10.0 INTRODUCTION TO SEQUENCES AND SERIES

Chapter 10 is an introduction to two special topics in calculus, sequences and series. The main idea underlying this chapter is that **polynomials are easy**, and that even the hard functions such as sin(x) and log(x) can be represented as "big polynomials."

Polynomials are easy. It is easy to do arithmetic (evaluate, add, subtract, multiply, and even divide) with polynomials. It is easy to do calculus (differentiate and integrate) with polynomials. And, strangely enough, every polynomial is <u>completely</u> determined by its value and the values of all of its derivatives at $x = 0$: if we know the values of $P(0), P'(0), P''(0)$, etc., we can determine a formula for $P(x)$ that is valid for all x.

Unfortunately, many of the important functions we need for applications (sin, cos, exp, log) are not polynomials: sin and cos have too many wiggles; exp grows too fast; and log has an asymptote. However, even these "hard" functions are "almost" polynomials and share many properties with polynomials:

For some values of x (to be specified in later sections), many important functions can be represented as "big polynomials" called power series:

$$\sin(x) = x - \frac{x^3}{2\bullet 3} + \frac{x^5}{2\bullet 3\bullet 4\bullet 5} - \frac{x^7}{2\bullet 3\bullet 4\bullet 5\bullet 6\bullet 7} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \qquad \text{for } n = 0, 1, 2, \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2\bullet 3\bullet 4} - \frac{x^6}{2\bullet 3\bullet 4\bullet 5\bullet 6} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

$$\exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\bullet 3} + \frac{x^4}{2\bullet 3\bullet 4} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n + \ldots$$

the " . . . " at the end means the pattern of the terms continues "forever"

In this chapter we examine

- what it means to sum an **infinite** number of terms,
- how to do algebra and calculus with series ("big polynomials"),
- how to represent functions as series ("big polynomials"), and
- how to use such series ("big polynomials") to calculate derivatives, integrals, and even solve differential equations.

First, however, we need to lay a foundation, and that foundation is the study of lists of numbers, their properties and behavior.

Section 10.1 focuses on this foundation material. It introduces lists of numbers, called sequences, and examines some specific sequences we will need later. It also introduces the idea of the convergence of a sequence and examines some ways we can determine whether or not a sequence converges.

Sections 10.2 to 10.7 focus on what it means to add up an infinite number of numbers, an infinite series, and on how we can determine whether the resulting sum is a finite number.

Sections 10.8 to 10.11 generalize the idea of an infinite series of numbers to infinite series that contain a variable. These series that contain powers of a variable are called power series. These sections discuss how we can represent and approximate functions such as $\sin(x)$ and e^x with power series, how accurate these approximations are for commonly needed functions, and how we can use them with derivatives and integrals.

PROBLEMS

These problems illustrate, at an elementary level, some of the problems and concepts we will examine more deeply in this chapter. They are intended to start you thinking in certain ways that are useful and necessary for Chapter 10.

Patterns in lists of numbers

For problems 1 – 6, the first four numbers a_1, a_2, a_3, and a_4 in a list are given. (a) Write the next two numbers in the list, (b) write a formula for the 5th number a_5 in the list and (c) write a formula for the nth number a_n in the list.

1. 2, 4, 8, 16, ___ , ___
2. 3, 9, 27, 81, ___ , ___
3. –1, +1, –1, +1, ___ , ___
4. 1, 1/2, 1/3, 1/4, ___ , ___
5. 1, 2, 6, 24, ___ , ___
6. 1, 4, 9, 16, ___ , ___

For problems 7 – 11, evaluate each of the four given numbers and write the next two numbers in the list.

7. 1, 1 + 1/2, 1 + 1/2 + 1/3, 1 + 1/2 + 1/3 + 1/4, _____ , _____
8. 1, 1 + 1/2, 1 + 1/2 + 1/4, 1 + 1/2 + 1/4 + 1/6, _____ , _____
9. 1, 1 – 1/2, 1 – 1/2 + 1/4, 1 – 1/2 + 1/4 – 1/8, ______ , ______
10. 1, 1 + 2, 1 + 2 + 4, 1 + 2 + 4 + 8, ____ , ____
11. 1, 1 – 1, 1 – 1 + 1, 1 – 1 + 1 – 1, ____ , ____

Lists and graphs

For problems 12 – 15, (a) fill in the next two entries in the table and (b) graph the function for x = 1, 2, ... 6. These particular functions are defined only for integer values of x.

12.

x	f(x)
1	2
2	4
3	8
4	16
5	
6	

13.

x	g(x)
1	–1
2	+1
3	–1
4	+1
5	
6	

14.

x	s(x)
1	1 + 1/2
2	1 + 1/2 + 1/3
3	1 + 1/2 + 1/3 + 1/4
4	1 + 1/2 + 1/3 + 1/4 + 1/5
5	
6	

15.

x	t(x)
1	1 – 1/2
2	1 – 1/2 + 1/4
3	1 – 1/2 + 1/4 – 1/8
4	1 – 1/2 + 1/4 – 1/8 + 1/16
5	
6	

Polynomials and sine, cosine, and the exponential function

16. (a) Fill in the table for $P(x) = x - \frac{x^3}{2 \cdot 3}$ and sin(x).

(b) Graph $y = P(x)$ and $y = \sin(x)$ for $-2 \le x \le 2$.

(c) Repeat parts (a) and (b) for $P(x) = x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5}$.

x	P(x)	sin(x)	$\lvert P(x) - \sin(x) \rvert$
0			
0.1			
0.2			
0.3			
1.0			
2.0			

17. (a) Fill in the table for $P(x) = 1 - \frac{x^2}{2}$ and cos(x).

(b) Graph $y = P(x)$ and $y = \cos(x)$ for $-2 \le x \le 2$.

(c) Repeat parts (a) and (b) for $P(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 3 \cdot 4}$.

x	P(x)	cos(x)	$\lvert P(x) - \cos(x) \rvert$
0			
0.1			
0.2			
0.3			
1.0			
2.0			

18. (a) Fill in the table for $P(x) = 1 + x + \frac{x^2}{2.}$ and e^x.

(b) Graph $y = P(x)$ and $y = e^x$. for $-2 \le x \le 2$.

(c) Repeat parts (a) and (b) for $P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$

x	P(x)	e^x	$\lvert P(x) - e^x \rvert$
0			
0.1			
0.2			
0.3			
1.0			
2.0			

Polynomials and their values at $x = 0$

These problems illustrate how we can determine a formula for a polynomial when we know the values of the polynomial and its derivatives at $x = 0$.

In problems 19 – 24, $P(x) = Ax + B$ is a linear polynomial, and the values of $P(0)$ and $P'(0)$ are given. Find the values of A and B and write a formula for $P(x)$.

19. $P(0) = 5$, $P'(0) = 3$
20. $P(0) = -2$, $P'(0) = 7$
21. $P(0) = 4$, $P'(0) = -1$
22. $P(0) = 8$, $P'(0) = 5$
23. $P(0) = 4$, $P'(0) = 0$
24. $P(0) = -3$, $P'(0) = -2$
25. How are the values of A and B related to the values of $P(0)$ and $P'(0)$?

In problems 26 – 31, $P(x) = Ax^2 + Bx + C$ is a quadratic polynomial, and the values of $P(0)$, $P'(0)$, and $P''(0)$ are given. Find the values of A, B, and C and write a formula for $P(x)$.

26. $P(0) = 5, P'(0) = 3, P''(0) = 4$
27. $P(0) = -2, P'(0) = 7, P''(0) = 6$
28. $P(0) = 4, P'(0) = -1, P''(0) = -2$
29. $P(0) = 8, P'(0) = 5, P''(0) = 10$
30. $P(0) = 4, P'(0) = 0, P''(0) = -4$
31. $P(0) = -3, P'(0) = -2, P''(0) = 4$
32. How are the values of A, B, and C related to the values of $P(0), P'(0)$, and $P''(0)$?

In problems 33 – 38, $P(x) = Ax^3 + Bx^2 + Cx + D$ is a cubic polynomial, and the values of $P(0)$, $P'(0)$, $P''(0)$, and $P'''(0)$ are given. Find the values of A, B, C, and D and write a formula for $P(x)$.

33. $P(0) = 5, P'(0) = 3, P''(0) = 4, P'''(0) = 6$
34. $P(0) = -2, P'(0) = 7, P''(0) = 6, P'''(0) = 18$
35. $P(0) = 4, P'(0) = -1, P''(0) = -2, P'''(0) = -12$
36. $P(0) = 8, P'(0) = 5, P''(0) = 10, P'''(0) = 12$
37. $P(0) = 4, P'(0) = 0, P''(0) = -4, P'''(0) = 36$
38. $P(0) = -3, P'(0) = -2, P''(0) = 4, P'''(0) = 36$
39. How are the values of A, B, C, and D related to the values of $P(0), P'(0), P''(0)$, and $P'''(0)$?