta) = 1 - x^2$$

Ordinarily we would write:

$$\cos(\theta) = \pm\sqrt{1 - x^2}$$

but from the discussion in the preceding margin note we know that $\cos(\theta) \geq 0$.

so $\cos(\theta) = \sqrt{1 - x^2}$. Therefore:

$$\frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + C = \frac{1}{2}\arcsin(x) + \frac{1}{2} \cdot x \cdot \sqrt{1 - x^2} + C$$

which agrees with our original geometric result.

Another Change of Variable

The preceding discussion suggests a new type of substitution: instead of setting u = function of x, replace x with a (trigonometric) function of θ. Each trigonometric substitution will involve four major steps:

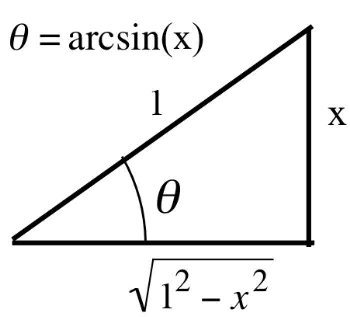
1. Choose a substitution to make x = a trigonometric function of θ.
2. Rewrite the original integral in terms of θ and $d\theta$.
3. Find an antiderivative of the new integrand.
4. Rewrite this antiderivative in terms of the original variable x.

The discussion that follows examines how these steps play out in three situations. The first step requires you to make a decision. The other steps follow from that decision. For most students, the key to success with the Trigonometric Substitution technique is to **think triangles**.

Step 1: Choosing the substitution

The Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$ played an important role in our first application of the trigonometric substitution technique. The familiar Pythagorean Theorem can help guide you to a correct choice of substitution so that a trigonometric identity will always come to the rescue when converting from θ back to x in Step 4. Thinking of a right triangle, we can state the Pythagorean Theorem as:

$$(\text{side})^2 + (\text{other side})^2 = (\text{hypotenuse})^2$$
$$\Rightarrow \ (\text{other side})^2 = (\text{hypotenuse})^2 - (\text{side})^2$$



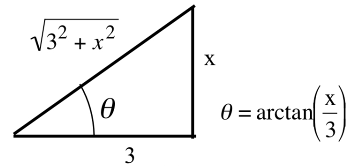
A right triangle representing our initial substitution $x = \sin(\theta)$ appears in the margin. Here x is the length of the side opposite θ and 1 is the length of the hypotenuse, so that the "other side" has length $\sqrt{1 - x^2}$, the radical expression that appeared in the original integral.

We now investigate the trigonometric substitutions for three representative patterns: $3^2 + x^2$, $3^2 - x^2$ and $x^2 - 3^2$.

There is nothing special about the number 3 in these examples: we would have used 5 or π or $\sqrt{17}$.

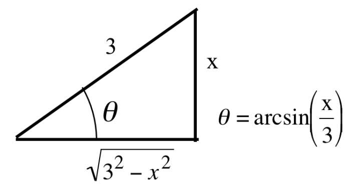
- The pattern $\mathbf{3^2 + x^2}$ matches the Pythagorean pattern if 3 and x are sides of a right triangle (see margin). Then:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{3} \Rightarrow x = 3\tan(\theta)$$



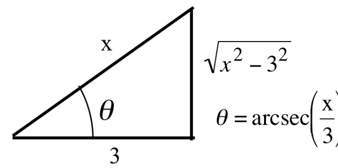
- The pattern $\mathbf{3^2 - x^2}$ matches the Pythagorean pattern if 3 is the hypotenuse and x is a side of a right triangle (see margin). Then:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{3} \Rightarrow x = 3\sin(\theta)$$



- The pattern $\mathbf{x^2 - 3^2}$ matches the Pythagorean pattern if x is the hypotenuse and 3 is a side of a right triangle (see margin). Then:

$$\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{x}{3} \Rightarrow x = 3\sec(\theta)$$



Step 2: Rewrite the original integral in terms of θ and $d\theta$

Once you have made the choice for the substitution, several things follow automatically: you can easily calculate dx, you can solve for θ, and you can rewrite the original pattern of interest as a function of θ.

- With the pattern $\mathbf{3^2 + x^2}$, differentiation yields:

$$x = 3\tan(\theta) \Rightarrow dx = 3\sec^2(\theta)\,d\theta$$

while solving the substitution equation for θ yields:

$$x = 3\tan(\theta) \Rightarrow \tan(\theta) = \frac{x}{3} \Rightarrow \theta = \arctan\left(\frac{x}{3}\right)$$

Here $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ because that is the range of the arctangent function. This means that $\sec(\theta) > 0$ always holds, so that $\sqrt{\sec^2(\theta)} = \sec(\theta)$.

and the original pattern becomes:

$$3^2 + x^2 = 3^2 + 3^2\tan^2(\theta) = 3^2\left[1 + \tan^2(\theta)\right] = 3^2\sec^2(\theta)$$

- With the pattern $\mathbf{3^2 - x^2}$, differentiation yields:

$$x = 3\sin(\theta) \Rightarrow dx = 3\cos(\theta)\,d\theta$$

$$x = 3\sin(\theta) \Rightarrow \sin(\theta) = \frac{x}{3} \Rightarrow \theta = \arcsin\left(\frac{x}{3}\right)$$

$$3^2 - x^2 = 3^2 - 3^2\sin^2(\theta) = 3^2\left[1 - \sin^2(\theta)\right] = 3^2\cos^2(\theta)$$

Here $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ because that is the range of the arcsine function. This means that $\cos(\theta) \geq 0$ always holds, so that $\sqrt{\cos^2(\theta)} = \cos(\theta)$.

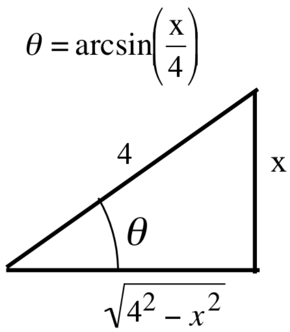
- With the pattern $\mathbf{x^2 - 3^2}$, differentiation yields:

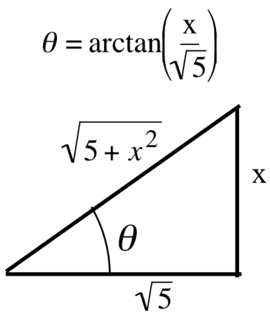
$$x = 3\sec(\theta) \Rightarrow dx = 3\sec(\theta)\tan(\theta)\,d\theta$$

$$x = 3\sec(\theta) \Rightarrow \sec(\theta) = \frac{x}{3} \Rightarrow \theta = \text{arcsec}\left(\frac{x}{3}\right)$$

$$x^2 - 3^2 = 3^2\sec^2(\theta) - 3^2 = 3^2\left[\sec^2(\theta) - 1\right] = 3^2\tan^2(\theta)$$

Here $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta < \pi$ because that is the range of the arcsecant function. We will want $\tan(\theta) \geq 0$ to avoid absolute values when simplifying $\sqrt{\tan^2(\theta)}$; to do so, we will need to assume that $x \geq 3$ so that $0 \leq \theta < \frac{\pi}{2}$.





Example 1. For the patterns $16 - x^2$ and $5 + x^2$, select an appropriate substitution for x, calculate dx and θ, and use the substitution to simplify the pattern.

Solution. The pattern $16 - x^2$ matches the Pythagorean pattern if 4 is a hypotenuse and x is the side of a right triangle. Then:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{4} \Rightarrow x = 4\sin(\theta) \Rightarrow dx = 4\cos(\theta)\,d\theta$$

while $\theta = \arcsin\left(\frac{x}{4}\right)$ and $16 - x^2$ becomes:

$$16 - [4\sin(\theta)]^2 = 16 - 16\sin^2(\theta) = 16\left[1 - \sin^2(\theta)\right] = 16\cos^2(\theta)$$

The pattern $5 + x^2$ matches the Pythagorean pattern if x and $\sqrt{5}$ are the sides of a right triangle. Then:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{5}} \Rightarrow x = \sqrt{5}\tan(\theta) \Rightarrow dx = \sqrt{5}\sec^2(\theta)\,d\theta$$

while $\theta = \arctan\left(\frac{x}{\sqrt{5}}\right)$ and $5 + x^2$ becomes:

$$5 + \left[\sqrt{5}\tan(\theta)\right]^2 = 5 + 5\tan^2(\theta) = 5\left[1 + \tan^2(\theta)\right] = 5\sec^2(\theta)$$

Drawing a right triangle for each pattern (as in the margin) will help you visualize the appropriate substitution. ◀

Practice 1. For the patterns $25 + x^2$ and $x^2 - 13$, decide on the appropriate substitution for x, calculate dx and θ, and use the substitution to simplify the pattern.

Example 2. Use $x = 5\tan(\theta)$ to rewrite the integral $\int \frac{1}{\sqrt{25 + x^2}}\,dx$.

Solution. Differentiating $x = 5\tan(\theta)$ yields $dx = 5\sec^2(\theta)\,d\theta$ and:

$$25 + x^2 = 25 + 25\tan^2(\theta) = 25\left[1 + \tan^2(\theta)\right] = 25\sec^2(\theta)$$

so that $\sqrt{25 + x^2} = \sqrt{25\sec^2(\theta)} = 5\sec(\theta)$ and the integral becomes:

$$\int \frac{1}{\sqrt{25 + x^2}}\,dx = \int \frac{1}{5\sec(\theta)} \cdot 5\sec^2(\theta)\,d\theta = \int \sec(\theta)\,d\theta$$

which is an integral we already know how to evaluate. ◀

Practice 2. Use $x = 5\sin(\theta)$ to rewrite the integral $\int \frac{1}{\sqrt{25 - x^2}}\,dx$.

Step 3: Find an antiderivative of the new integrand

After changing the variable, the new integrand typically involves products of powers of trigonometric functions and we can use any of our previous methods (another change of variable, integration by parts, a trigonometric identity or integral tables) to find an antiderivative.

Step 4: Rewrite this antiderivative in terms of the original variable

Once we have found an antiderivative, usually involving a trigonometric function of θ, we can replace θ with the appropriate inverse trigonometric function of x and simplify. Because the antiderivatives commonly contain trigonometric functions, we frequently need to simplify a trigonometric function of an inverse trigonometric function, and at this stage it is often *very* helpful to refer back to the right triangle we used at the beginning of the substitution process.

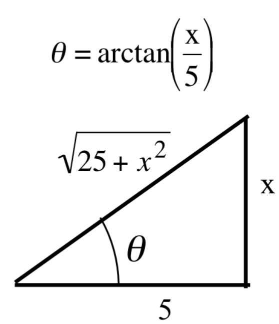
Example 3. Evaluate $\displaystyle\int \frac{1}{\sqrt{25+x^2}}\,dx$.

Solution. In Example 2 we used the substitution $x = 5\tan(\theta)$ to convert this integral to:

$$\int \frac{1}{\sqrt{25+x^2}}\,dx = \int \sec(\theta)\,d\theta = \ln\left(|\sec(\theta)+\tan(\theta)|\right) + C$$

We now need to rewrite $\sec(\theta)$ and $\tan(\theta)$ in this antiderivative in terms of x using the fact that $\theta = \arctan\left(\frac{x}{5}\right)$:

$$\ln\left(\left|\sec\left(\arctan\left(\frac{x}{5}\right)\right) + \tan\left(\arctan\left(\frac{x}{5}\right)\right)\right|\right) + C$$



Referring to the right triangle for this substitution (see margin):

$$\sec\left(\arctan\left(\frac{x}{5}\right)\right) = \frac{\sqrt{25+x^2}}{5} \quad \text{and} \quad \tan\left(\arctan\left(\frac{x}{5}\right)\right) = \frac{x}{5}$$

so, putting all of these pieces together:

$$\int \frac{1}{\sqrt{25+x^2}}\,dx = \ln\left(\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right) + C$$

Where did the absolute value signs go? Because:

$$-\frac{x}{5} \le \left|\frac{x}{5}\right| = \frac{\sqrt{x^2}}{5} < \frac{\sqrt{x^2+25}}{5}$$

we know that:

$$\frac{\sqrt{x^2+25}}{5} + \frac{x}{5} > 0$$

As always, we can check that this is indeed an antiderivative of the original integrand by differentiating. ◀

Practice 3. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$.

Variations on a Theme

While we will typically apply trigonometric substitution to integrands involving radicals of the form $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ or $\sqrt{x^2-a^2}$, we can adapt this technique for more general integrands by incorporating other methods, such as u-substitution and completing the square.

Example 4. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+2x+26}}\,dx$.

Solution. The polynomial inside the radical in this integrand is an irreducible quadratic so — just as we did with rational functions — we can complete the square:

$$x^2 + 2x + 26 = (x+1)^2 + 25$$

and next use the substitution $u = x + 1 \Rightarrow du = dx$:

$$\int \frac{1}{\sqrt{x^2+2x+26}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+5^2}}\,dx = \int \frac{1}{\sqrt{u^2+5^2}}\,du$$

The integrand is now ready for a trigonometric substitution. In fact, this resembles the integral in Example 3, so:

$$\begin{aligned}\int \frac{1}{\sqrt{u^2+5^2}}\,du &= \ln\left(\frac{\sqrt{25+u^2}}{5} + \frac{u}{5}\right) + C \\ &= \ln\left(\sqrt{25+u^2} + u\right) + C - \ln(5) \\ &= \ln\left(\sqrt{25+(x+1)^2} + (x+1)\right) + K\end{aligned}$$

Here we employ the logarithmic identity:

$$\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$$

We could have made a similar simplification in Example 3.

or $\ln\left(x+1+\sqrt{x^2+2x+26}\right) + K$. ◀

Wrap-up

When using trig substitution, remember to **think triangles**. The first and last steps (choosing the substitution and writing the answer in terms of x) are much easier if you have drawn the appropriate triangle for the problem. Of course, you also need to practice the method.

8.4 Problems

In Problems 1–8, determine the appropriate substitution for the given integrand.

1. $\sqrt{49-x^2}$
2. $\sqrt{x^2-36}$
3. $\left(81+x^2\right)^{\frac{5}{2}}$
4. $\sqrt{8+x^2}$
5. $\sqrt{x^2-7}$
6. $\sqrt{99-x^2}$
7. $\dfrac{1}{x\sqrt{100-x^2}}$
8. $\dfrac{1}{x^2\sqrt{x^2-100}}$

In Problems 9–14, make the given substitution and simplify the result, then calculate dx.

9. $x = 3\cdot\sin(\theta)$ in $\dfrac{1}{\sqrt{9-x^2}}$
10. $x = 3\cdot\tan(\theta)$ in $\dfrac{1}{\sqrt{x^2+9}}$
11. $x = 3\cdot\sec(\theta)$ in $\dfrac{1}{\sqrt{x^2-9}}$
12. $x = 6\cdot\sin(\theta)$ in $\dfrac{1}{36-x^2}$

13. $x = \sqrt{2} \cdot \tan(\theta)$ in $\dfrac{1}{\sqrt{2+x^2}}$

14. $x = \sec(\theta)$ in $\dfrac{1}{\sqrt{x^2-1}}$

In Problems 13–18, (a) solve for θ as a function of x, (b) replace θ in $f(\theta)$ with you result from part (a), and (c) simplify.

15. $x = 3 \cdot \sin(\theta)$, $f(\theta) = \cos(\theta) \cdot \tan(\theta)$

16. $x = 3 \cdot \tan(\theta)$, $f(\theta) = \sin(\theta) \cdot \tan(\theta)$

17. $x = 3 \cdot \sec(\theta)$, $f(\theta) = \sqrt{1+\sin^2(\theta)}$

18. $x = 5 \cdot \sin(\theta)$, $f(\theta) = \dfrac{\cos(\theta)}{1+\sec(\theta)}$

19. $x = 5 \cdot \tan(\theta)$, $f(\theta) = \dfrac{\cos^2(\theta)}{1+\cot(\theta)}$

20. $x = 5 \cdot \sec(\theta)$, $f(\theta) = \cos(\theta) + 7 \cdot \tan^2(\theta)$

In Problems 21–44, evaluate the integral. (More than one method works for some of the integrals.)

21. $\displaystyle\int \frac{1}{x\sqrt{9-x^2}}\,dx$

22. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

23. $\displaystyle\int \frac{1}{\sqrt{x^2+49}}\,dx$

24. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$

25. $\displaystyle\int \sqrt{36-x^2}\,dx$

26. $\displaystyle\int \sqrt{1-36x^2}\,dx$

27. $\displaystyle\int \frac{1}{\sqrt{36+x^2}}\,dx$

28. $\displaystyle\int \frac{1}{x\sqrt{25-x^2}}\,dx$

29. $\displaystyle\int \frac{x^2}{\sqrt{49-x^2}}\,dx$

30. $\displaystyle\int \frac{\sqrt{25-x^2}}{x^2}\,dx$

31. $\displaystyle\int \frac{x}{\sqrt{25-x^2}}\,dx$

32. $\displaystyle\int \frac{1}{x^2+49}\,dx$

33. $\displaystyle\int \frac{x}{x^2+49}\,dx$

34. $\displaystyle\int \frac{1}{49x^2+25}\,dx$

35. $\displaystyle\int \frac{1}{(x^2-9)^{\frac{3}{2}}}\,dx$

36. $\displaystyle\int \frac{1}{(4x^2-1)^{\frac{3}{2}}}\,dx$

37. $\displaystyle\int \frac{5}{2x\sqrt{x^2-25}}\,dx$

38. $\displaystyle\int \frac{1}{x\sqrt{3-x^2}}\,dx$

39. $\displaystyle\int \frac{1}{25-x^2}\,dx$

40. $\displaystyle\int \frac{1}{a^2+x^2}\,dx$

41. $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\,dx$

42. $\displaystyle\int \frac{1}{x\sqrt{a^2+x^2}}\,dx$

43. $\displaystyle\int \frac{1}{x^2\sqrt{a^2+x^2}}\,dx$

44. $\displaystyle\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\,dx$

In Problems 45–50, complete the square and make an appropriate substitution (as necessary), then evaluate the integral.

45. $\displaystyle\int \frac{1}{\sqrt{(x+1)^2+9}}\,dx$

46. $\displaystyle\int \frac{1}{\sqrt{(x+3)^2+1}}\,dx$

47. $\displaystyle\int \frac{1}{x^2+10x+29}\,dx$

48. $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$

49. $\displaystyle\int \frac{1}{\sqrt{x^2+4x+3}}\,dx$

50. $\displaystyle\int \frac{1}{\sqrt{x^2-6x-16}}\,dx$

51. The integral $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$ arises when finding antiderivatives using partial fractions.

(a) Evaluate this integral using an appropriate trigonometric substitution.

(b) Now evaluate it using integration by parts.

(c) Which method is easier?

52. Evaluate $\displaystyle\int \frac{1}{(x^2-8x+25)^2}\,dx$.

53. Evaluate $\displaystyle\int \frac{8x}{(x^2+25)^2}\,dx$.

8.4 Practice Answers

1. For the pattern $25 + x^2$, use $x = 5 \cdot \tan(\theta) \Rightarrow dx = 5 \cdot \sec^2(\theta)\,d\theta$, while $\theta = \arctan\left(\frac{x}{5}\right)$ and:

$$25 + x^2 = 25 + 25\tan^2(\theta) = 25\left[1 + \tan^2(\theta)\right] = 25\sec^2(\theta)$$

For $x^2 - 13$, use $x = \sqrt{13}\sec(\theta) \Rightarrow dx = \sqrt{13}\sec(\theta) \cdot \tan(\theta)\,d\theta$, while $\theta = \operatorname{arcsec}\left(\frac{x}{\sqrt{13}}\right)$ and:

$$x^2 - 13 = 13\sec^2(\theta) - 13 = 13\left[\sec^2(\theta) - 1\right] = 13\tan^2(\theta)$$

2. $x = 5\sin(\theta) \Rightarrow dx = 5\cos(\theta)\,d\theta$, while $\theta = \arcsin\left(\frac{x}{5}\right)$ so:

$$25 - x^2 = 25\left[1 - \sin^2(\theta)\right] = 25\cos^2(\theta)$$

and the integral becomes:

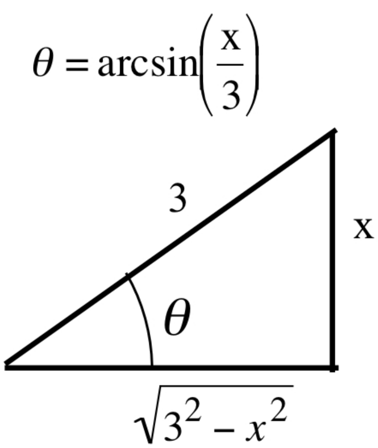
$$\int \frac{1}{\sqrt{25 - x^2}}\,dx = \int \frac{1}{\sqrt{25\cos^2(\theta)}} \cdot 5\cos(\theta)\,d\theta = \frac{5}{5}\int \frac{\cos(\theta)}{\cos(\theta)}\,d\theta$$
$$= \int 1\,d\theta = \theta + C = \arcsin\left(\frac{x}{5}\right) + C$$

3. Use $x = 3\sin(\theta) \Rightarrow dx = 3\cos(\theta)\,d\theta$ while $\theta = \arcsin\left(\frac{x}{2}\right)$ and:

$$9 - x^2 = 9\left[1 - \sin^2(\theta)\right] = 9\cos^2(\theta)$$

Then the integral becomes:

$$\int \frac{1}{x^2\sqrt{9 - x^2}}\,dx = \int \frac{1}{9\sin^2(\theta)\sqrt{9\cos^2(\theta)}} \cdot 3\cos(\theta)\,d\theta$$
$$= \int \frac{3\cos(\theta)}{27\sin^2(\theta)\cos(\theta)}\,d\theta = \frac{1}{9}\int \csc^2(\theta)\,d\theta$$
$$= -\frac{1}{9}\cot(\theta) + C = -\frac{1}{9}\cot\left(\arcsin\left(\frac{x}{2}\right)\right) + C$$
$$= -\frac{1}{9} \cdot \frac{\sqrt{9 - x^2}}{x} + C$$



8.5 Integrals of Trigonometric Functions

In the previous section, we learned how to turn integrands involving various radical and rational expressions containing the variable x into functions consisting of products of powers of trigonometric functions of θ. An overwhelming number of combinations of trigonometric functions can appear in these integrals, but fortunately most fall into a few general patterns—and most can be integrated using reduction formulas and integral tables. This section examines some of these patterns and illustrates how to obtain some of their integrals.

Integrals of functions of this type also arise in other mathematical applications, such as Fourier series.

Products of $\sin(ax)$ *and* $\cos(bx)$

We can handle the integrals $\int \sin(ax) \cdot \sin(bx)\,dx$, $\int \cos(ax) \cdot \cos(bx)\,dx$ and $\int \sin(ax) \cdot \cos(bx)\,dx$ by referring to the trigonometric identities for sums and differences of sine and cosine:

$$\begin{aligned}
\sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B)\\
\sin(A-B) &= \sin(A)\cos(B) - \cos(A)\sin(B)\\
\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B)\\
\cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B)
\end{aligned}$$

By adding or subtracting pairs of identities, we can write products such as $\sin(ax)\cos(bx)$ as a sum or difference of single sines or cosines. For example, by adding the first two identities, we get:

$$\begin{aligned}
& 2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)\\
\Rightarrow \quad & \sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]
\end{aligned}$$

Using this last identity (for $a \neq b$):

$$\begin{aligned}
\int \sin(ax)\cos(bx)\,dx &= \int \frac{1}{2}\Big[\sin\left((a+b)x\right) + \sin\left((a-b)x\right)\Big]\,dx\\
&= \frac{1}{2}\left[-\frac{\cos\left((a+b)x\right)}{a+b} - \frac{\cos\left((a-b)x\right)}{a-b}\right] + C
\end{aligned}$$

The other integrals of products of sine and cosine follow similarly.

If $a \neq b$, then:

$$\begin{aligned}
\int \sin(ax)\sin(bx)\,dx &= \frac{1}{2}\left[\frac{\sin\left((a-b)x\right)}{a-b} - \frac{\sin\left((a+b)x\right)}{a+b}\right] + C\\
\int \cos(ax)\cos(bx)\,dx &= \frac{1}{2}\left[\frac{\sin\left((a-b)x\right)}{a-b} + \frac{\sin\left((a+b)x\right)}{a+b}\right] + C\\
\int \sin(ax)\cos(bx)\,dx &= -\frac{1}{2}\left[\frac{\cos\left((a-b)x\right)}{a-b} + \frac{\cos\left((a+b)x\right)}{a+b}\right] + C
\end{aligned}$$

If $a = b$, we have already developed the relevant integral patterns:

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C = \frac{x}{2} - \frac{\sin(ax)\cdot\cos(ax)}{2a} + C$$
$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C = \frac{x}{2} + \frac{\sin(ax)\cdot\cos(ax)}{2a} + C$$
$$\int \sin(ax)\cos(ax)\,dx = \frac{\sin^2(ax)}{2a} + C = \frac{1-\cos(2ax)}{4a} + C$$

The first and second of these integral formulas follow from the identities $\sin^2(ax) = \frac{1}{2} - \frac{1}{2}\cos^2(2ax)$ and $\cos^2(ax) = \frac{1}{2} + \frac{1}{2}\cos^2(2ax)$. The third can be obtained by changing the variable to $u = \sin(ax)$.

Powers of Sine and Cosine Alone: $\int \sin^n(x)\,dx$ *or* $\int \cos^n(x)\,dx$

See Problems 25 and 26 from Section 8.2.

We can find antiderviatives of $\sin^n(x)$ or $\cos^n(x)$ using integration by parts or reduction formulas that we obtained using integration by parts. For small values of n we can also find the antiderivatives directly.

For **even powers** of sine or cosine, we can reduce the exponent by repeatedly applying the identities $\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$ and $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$.

Example 1. Evaluate $\int \sin^4(x)\,dx$.

Solution. Applying the identity for $\sin^2(\theta)$, we can write $\sin^4(x)$ as:

$$\left[\sin^2(x)\right]^2 = \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]^2 = \frac{1}{4}\left[1 - 2\cos(2x) + \cos^2(2x)\right]$$

and integrating gives:

$$\begin{aligned}\int \sin^4(x)\,dx &= \frac{1}{4}\int \left[1 - 2\cos(2x) + \cos^2(2x)\right]\,dx \\ &= \frac{1}{4}\left[x - \sin(2x) + \frac{x}{2} + \frac{1}{8}\sin(4x)\right] + C \\ &= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C\end{aligned}$$

using the formula for $\int \cos^2(2u)\,du$. ◀

Practice 1. Evaluate $\int \cos^4(x)\,dx$.

For **odd powers** of sine or cosine we can split off one factor of sine or cosine and rewrite the remaining even power using the identities $\sin^2(\theta) = 1 - \cos^2(\theta)$ or $\cos^2(\theta) = 1 - \sin^2(\theta)$, then integrate by changing the variable.

Example 2. Evaluate $\int \sin^5(x)\,dx$.

Solution. First split off one power of sine, writing:

$$\sin^5(x) = \sin^4(x) \cdot \sin(x) = \left[\sin^2(x)\right]^2 \cdot \sin(x) = \left[1 - \cos^2(x)\right]^2 \sin(x)$$

and then integrate, using the substitution $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$:

$$\begin{aligned}\int \sin^5(x)\,dx &= \int \left[1-\cos^2(x)\right]^2 \sin(x)\,dx = -\int \left[1-u^2\right]^2\,du \\ &= -\int \left[1 - 2u^2 + u^4\right]\,du = -\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right] + C \\ &= -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + C\end{aligned}$$

The reduction formula obtained in Problem 25 of Section 8.2 yields:

$$\int \sin^5(x)\,dx = \frac{1}{5}\sin^4(x)\cos(x) - \frac{4}{15}\sin^2(x)\cos(x) - \frac{8}{15}\cos(x) + K$$

which looks nothing like the result above, but these functions (aside from the different constants of integration) are in fact equal. ◀

You should be able to *see* this by graphing the two functions, and *prove* this using trig identities.

Practice 2. Evaluate $\int \cos^5(x)\,dx$.

Patterns for $\int \sin^m(x)\cos^n(x)\,dx$

For integrands of the form $\sin^m(x)\cos^n(x)$, if the **exponent of sine is odd**, you can split off one factor of $\sin(x)$ and use the identity $\sin^2(x) = 1 - \cos^2(x)$ to rewrite the remaining even power of sine in terms of cosine, then change the variable using $u = \cos(x)$.

Example 3. Evaluate $\int \sin^3(x)\cos^6(x)\,dx$.

Solution. First split off a power of sine, writing:

$$\sin^3(x)\cos^6(x) = \sin(x)\sin^2(x)\cos^6(x) = \sin(x)\left[1-\cos^2(x)\right]\cos^6(x)$$

and then use the substitution $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$:

$$\begin{aligned}\int \sin^3(x)\cos^6(x) &= \int \sin(x)\left[1-\cos^2(x)\right]\cos^6(x)\,dx \\ &= \int -\left[1-u^2\right]u^6\,du = \int \left[u^8 - u^6\right]\,du \\ &= \frac{1}{9}u^9 - \frac{1}{7}u^7 + C = \frac{1}{9}\cos^9(x) - \frac{1}{7}\cos^7(x) + C\end{aligned}$$

You can verify this is the correct antiderivative by differentiating the result and comparing it to the original integrand. ◀

You may need to use some trig identities.

Practice 3. Evaluate $\int \sin^3(x)\cos^4(x)\,dx$.

If the **exponent of cosine** is odd, split off one $\cos(x)$ and use the identity $\cos^2(x) = 1 - \sin^2(x)$ to rewrite the remaining even power of cosine in terms of sine. Then use the change of variable $u = \sin(x)$.

If **both exponents are even**, use the identities $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$ and $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ to rewrite the integral in terms of powers of $\cos(2x)$, then proceed by integrating even powers of cosine.

Powers of Secant or Tangent Alone

You integrate any power of $\sec(x)$ and $\tan(x)$ by knowing that:

$$\int \sec(x)\,dx = \ln\left(|\sec(x) + \tan(x)|\right) + C$$
$$\text{and} \quad \int \tan(x)\,dx = \ln\left(|\sec(x)|\right) + C$$

and using the reduction formulas:

See Problems 27 and 28 in Section 8.2.

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\cdot\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$
$$\text{and} \quad \int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

Example 4. Evaluate $\int \sec^3(x)\,dx$.

Solution. Using the first reduction formula with $n = 3$:

$$\begin{aligned}\int \sec^3(x)\,dx &= \frac{1}{2}\sec(x)\cdot\tan(x) + \frac{1}{2}\int \sec(x)\,dx \\ &= \frac{1}{2}\sec(x)\cdot\tan(x) + \frac{1}{2}\ln\left(|\sec(x)+\tan(x)|\right) + C\end{aligned}$$

You could also write $\sec^3(x) = \sec(x)\cdot\sec^2(x)$ and work out this integral directly using integration by parts. ◀

Practice 4. Evaluate $\int \tan^3(x)\,dx$ and $\int \sec^5(x)\,dx$.

Patterns for $\int \sec^m(x)\tan^n(x)\,dx$

The patterns for evaluating $\int \sec^m(x)\tan^n(x)\,dx$ resemble those for $\int \sin^m(x)\cos^n(x)\,dx$: we treat the even and odd powers differently and we use the identities $\tan^2(\theta) = \sec^2(\theta) - 1$ and $\sec^2(\theta) = \tan^2(\theta) + 1$.

If the **exponent of secant is even**, factor off $\sec^2(x)$, replace the other even powers (if any) of secant using $\sec^2(x) = \tan^2(x) + 1$, and then make the change of variable $u = \tan(x) \Rightarrow du = \sec^2(x)\,dx$.

If the **exponent of tangent is odd**, factor off $\sec(x)\tan(x)$, replace any remaining even powers of tangent using $\tan^2(x) = \sec^2(x) - 1$, and then make the change of variable $u = \sec(x) \Rightarrow du = \sec(x)\tan(x)\,dx$.

If the **exponent of secant is odd and the exponent of tangent is even**, replace the even powers of tangent using $\tan^2(x) = \sec^2(x) - 1$. Then the integral contains only powers of secant, and you can use the strategy for integrating powers of secant alone.

Example 5. Evaluate $\int \sec(x)\tan^2(x)\,dx$.

Solution. Because the exponent of secant is odd and and the exponent of tangent is even, we can use the last method mentioned above and replace $\tan^2(x)$ with $\sec^2(x) - 1$. Then:

$$\begin{aligned}\int \sec(x)\tan^2(x)\,dx = \int \sec(x)\left[\sec^2(x)-1\right]\,dx &= \int \sec^3(x)\,dx - \int \sec(x)\,dx \\ &= \frac{1}{2}\sec(x)\cdot\tan(x) + \frac{1}{2}\ln\left(|\sec(x)+\tan(x)|\right) - \ln\left(|\sec(x)+\tan(x)|\right) + C \\ &= \frac{1}{2}\sec(x)\cdot\tan(x) - \frac{1}{2}\ln\left(|\sec(x)+\tan(x)|\right) + C\end{aligned}$$

where we used the result of Example 4 for $\int sec^3(x)\,dx$. ◀

Practice 5. Evaluate $\int \sec^4(x)\tan^2(x)\,dx$.

Wrap-Up

Even if you use integral tables (or computers) for most of your future work, it is important to realize that most of the integral patterns for products of powers of trigonometric functions can be obtained using some basic trigonometric identities and the techniques we have discussed in this and earlier sections.

8.5 Problems

In Problems 1–36, evaluate the integral. (More than one method works for some of the integrals.)

1. $\int \sin^2(3x)\,dx$
2. $\int \cos^2(5x)\,dx$
3. $\int e^x\sin(e^x)\cos(e^x)\,dx$
4. $\int \frac{1}{x}\sin^2\left(\ln(x)\right)\,dx$
5. $\int_0^\pi \sin^4(3x)\,dx$
6. $\int_0^\pi \cos^4(5x)\,dx$
7. $\int_0^\pi \cos^3(5x)\,dx$
8. $\int_0^\pi \sin^3(7x)\,dx$
9. $\int \sin(7x)\cos(7x)\,dx$
10. $\int \sin(7x)\cos^2(7x)\,dx$
11. $\int \sin(7x)\cos^3(7x)\,dx$
12. $\int \sin^2(\pi x)\cos(\pi x)\,dx$
13. $\int \sin^2(3x)\cos^2(3x)\,dx$
14. $\int \sin^2(\pi x)\cos^3(\pi x)\,dx$
15. $\int \sin^5(x)\cos^2(x)\,dx$
16. $\int \sin^2(x)\cos^5(x)\,dx$
17. $\int \sec^4(4x)\,dx$
18. $\int \tan^4(4x)\,dx$
19. $\int \tan^5(4x)\,dx$
20. $\int \sec^3(4x)\,dx$
21. $\int \sec^2(5x)\tan(5x)\,dx$
22. $\int \sec^2(5x)\tan^2(5x)\,dx$
23. $\int \sec^3(5x)\tan(5x)\,dx$
24. $\int \sec^3(5x)\tan^2(5x)\,dx$

25. $\int \sec^4(\theta)\tan(\theta)\,d\theta$

26. $\int \sec^4(\theta)\tan^2(\theta)\,d\theta$

27. $\int \sec^4(\theta)\tan^4(\theta)\,d\theta$

28. $\int \sec^4(\theta)\tan^{2015}(\theta)\,d\theta$

29. $\int \frac{\sin^2(x)}{\cos^2(x)}\,dx$

30. $\int \frac{1}{\cos^2(x)}\,dx$

31. $\int \cos^4(\theta)\tan^4(\theta)\,d\theta$

32. $\int \cos^4(\theta)\tan^2(\theta)\,d\theta$

33. $\int \sin(x)\cos(3x)\,dx$

34. $\int \sin(7x)\cos(3x)\,dx$

35. $\int \sin(x)\sin(3x)\,dx$

36. $\int \cos(7x)\cos(3x)\,dx$

37. Show that if n is a positive, **odd** integer, then:
$$\int_0^{2\pi} \sin^n(x)\,dx = 0$$

38. Using integral tables or reduction formulas, it is straightforward to show that:
$$\int_0^{2\pi} \sin^2(x)\,dx = \pi$$
$$\int_0^{2\pi} \sin^4(x)\,dx = \frac{3}{4}\pi$$
$$\int_0^{2\pi} \sin^6(x)\,dx = \frac{3}{4}\cdot\frac{5}{6}\pi$$
Evaluate $\int_0^{2\pi} \sin^8(x)\,dx$, then make a prediction about the value of $\int_0^{2\pi} \sin^{10}(x)\,dx$ and evaluate that integral.

The definite integrals of various combinations of sine and cosine on the interval $[0, 2\pi]$ exhibit a number of interesting patterns. For now, these are simply curiosities and a source of additional practice problems, but the patterns are very important as the foundation for an applied topic, Fourier series, which you may encounter in more advanced courses. Problems 39–41 ask you to show that the definite integral on $[0, 2\pi]$ of $\sin(mx)$ multiplied by almost any other $\sin(nx)$ or $\cos(nx)$ is 0. The only nonzero value comes when $\sin(mx)$ is multiplied by itself.

39. Show that if m and n are integers with $m \neq n$, then:
$$\int_0^{2\pi} \sin(mx)\cdot\sin(nx)\,dx = 0$$

40. Show that if m and n are integers, then:
$$\int_0^{2\pi} \sin(mx)\cdot\cos(nx)\,dx = 0$$
(Consider $m = n$ and $m \neq n$ separately.)

41. Show that if $m \neq 0$ is an integer, then:
$$\int_0^{2\pi} \sin(mx)\cdot\sin(mx)\,dx = \pi$$

Problems 42–47, concern the following function $P(x)$, a **trigonometric polynomial**:
$$P(x) = 5\sin(x) + 7\cos(x) - 4\sin(2x) + 8\cos(2x) - 2\sin(3x)$$

In Problems 42–45, use the *results* of Problems 39–41 to quickly evaluate each integral.

42. $a_1 = \frac{1}{\pi}\int_0^{2\pi} \sin(1x)\cdot P(x)\,dx$

43. $a_2 = \frac{1}{\pi}\int_0^{2\pi} \sin(2x)\cdot P(x)\,dx$

44. $a_3 = \frac{1}{\pi}\int_0^{2\pi} \sin(3x)\cdot P(x)\,dx$

45. $a_4 = \frac{1}{\pi}\int_0^{2\pi} \sin(4x)\cdot P(x)\,dx$

46. Describe how the values of a_k in Problems 42–45 are related to the coefficients of $P(x)$, then make up your own trigonometric polynomial $Q(x)$ and see if your description holds for the a_k values calculated from $Q(x)$.

47. Just by knowing the a_k values we can "rebuild" part of P(x). Find a similar method for getting the coefficients of the cosine terms of $P(x)$: $b_k = ??$

8.5 Practice Answers

1. Using $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$, we can write $\cos^4(x)$ as:

$$\left[\cos^2(x)\right]^2 = \left[\frac{1}{2} + \frac{1}{2}\cos(2x)\right]^2 = \frac{1}{4}\left[1 + 2\cos(2x) + \cos^2(2x)\right]$$

and integrating gives:

$$\begin{aligned}\int \cos^4(x)\,dx &= \frac{1}{4}\int \left[1 + 2\cos(2x) + \cos^2(2x)\right]\,dx \\ &= \frac{1}{4}\left[x + \sin(2x) + \frac{x}{2} + \frac{1}{8}\sin(4x)\right] + C \\ &= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C\end{aligned}$$

2. First split off one power of sine, writing:

$$\cos^5(x) = \cos^4(x)\cdot\cos(x) = \left[\cos^2(x)\right]^2\cdot\cos(x) = \left[1 - \sin^2(x)\right]^2\cos(x)$$

and then integrate, using $u = \sin(x) \Rightarrow du = \cos(x)\,dx$:

$$\begin{aligned}\int \cos^5(x)\,dx = \int \left[1 - \sin^2(x)\right]^2\cos(x)\,dx &= \int \left[1 - u^2\right]^2\,du \\ = \int \left[1 - 2u^2 + u^4\right]\,du &= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right] + C \\ = \sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C\end{aligned}$$

3. First split off a power of sine, writing:

$$\sin^3(x)\cos^4(x) = \sin(x)\sin^2(x)\cos^4(x) = \sin(x)\left[1 - \cos^2(x)\right]\cos^4(x)$$

and then use the substitution $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$:

$$\begin{aligned}\int \sin^3(x)\cos^4(x) &= \int \sin(x)\left[1 - \cos^2(x)\right]\cos^4(x)\,dx \\ &= \int -\left[1 - u^2\right]u^4\,du = \int \left[u^6 - u^4\right]\,du \\ &= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + C\end{aligned}$$

4. For the first integral, write:

$$\tan^3(x) = \tan(x)\cdot\tan^2(x) = \tan(x)\left[\sec^2(x) - 1\right]$$

so that the integral becomes:

$$\begin{aligned}\int \tan^3(x)\,dx &= \int \left[\tan(x)\sec^2(x) - \tan(x)\right]\,dx \\ &= \frac{1}{2}\tan^2(x) - \ln\left(|\sec(x)|\right) + C \\ &= \frac{1}{2}\tan^2(x) + \ln\left(|\cos(x)|\right) + C\end{aligned}$$

For the second integral, use the reduction formula (twice):

$$\begin{aligned}\int \sec^5(x)\,dx &= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{4}\int \sec^3(x)\,dx \\ &= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{4}\left[\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx\right] \\ &= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x) \\ &\qquad\qquad + \frac{3}{8}\ln\left(|\sec(x)+\tan(x)|\right) + C\end{aligned}$$

5. First write:

$$\begin{aligned}\sec^4(x)\tan^2(x) &= \sec^2(x)\cdot\sec^2(x)\cdot\tan^2(x) \\ &= \sec^2(x)\left[1+\tan^2(x)\right]\tan^2(x)\end{aligned}$$

and then use the substitution $u = \tan(x) \Rightarrow du = \sec^2(x)\,dx$:

$$\begin{aligned}\int \sec^4(x)\tan^2(x) &= \int \left[1+u^2\right]u^2\,du = \int \left[u^2+u^4\right]\,du \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \frac{1}{3}\tan^3(x) + \frac{1}{5}\tan^5(x) + C\end{aligned}$$

8.6 Integration Tactics

This section, currently in preparation, will be included in the official first printing of this edition. It will present a review of the integrations tactics presented earlier in the chapter, outline strategies for selecting an appropriate integration method to attempt to find an antiderivative of a given integrand, explore some more advanced (optional) integration techniques, and include integrands involving hyperbolic and inverse hyperbolic functions.

For the problems starting on the next page, use any integration technique you have learned so far to evaluate the given integrals.

8.6 Problems

1. $\int \sqrt{1-x}\,dx$

2. $\int (1+2x)^{2017}\,dx$

3. $\int x\sqrt{a^2-x^2}\,dx$

4. $\int x\sqrt{1-x}\,dx$

5. $\int \sqrt{a+bx}\,dx$

6. $\int \frac{1}{1+4x^2}\,dx$

7. $\int \frac{x}{1-x^4}\,dx$

8. $\int \frac{\sec^2(\theta)}{1+\tan(\theta)}\,d\theta$

9. $\int \frac{x^4+x^3+1}{x^3}\,dx$

10. $\int \frac{y^2}{\sqrt{1-y^3}}\,dy$

11. $\int \frac{\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{3}{2}}}{x^{\frac{1}{3}}}\,dx$

12. $\int \frac{4x}{(a^2-x^2)^2}\,dx$

13. $\int \left(\sqrt{a}-\sqrt{x}\right)^2\,dx$

14. $\int \frac{3}{1-4x}\,dx$

15. $\int \frac{x^2}{\sqrt{1+x^3}}\,dx$

16. $\int \frac{2x-1}{2x+3}\,dx$

17. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

18. $\frac{1}{4}\int \left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)^2\,dx$

19. $\int \frac{x^3-x^2+2x-1}{1+x^2}\,dx$

20. $\int \frac{\sec^2(\varphi)}{\sqrt{1+\tan(\varphi)}}\,d\varphi$

21. $\int \frac{x^3-5x+7}{3x-4}\,dx$

22. $\int \frac{1}{e^t}\,dt$

23. $\int \frac{\sin(2\theta)}{a+b\cos(2\theta)}\,d\theta$

24. $\int \frac{1}{1+\sin(\varphi)}\,d\varphi$

25. $\int \frac{\sin(\theta)}{\cos^5(\theta)}\,d\theta$

26. $\int \frac{1}{4x^2+9}\,dx$

27. $\int \frac{1}{1-2x+2x^2}\,dx$

28. $\int \frac{x+1}{x^2+4}\,dx$

29. $\int \frac{x^2+x}{x^2+1}\,dx$

30. $\int \frac{e^y}{\sqrt{1-e^{2y}}}\,dy$

31. $\int \frac{e^{2t}}{\sqrt{1-e^{2t}}}\,dt$

32. $\int \frac{\sin(3\alpha)}{1+\cos^2(3\alpha)}\,d\alpha$

33. $\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$

34. $\int \frac{1}{x\left[1+(\ln(x))^2\right]}\,dx$

35. $\int \frac{x}{(1+x^2)^2}\,dx$

36. $\int x^{2017}\ln(x)\,dx$

37. $\int x^2\arcsin(x)\,dx$

38. $\int t\tan^2(t)\,dt$

39. $\int x^3\sqrt{a^2-x^2}\,dx$

40. $\int x^3\sqrt{a^2-x^4}\,dx$

41. $\int \cos(\theta)\cdot\ln\left(\sin(\theta)\right)\,d\theta$

42. $\int \frac{x^2}{(9-x^2)^{\frac{3}{2}}}\,dx$

43. $\int \frac{x^2}{(1+x^2)^2}\,dx$

44. $\int \frac{y^3}{\sqrt{25-y^2}}\,dy$

45. $\int \frac{\sqrt{x}}{1+x}\,dx$

46. $\int \frac{1}{1-\sqrt{x}}\,dx$

47. $\int \frac{x+3}{\sqrt{1+2x}}\,dx$

48. $\int \frac{x}{(x+1)^2}\,dx$

49. $\int e^{2x}\sin\left(e^x\right)\,dx$

50. $\int x^3 e^{x^2}\,dx$

51. $\int \frac{x^3}{\left(a^2+x^2\right)^2}\,dx$

52. $\int \cos\left(\sqrt{t}\right)\,dt$

53. $\int x^2\sqrt{1-x}\,dx$

54. $\int \frac{\sec^2(\theta)\tan^2(\theta)}{\sqrt{1+\tan(\theta)}}\,d\theta$

55. $\int \frac{x^3-x}{2+3x}\,dx$

56. $\int x\cos(3x)\,dx$

57. $\int \left[1+\cos\left(\frac{\theta}{2}\right)\right]^3 \sin\left(\frac{\theta}{2}\right)\,d\theta$

58. $\int \sin^2(2\varphi)\cos(2\varphi)\,d\varphi$

59. $\int \frac{1-e^{-x}}{1+e^{-x}}\,dx$

60. $\int \frac{1}{(1-x)^4}\,dx$

61. $\int \frac{x^2}{(1-x)^4}\,dx$

62. $\int \frac{1}{x^2+8x+20}\,dx$

63. $\int \cot(\theta)\cdot\ln\left(\sin(\theta)\right)\,d\theta$

64. $\int \frac{1}{x\left[\ln(x)\right]^2}\,dx$

65. $\int \frac{x^2}{\sqrt{1-x^6}}\,dx$

66. $\int e^{2x}\sqrt{1-e^{2x}}\,dx$

67. $\int \cos\left(\frac{x}{2}\right)\,dx$

68. $\int e^{\tan(\theta)}\sec^2(\theta)\,d\theta$

69. $\int \frac{x^3+x^2+x}{x^2+9}\,dx$

70. $\int \left(a^{\frac{2}{3}}-y^{\frac{2}{3}}\right)^3\,dy$

71. $\int \frac{x+x^3}{\sqrt{4-x^4}}\,dx$

72. $\int \frac{x^3}{\left(1-x^2\right)^2}\,dx$

73. $\int \theta\sec^2(\theta)\,d\theta$

74. $\int \frac{\arctan(x)}{1+x^2}\,dx$

75. $\int e^{\sqrt{x}}\,dx$

76. $\int_0^a y\left(b-\frac{b}{a}y\right)^3\,dy$

77. $\int \frac{x-x^2}{1+x^2}\,dx$

78. $\int e^{-x}\left(1+e^{-x}\right)^3\,dx$

79. $\left(1-x^3\right)^2\,dx$

80. $\int x\,(a-x)^{\frac{5}{2}}\,dx$

81. $\int \frac{1}{x^2-6x+10}\,dx$

82. $\int \frac{x^3}{1+x^8}\,dx$

83. $\int \frac{e^x}{1-3e^x}\,dx$

84. $\int \frac{\sqrt{1+\ln(x)}}{x}\,dx$

85. $\int \left[\ln(x)\right]^2\,dx$

86. $\int \frac{1}{\sqrt{x}(1+x)}\,dx$

87. $\int x^5 e^{-x^3}\,dx$

88. $\int \frac{x^2}{x^2-4}\,dx$

89. $\int \frac{2x^2+x-1}{x^3+x^2-4x-4}\,dx$

90. $\int \frac{x}{1-x^4}\,dx$

91. $\int \frac{x^3}{x^2+3x+2}\,dx$

92. $\int \frac{x^4}{x^3+2x^2-x-2}\,dx$

93. $\int \frac{e^{3t}}{1-e^{2t}}\,dt$

94. $\int \frac{x^3-1}{x(x+1)^3}\,dx$

95. $\int \frac{1}{x^3-x^2}\,dx$

96. $\int \frac{x^3-1}{x(x-2)^2}\,dx$

97. $\displaystyle\int \frac{1}{(x^2+x)(x-1)^2}\,dx$

98. $\displaystyle\int \frac{1}{e^{2y}-2e^y}\,dy$

99. $\displaystyle\int \frac{x^2}{x^4+12x^3+52x^2+96x+64}\,dx$

100. $\displaystyle\int \frac{x^2+4x+10}{x^3+2x^2+5x}\,dx$

101. $\displaystyle\int \frac{x^2}{x^2-4x+5}\,dx$

102. $\displaystyle\int \frac{1}{x^3+4x^2+8x}\,dx$

103. $\displaystyle\int \frac{x^2}{25-x^4}\,dx$

104. $\displaystyle\int \frac{x}{1-x^8}\,dx$

105. $\displaystyle\int \frac{x}{x^2+x+1}\,dx$

106. $\displaystyle\int \frac{x}{x^3+x^2+4x+4}\,dx$

107. $\displaystyle\int \frac{x^3}{(1+x^2)^2}\,dx$

108. $\displaystyle\int \frac{x}{x^2-2x+2}\,dx$

109. $\displaystyle\int \frac{1}{\tan(\theta)-\cot(\theta)}\,d\theta$

110. $\displaystyle\int \frac{x^2-2}{1+6x-x^3}\,dx$

111. $\displaystyle\int \frac{x^4+1}{(1+x^2)^2}\,dx$

112. $\displaystyle\int_0^8 \frac{1}{1+\sqrt[3]{x}}\,dx$

113. $\displaystyle\int_0^{\ln(2)} \sqrt{e^t-1}\,dt$

114. $\displaystyle\int_0^4 \frac{1}{1+\sqrt{x}}\,dx$

115. $\displaystyle\int_2^3 \frac{x}{1-x^4}\,dx$

116. $\displaystyle\int \sqrt{25-9x^2}\,dx$

117. $\displaystyle\int \frac{1}{\sqrt{10x-x^2}}\,dx$

118. $\displaystyle\int \frac{x^2}{\sqrt{4x-x^2}}\,dx$

119. $\displaystyle\int \frac{1}{(9x^2-36x+32)^{\frac{3}{2}}}\,dx$

120. $\displaystyle\int \frac{\left(1-x^2\right)^{\frac{3}{2}}}{x^6}\,dx$

121. $\displaystyle\int_0^{\frac{\pi}{2}} \cos(2\varphi)\cos(5\varphi)\,d\varphi$

122. $\displaystyle\int \frac{\cos^3(\theta)}{\sin^4(\theta)}\,d\theta$

123. $\displaystyle\int \sin(\varphi)\sin(3\varphi)\sin(5\varphi)\,d\varphi$

124. $\displaystyle\int \tan(x)\sqrt{\sec(x)}\,dx$

125. $\displaystyle\int x\cosh(x)\,dx$

126. $\displaystyle\int x^2\sinh(x)\,dx$

127. $\displaystyle\int e^x\sinh(x)\,dx$

128. $\displaystyle\int \tanh(x)\,dx$

129. $\displaystyle\int \coth(x)\,dx$

130. $\displaystyle\int \operatorname{sech}(x)\,dx$

131. $\displaystyle\int \operatorname{argtanh}(x)\,dx$

132. $\displaystyle\int \operatorname{argsinh}(x)\,dx$

133. $\displaystyle\int x\operatorname{argsinh}(x)\,dx$

8.7 *MacLaurin Polynomials*

In this chapter you have learned to find antiderivatives of a wide variety of elementary functions, but many more such functions fail to have an antiderivative that can be expressed in terms of other elementary functions. By this point, you should easily be able to find antiderviatives for $\sin(x)$, $x \cdot \sin(x)$, $x \cdot \sin(x^2)$ or $x \cdot e^{-x^2}$, but no matter how hard you try, you won't be able to find antiderivatives for $\sin(x^2)$, $\sin(x^3)$ or e^{-x^2}.

An **elementary function** is a function that can be expressed using a finite number of compositions or combinations of exponential functions, logarithms, trigonometric and inverse trig functions, polynomials and constants, using sums, products and exponentiation.

In Section 2.8, you used linear approximations (tangent lines) to approximate values of complicated functions. In this section, we will investigate how to use polynomials — slightly more complicated, but still relatively "nice" functions — to approximate integrands such as $\sin(x^2)$ and then approximate values of definite integrals such as $\int_0^1 \sin(x^2)\,dx$. This approach will motivate the exploration of many of the concepts to follow in the next two chapters.

Proving that you can't find elementary antiderviatives of integrands such as $\sin(x^2)$ turns out to be quite complicated.

Polynomials

Consider any (non-vertical) line: if you know its y-intercept, b, and its slope, m, you can write down an equation of the line: $y = b + mx$. If we write $y = f(x)$, then $f(0) = b + m \cdot 0 = b$ and $f'(0) = m$, so an equation of *any* linear function is completely determined by its value at $x = 0$ and the value of its first derivative at $x = 0$. It turns out that if you know the values of any polynomial $P(x)$ and all of its derivatives at $x = 0$, you can use those values to find a formula for $P(x)$.

Example 1. If $P(x)$ is a cubic polynomial with $P(0) = 7$, $P'(0) = 5$, $P''(0) = 16$ and $P'''(0) = 18$, find a formula for $P(x)$.

Because $P(x)$ is a cubic, its derivatives of order four and higher are all 0.

Solution. Because $P(x)$ is a cubic polynomial, we can write it as:

$$P(x) = A + Bx + Cx^2 + Dx^3$$

for some numbers A, B, C and D. We know that $P(0) = 7$, and substituting $x = 0$ into the formula above tells us that $P(0) = A$, so $A = 7$. We also know that $P'(0) = 5$, while:

$$P'(x) = B + 2Cx + 3Dx^2 \;\Rightarrow\; P'(0) = B$$

so $B = 5$. Similarly, we know that $P''(0) = 16$ while:

$$P''(x) = 2C + 3 \cdot 2Dx \;\Rightarrow\; P''(0) = 2C$$

so $2C = 16 \Rightarrow C = \frac{16}{2} = 8$. Finally, we know that $P'''(0) = 18$ while:

$$P'''(x) = 3 \cdot 2 \cdot D \;\Rightarrow P'''(0) = 6D \;\Rightarrow\; 6D = 18 \;\Rightarrow D = 3$$

Therefore $P(x) = 7 + 5x + 8x^2 + 3x^3$. (You should verify that this cubic polynomial and its derivatives have the values specified above.) ◀

Practice 1. If $P(x)$ is a fourth-degree polynomial with $P(0) =$ ˘3, $P'(0) = 4$, $P''(0) = 10$, $P'''(0) = 12$ and $P^{(4)}(0) = 24$, find a formula for $P(x)$.

Now consider a general polynomial of order 5 with coefficients a_0, a_1, a_2, a_3, a_4, a_5:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

Observe that $P(0) = a_0$ and then differentiate to get:

$$P'(x) = a_1 + 2 \cdot a_2 x + 3 \cdot a_3 x^2 + 4 \cdot a_4 x^3 + 5 \cdot a_5 x^4$$

Putting $x = 0$ into this new equation tells us that $P'(0) = a_1$. Differentiating again yields:

$$P''(x) = 2 \cdot a_2 + 3 \cdot 2 \cdot a_3 x + 4 \cdot 3 \cdot a_4 x^2 + 5 \cdot 4 \cdot a_5 x^3$$

so that $P''(0) = 2 \cdot a_2$. Differentiating again:

$$P'''(x) = 3 \cdot 2 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4 x + 5 \cdot 4 \cdot 3 \cdot a_5 x^2$$

so that $P'''(0) = 3 \cdot 2 \cdot a_3$. Continuing this process, $P^{(4)}(0) = 4 \cdot 3 \cdot 2 \cdot a_4$ and $P^{(5)}(0) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot a_5$. In general, we can write $P^{(k)}(0) = k! \cdot a_k$, where:

We call this expression $k!$ "k **factorial**," defined for any positive integer k.

$$k! = k \cdot (k-1) \cdot (k-2) \cdots 3 \cdot 2 \cdot 1$$

and $k = 1, 2, 3, 4$ or 5. If we define $0! = 1$, then the equation $P^{(k)}(0) = k! \cdot a_k$ holds for $k = 0$ as well (the 0-th derivative of a function is just the function itself). And for any integer $k \geq 6$, $a_k = 0$ and $P^{(k)}(x) = 0$, so for any integer $k \geq 0$ we have:

$$P^{(k)}(0) = k! \cdot a_k \;\Rightarrow\; a_k = \frac{P^{(k)}(0)}{k!}$$

Practice 2. If $P(x)$ is a fourth-degree polynomial with $P(0) =$ ˘3, $P'(0) = 4$, $P''(0) = 10$, $P'''(0) = 12$ and $P^{(4)}(0) = 24$, find a formula for $P(x)$ using the above formula, then compare with your answer to Practice 1.

There was nothing special about the degree $n = 5$ of the polynomial in the preceding discussion. The formula $a_k = \dfrac{P^{(k)}(0)}{k!}$ holds for *any* polynomial of the form:

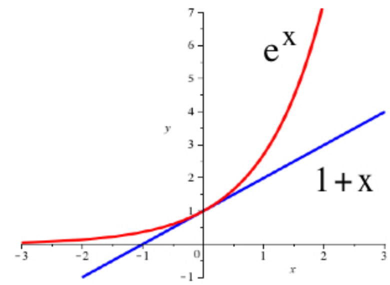
$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

And there was nothing special about the "base point" $x = 0$: we can develop similar formulas if we know the values of a function and all of its derivatives at $x = 1$ or $x = -3$ or $x = \sqrt{7}$ (but we'll stick with $x = 0$ for now, to keep things simple).

Using Polynomials to Approximate Functions

Given any function $f(x)$, we know that a tangent-line approximation to this function at $x = a$ is:

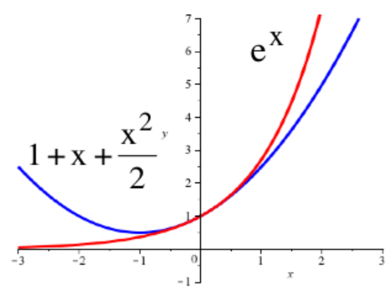
$$L(x) = f(a) + f'(a) \cdot (x - a)$$



If $a = 0$, this becomes $L(x) = f(0) + f'(0) \cdot x$. For $f(x) = e^x$, $f(0) = 1$ and $f'(x) = e^x \Rightarrow f'(0) = 1$, so $L(x) = 1 + x$ (see margin for a graph comparing $L(x)$ and $f(x)$). For values of x very close to 0, we can see that $L(x) = 1 + x \approx e^x = f(x)$ is a decent approximation, but for values of x not close to 0, we need a better approximation.

For the linear approximation $L(x)$, its 0-th derivative agrees with the 0-th derivative of $f(x)$ at $x = 0$: $L(0) = 1 = e^0 = f(0)$. Likewise, the first derivatives agree at $x = 0$: $L'(0) = 1 = e^0 = f'(0)$. But the second derivatives do not agree: $L''(0) = 0 \neq 1 = e^0 = f''(0)$. Can we find a reasonably simple function whose 0-th, first and second derivatives at $x = 0$ match those of $f(x) = e^x$? The next simplest function after a linear function is a quadratic function, so let's try $Q(x) = A + Bx + Cx^2$, for which $Q'(x) = B + 2Cx$ and $Q''(x) = 2C$. We need:

$$\begin{aligned} Q(0) = f(0) &\Rightarrow A = e^0 = 1 \\ Q'(0) = f'(0) &\Rightarrow B = e^0 = 1 \\ Q''(0) = f''(0) &\Rightarrow 2C = e^0 = 1 \Rightarrow C = \frac{1}{2} \end{aligned}$$



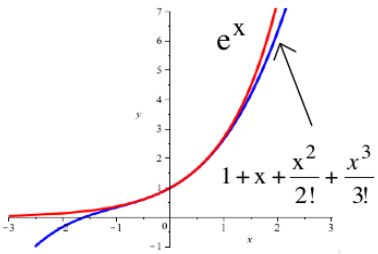
so $Q(x) = 1 + x + \frac{1}{2}x^2$ (see margin for a graph of $Q(x)$ and $f(x)$). While $L(x)$ did a decent job of approximating $f(x) = e^x$ on the interval $[-0.2, 0.2]$, our quadratic approximation $Q(x)$ appears to do a nice job on the interval $[-1, 1]$, but could we do better?

A cubic polynomial is not much more complicated than a quadratic, so let's look for something of the form $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. We want derivatives 0 through 3 of our polynomial to match derivatives 0 through 3 of the function $f(x) = e^x$. We know that:

$$f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow f''(x) = e^x \Rightarrow f'''(x) = e^x$$

so that $f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$ and $f'''(0) = 1$. We therefore need $P(0) = 1$, $P'(0) = 1$, $P''(0) = 1$ and $P'''(0) = 1$. From our work earlier in this section we know this requires that $a_k = \dfrac{P^{(k)}(0)}{k!}$ for $k = 0$, 1, 2 and 3. Thus:

$$\begin{aligned} a_0 &= \frac{P(0)}{0!} = \frac{1}{1} = 1 \\ a_1 &= \frac{P'(0)}{1!} = \frac{1}{1} = 1 \end{aligned}$$



$$a_2 = \frac{P''(0)}{2!} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$
$$a_3 = \frac{P'''(0)}{3!} = \frac{1}{3 \cdot 2 \cdot 1} = \frac{1}{6}$$

which tells us that $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ (see margin for a graphs of $P(x)$ and $f(x)$). This polynomial appears to approximate $f(x) = e^x$ quite well on an even bigger interval.

Could we do even better? You may notice a pattern in our work above. If we used a fourth-order polynomial, then we would also need:

$$a_4 = \frac{P^{(4)}(0)}{4!} = \frac{f^{(4)}(0)}{4!} = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{24}$$

so $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$ would approximate e^x even better. In general, for any positive integer n, we would have:

$$e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n$$

Named after Scottish mathematician Colin MacLaurin (1698–1746).

We call this a **MacLaurin polynomial** of order n for $f(x) = e^x$.

Example 2. Find a MacLaurin polynomial with three nonzero terms for $f(x) = \sin(x)$.

Solution. We want to find a polynomial of the form:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

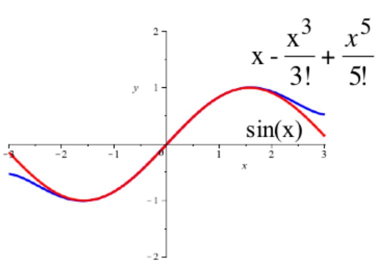
so that $P^{(k)}(0) = f^{(k)}(0)$ for three nonzero values. We know that:

$$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x)$$
$$\Rightarrow f'''(x) = -\cos(x) \Rightarrow f^{(4)}(x) = \sin(x)$$

and that this pattern then repeats. This tells us that:

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{(4)}(0) = 0,\ f^{(5)}(0) = 1$$

so that $P(0) = f(0) = 0$, $P'(0) = f'(0) = 1$, $P''(0) = f''(0) = 0$, $P'''(0) = f'''(0) = -1$, $P^{(4)}(0) = f^{(4)}(0) = 0$ and $P^{(5)}(0) = f^{(5)}(0) = 1$. Using the formula $a_k = \frac{P^{(k)}(0)}{k!}$ yields:



$$P(x) = \frac{0}{0!} + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

so that $P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ should do the job. (See the margin for a graph of this polynomial compared to $\sin(x)$.) ◀

Practice 3. Find a MacLaurin polynomial with five nonzero terms for $f(x) = \sin(x)$.

Practice 4. Find a MacLaurin polynomial with three nonzero terms for $f(x) = \cos(x)$.

Applications of MacLaurin Polynomials

In Section 7.6, we developed a concrete definition of the exponential function $\exp(x) = e^x$, but this definition did not provide us with a useful way to compute values of e^x (other than $e^0 = 1$). To estimate the value of $e = e^1$ using the definition of e^x, we would need to use the numerical techniques of Section 4.9 to evaluate $L(b) = \int_1^b \frac{1}{t}\,dt$ for various values of b until we found a value for which $L(b) \approx 1$.

In this section, however, we have found polynomials that closely approximate e^x, so we can evaluate one of these polynomials at $x = 1$ to approximate e:

$$e = e^1 \approx 1 + 1 + \frac{1}{2!} \cdot 1^2 + \frac{1}{3!} \cdot 1^3 + \frac{1}{4!} \cdot 1^4 + \frac{1}{5!} \cdot 1^5 + \frac{1}{6!} \cdot 1^6 \approx 2.178$$

Using higher-degree MacLaurin polynomials for e^x will result in even better approximations of e.

Practice 5. Use a MacLaurin polynomial to approximate $\frac{1}{e}$ and $\sqrt{e}$.

Practice 6. Use a MacLaurin polynomial to approximate $\sin(1)$ and compare you approximation to what your calculator reports for $\sin(1)$.

In Section 4.9, we learned various numerical integration techniques to approximate values of definite integrals. With enough computing power available, techniques such as the Trapezoidal Rule and Simpson's Rule allow you to approximate values of definite integrals such as $\int_0^1 e^{-x^2}\,dx$ or $\int_0^1 \sin(x^2)\,dx$ (for which it is impossible to use the Fundamental Theorem of Calculus unless we can think of an antiderivative of the integrand). Unfortunately, these approximation methods require you (or a computer) to evaluate the integrand at many different values of x. Now that we know how to approximate transcendental functions with polynomials, we might first approximate a complicated integrand with a "nice" polynomial and then integrate the polynomial instead of the transcendental function.

Example 3. Approximate the value of $\int_0^1 e^{-x^2}\,dx$.

Solution. We don't know how to find an antiderivatve for e^{-x^2}, so we can't use the Fundamental Theorem of Calculus. But we already know a MacLaurin polynomial for e^u:

$$e^u \approx 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \frac{1}{5!}u^5$$

into which we can substitute $u = -x^2$ to get:

$$e^{-x^2} \approx 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10}$$

and use this polynomial to approximate the integrand:

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &\approx \int_0^1 \left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \frac{1}{120}x^{10}\right] dx \\ &= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 - \frac{1}{1320}x^{11}\right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} \approx 0.7467\end{aligned}$$

The Trapezoidal Rule with $n = 25$ yields a similar approximation, but requires many more computations. ◀

Practice 7. Approximate the value of $\int_0^1 \sin(x^2)\,dx$.

How Good Are These Approximations?

Section 4.9 included (without proof) error bounds for the Trapezoidal Rule and Simpson's Rule that provided guarantees for how closely the results of these numerical methods approximated the exact values of the integrals we were trying to compute. We can—and will—state (and prove) a similar error bound for the MacLaurin polynomial approximations we have learned how to use in this section, but we will defer that discussion until Chapter 10.

8.7 Problems

In Problems 1–6, $P(x) = Ax + B$ is a linear polynomial, with the values of $P(0)$ and $P'(0)$ given. Find A and B and then write a formula for $P(x)$.

1. $P(0) = 5$, $P'(0) = 3$
2. $P(0) = -2$, $P'(0) = 7$
3. $P(0) = 4$, $P'(0) = -1$
4. $P(0) = 8$, $P'(0) = 5$
5. $P(0) = 4$, $P'(0) = 0$
6. $P(0) = -3$, $P'(0) = -2$
7. If $P(x) = A + Bx$, how are the values of A and B related to the values of $P(0)$ and $P'(0)$?

In 8–13, $P(x) = A + Bx + Cx^2$ is a quadratic polynomial, with values of $P(0)$, $P'(0)$ and $P''(0)$ given. Find A, B and C, then write a formula for $P(x)$.

8. $P(0) = 5$, $P'(0) = 3$, $P''(0) = 4$
9. $P(0) = -2$, $P'(0) = 7$, $P''(0) = 6$
10. $P(0) = 4$, $P'(0) = -1$, $P''(0) = -2$
11. $P(0) = 8$, $P'(0) = 5$, $P''(0) = 10$
12. $P(0) = 4$, $P'(0) = 0$, $P''(0) = -4$
13. $P(0) = -3$, $P'(0) = -2$, $P''(0) = 4$
14. If $P(x) = A + Bx + Cx^2$, how are the values of A, B and C related to $P(0)$, $P'(0)$ and $P''(0)$?

In Problems 15–20, $P(x) = A + Bx + Cx^2 + Dx^3$ is a cubic polynomial, with values of $P(0)$, $P'(0)$, $P''(0)$ and $P'''(0)$ given. Find the values of A, B, C and D, and then write a formula for $P(x)$.

15. $P(0) = 5$, $P'(0) = 3$, $P''(0) = 4$, $P'''(0) = 6$
16. $P(0) = -2$, $P'(0) = 7$, $P''(0) = 6$, $P'''(0) = 18$
17. $P(0) = 4$, $P'(0) = -1$, $P''(0) = -2$, $P'''(0) = -12$
18. $P(0) = 8$, $P'(0) = 5$, $P''(0) = 10$, $P'''(0) = 12$
19. $P(0) = 4$, $P'(0) = 0$, $P''(0) = -4$, $P'''(0) = 36$
20. $P(0) = -3$, $P'(0) = -2$, $P''(0) = 4$, $P'''(0) = 36$

21. If $P(x) = A + Bx + Cx^2 + Dx^3$, how are the values of A, B, C and D related to the values of $P(0)$, $P'(0)$, $P''(0)$ and $P'''(0)$?

In Problems 22–28, fill in the table below for $f(x)$ and $P(x)$, then graph $f(x)$ and $P(x)$ for $-2 \le x \le 2$.

x	$f(x)$	$P(x)$	$\lvert f(x) - P(x) \rvert$
0.0			
0.1			
0.2			
0.3			
1.0			
2.0			

22. $f(x) = \sin(x)$, $P(x) = x - \dfrac{1}{2 \cdot 3}x^3$
23. $f(x) = \sin(x)$, $P(x) = x - \dfrac{1}{2 \cdot 3}x^3 + \dfrac{1}{2 \cdot 3 \cdot 4 \cdot 5}x^5$
24. $f(x) = \cos(x)$, $P(x) = 1 - \dfrac{1}{2}x^2$
25. $f(x) = \cos(x)$, $P(x) = 1 - \dfrac{1}{2}x^2 + \dfrac{1}{2 \cdot 3 \cdot 4}x^4$
26. $f(x) = e^x$, $P(x) = 1 + x + \dfrac{1}{2}x^2$
27. $f(x) = e^x$, $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2 \cdot 3}x^3$
28. $f(x) = e^x$, $P(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 3 \cdot 4}x^4$

In 29–38, find a MacLaurin polynomial with four nonzero terms that approximates the given function.

29. e^{2x}
30. $\sin(2x)$
31. $\dfrac{1}{1-x}$
32. $\dfrac{1}{1+x}$
33. $\ln(1+x)$
34. $\sqrt{1+x}$
35. $\cos\left(x^2\right)$
36. e^{-x^2}
37. $x^3 \cdot \sin\left(x^2\right)$
38. $x \cdot e^{-x^2}$

In 39–42, use a MacLaurin polynomial with four nonzero terms to approximate the value of the definite integral.

39. $\displaystyle\int_0^1 \sin\left(x^3\right)\,dx$
40. $\displaystyle\int_0^{\frac{1}{2}} \cos\left(x^2\right)\,dx$
41. $\displaystyle\int_0^{\frac{1}{2}} e^{-x^3}\,dx$
42. $\displaystyle\int_0^1 x \cdot \sin\left(x^3\right)\,dx$

8.7 Practice Answers

1. With $P(x) = A + Bx + Cx^2 + Dx^3 + Ex^4$, $P'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 \Rightarrow P''(x) = 2C + 6Dx + 12Ex^2 \Rightarrow P'''(x) = 6D + 24Ex \Rightarrow P^{(4)}(x) = 24E$ so $3 = P(0) = A \Rightarrow A = 3$, $4 = P'(0) = B \Rightarrow B = 4$, $10 = P''(0) = 2C \Rightarrow C = 5$, $12 = P'''(0) = 6D \Rightarrow D = 2$ and $24 = P^{(4)}(0) = 24E \Rightarrow E = 1$, hence:

$$P(x) = 3 + 4x + 5x^2 + 2x^3 + x^4$$

2. With $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, $a_0 = \dfrac{P(0)}{0!} = \dfrac{3}{1} = 3$, $a_1 = \dfrac{P'(0)}{1!} = \dfrac{4}{1} = 4$, $a_2 = \dfrac{P''(0)}{2!} = \dfrac{10}{2} = 5$, $a_3 = \dfrac{P'''(0)}{3!} = \dfrac{12}{6} = 2$ and $a_4 = \dfrac{P^{(4)}(0)}{4!} = \dfrac{24}{24} = 1$ so:

$$P(x) = 3 + 4x + 5x^2 + 2x^3 + x^4$$

which agrees with the result of Practice 1.

3. Continuing with the differentiation process from Example 2:

$$f^{(5)}(x) = \cos(x) \Rightarrow f^{(6)}(x) = -\sin(x) \Rightarrow f^{(7)}(x) = -\cos(x)$$
$$\Rightarrow f^{(8)}(x) = \sin(x) \Rightarrow f^{(9)}(x) = \cos(x)$$

so that $f^{(6)}(0) = 0$, $f^{(7)}(0) = -1$, $f^{(8)}(0) = 0$ and $f^{(9)}(0) = 1$. Hence $a_6 = \dfrac{0}{6!} = 0$, $a_7 = \dfrac{-1}{7!} = -\dfrac{1}{5040}$, $a_8 = \dfrac{0}{8!} = 0$ and $a_9 = \dfrac{1}{9!} = 362880$, yielding the polynomial:

$$P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9$$

4. Differentiating $f(x) = \cos(x)$ yields $f'(x) = -\sin(x) \Rightarrow f''(x) = -\cos(x) \Rightarrow f'''(x) = \sin(x) \Rightarrow f^{(4)}(x) = \cos(x)$ so that $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$ and $f^{(4)}(0) = 1$, hence $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{-1}{2!} = -\frac{1}{2}$, $a_3 = 0$ and $a_4 = \frac{1}{4!} = \frac{1}{24}$. A MacLaurin polynomial is thus:

Notice that differentiating the result of Example 2 yields the same result.

$$P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

5. Using the MacLaurin polynomial for e^x from the discussion preceding Practice 5 with $x = -1$ yields:

$$\frac{1}{e} = e^{-1} \approx 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \frac{(-1)^5}{5!} + \frac{(-1)^6}{6!}$$
$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = \frac{53}{144} \approx 0.368$$

while using $x = \frac{1}{2}$ approximates $\sqrt{e} = e^{\frac{1}{2}}$ by:

$$1 + \left(\frac{1}{2}\right) + \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \frac{1}{4!}\left(\frac{1}{2}\right)^4 + \frac{1}{5!}\left(\frac{1}{2}\right)^5 + \frac{1}{6!}\left(\frac{1}{2}\right)^6$$
$$= 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840} + \frac{1}{46080} \approx 1.647872$$

6. Using the result of Practice 3:

$$\sin(1) \approx 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \frac{1}{362880} \approx 0.8414710097$$

which agrees to five decimal places with 0.8414709848.

7. Substituting $u = x^2$ into the MacLaurin polynomial from Practice 3 yields:

$$\sin(x^2) \approx x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10} - \frac{1}{5040}x^{14} + \frac{1}{362880}x^{18}$$

Integrating this polynomial from $x = 0$ to $x = 1$ yields:

$$\left[\frac{1}{3}x^3 - \frac{1}{42}x^7 + \frac{1}{1320}x^{11} - \frac{1}{75600}x^{15} + \frac{1}{6894720}x^{19}\right]_0^1$$

which evaluates to (approximately) 0.3102683028.