10.11 APPROXIMATION USING TAYLOR POLYNOMIALS

The previous two sections focused on obtaining power series representations for functions, finding their intervals of convergence, and using those power series to approximate values of functions, limits, and integrals. In the cases where the power series resulted in an alternating numerical series, we were also able to use the Estimation Bound for Alternating Series (Section 10.6) to get a bound on the "error:"

"error" = | {exact value} – {partial sum approximation} | < | next term in the series | .

If the power series did not result in an alternating numerical series, we did not have a bound on the size of the error of the approximation.

In this section we introduce Taylor Polynomials (partial sums of the Taylor Series) and obtain a bound on the approximation error, the value |{ exact value of f(x) } – { Taylor Polynomial approximation of f(x) }| . The bound we get is valid even if the Taylor series is not an alternating series, and the pattern for the error bound looks very much like **the next term in the series**, the first unused term in the partial sum of the Taylor series. In mathematics, this error bound is important for determining which functions are approximated by their Taylor series. In computer and calculator applications, the error bound is important to designers to ensure that their machines calculate enough digits of functions such as e^x and sin(x) for various values of x. In general, knowing this error bound can help us work efficiently by allowing us to use only the number of terms we really need.

We also examine graphically how well the Taylor Polynomials of f(x) approximate f(x)

Taylor Polynomials

If we add a finite number of terms of a power series, the result is a polynomial.

Definition

For a function f, the **n^{th} degree Taylor Polynomial** (centered at c), written **$P_n(x)$**, is the partial sum of the terms up to the n^{th} degree of the Taylor Series for f:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Example 1: Write the first four Taylor Polynomials , $P_0(x)$ to $P_3(x)$, centered at 0 for e^x, and then graph them for $-1 < x < 1$.

Solution: The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so

$$P_0(x) = 1,\ P_1(x) = 1 + x,\ P_2(x) = 1 + x + \frac{x^2}{2}\ ,\ \text{and}\ P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}\ .$$

The graphs of e^x and $P_1(x)$, $P_2(x)$, and $P_3(x)$ are shown in Fig. 1.

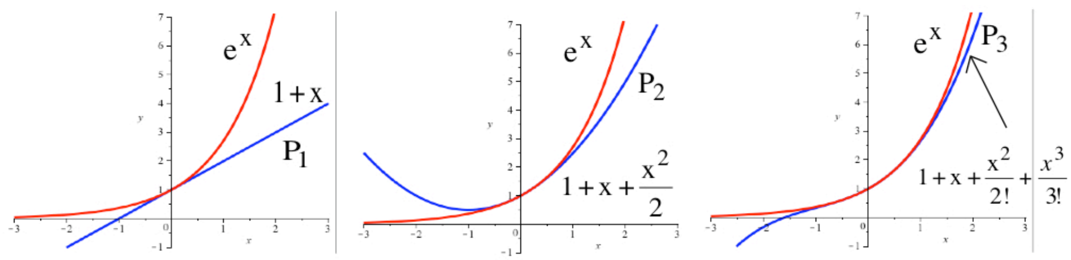

Fig. 1

Notice that $P_0(x)$ and e^x agree in value when $x = 0$,

$P_1(x)$, e^x , and their first derivatives agree in value when $x = 0$,

$P_2(x)$, e^x , their first derivatives, and their second derivatives agree in value when $x = 0$.

Practice 1: Write the Taylor Polynomials $P_0(x)$, $P_2(x)$, and $P_4(x)$ centered at 0 for $\cos(x)$, and then graph them for $-\pi < x < \pi$. Write the Taylor Polynomials $P_1(x)$ and $P_3(x)$.

When we center the Taylor Polynomial at $x = c \neq 0$, the Taylor Polynomials approximate the function and its derivatives well for x close to c.

Example 2: Write the Taylor Polynomials $P_0(x)$, $P_2(x)$, and $P_4(x)$ **centered at 3π/2** for $\sin(x)$, and then graph them for $2 < x < 8$.

Solution: The Taylor series, centered at $3\pi/2$, for $\sin(x)$ is

$$\sin(x) = -1 + \frac{1}{2!}(x - 3\pi/2)^2 - \frac{1}{4!}(x - 3\pi/2)^4 + \frac{1}{6!}(x - 3\pi/2) + \ldots = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n)!}(x - 3\pi/2)^{2n}\ .$$

Then $P_0(x) = -1$, $P_2(x) = -1 + \frac{1}{2}(x - 3\pi/2)^2$, and $P_4(x) = -1 + \frac{1}{2}(x - 3\pi/2)^2 - \frac{1}{24}(x - 3\pi/2)^4$.

The graphs of $\sin(x)$, $P_0(x)$, $P_2(x)$, and $P_4(x)$ are shown in Fig. 2.

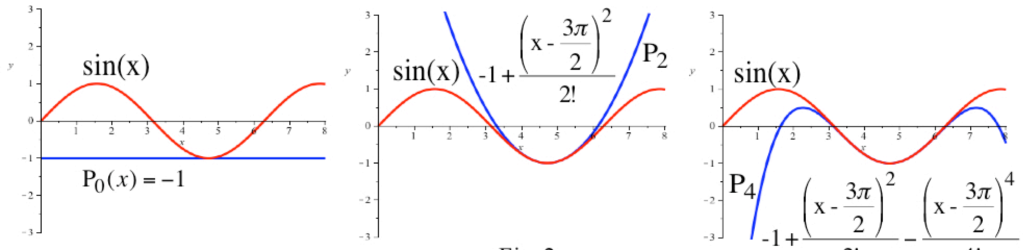

Fig. 2

Practice 2: Write the Taylor Polynomials $P_0(x)$, $P_1(x)$, and $P_3(x)$ **centered at $\pi/2$** for $\cos(x)$, and then graph them for $-1 < x < 4$.

The Remainder

Approximation formulas such as the Taylor Polynomials are useful by themselves, but in many applied situations we want to know how good the approximation is or how many terms of a series are required to obtain a needed level of accuracy. If 2 terms of a series give you the needed level of accuracy for your application, it is a waste of time and money to use 100 terms. On the other hand, sometimes even 100 terms may not give the accuracy you need. Fortunately, it is possible to obtain a guarantee on how close a particular Taylor Polynomial approximation is to the exact value. Then we can work efficiently and use the number of terms that we need. The next theorem gives a pattern for the amount of "error" in our Taylor Polynomial approximation and can be used to obtain a bound on the size of the "error."

Taylor's Formula with Remainder

If f has $n+1$ derivatives in an interval I containing c, and x is in I,

then there is a number **z** , strictly <u>between</u> c and x, so that

$$f(x) = P_n(x) + R_n(x) \qquad \text{where} \quad R_n(x) = \frac{f^{(n+1)}(\mathbf{z})}{(n+1)!}(x-c)^{n+1} .$$

This says that $f(x)$ is equal to the n^{th} degree Taylor Polynomial plus a Remainder, and the Remainder $R_n(x)$ has the form given in the theorem. Notice that the pattern for R_n looks like the pattern for the $(n+1)^{st}$ term of the Taylor series for $f(x)$ except that it contains $f^{(n+1)}(\mathbf{z})$ instead of $f^{(n+1)}(\mathbf{c})$. This particular pattern for $R_n(x)$ is called the Lagrange form of the remainder, and is named for the French–Italian mathematician and astronomer Joseph Lagrange (1736–1813).

The main idea of the proof of the Taylor's Formula with Remainder is straightforward, but the technical details are rather complicated. The main idea and the technical details are given in the Appendix.

The pattern for the remainder, $\frac{f^{(n+1)}(\mathbf{z})}{(n+1)!}(x-c)^{n+1}$, contains three pieces, $(n+1)!$, $(x-c)^{n+1}$, and $f^{(n+1)}(\mathbf{z})$ for some z between x and c. The Taylor Remainder Formula is typically used in two ways:

In one type of use, the Taylor Polynomial is given so x, c, and n are known, and we can evaluate $(n+1)!$ and $(x-c)^{n+1}$ exactly. That leaves the piece $f^{(n+1)}(\mathbf{z})$ for some z between x and c. If we can find a bound for the value of $|f^{(n+1)}(\mathbf{z})|$ for all z between x and c, then we can put it together with the values of $(n+1)!$ and $(x-c)^{n+1}$ to obtain a bound for the remainder term $R_n(x)$.

In the other common usage, the amount of acceptable "error" is given, so x, c, and $R_n(x)$ are known, and we need to find a value of n that guarantees the required accuracy.

Corollary: A Bound for the Remainder $R_n(x)$

If f has n+1 derivatives in an interval I containing c, and x is in I, and $|f^{(n+1)}(z)| \le M$ for **all** z between x and c,

then "error" $= |f(x) - P_n(x)| = |R_n(x)| \le M\cdot\frac{|x-c|^{n+1}}{(n+1)!}$.

Example 3: We plan to approximate the values of e^x with $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$. Find a bound for the "error" of the approximation, $R_3(x)$, if x is in the interval

(a) $[-1, 1]$, (b) $[-3, 2]$ and (c) $[-0.2, 0.3]$.

Solution: $f(x) = e^x$, $c = 0$ (a Maclaurin series), $n = 3$, and $f^{(n+1)}(x) = f^{(4)}(x) = e^x$.

(a) For x in the interval $[-1, 1]$:

$|(x-c)^{n+1}| = |x^4| \le |1^4| = 1$. $(n+1)! = 4! = 24$.

For x in $[-1, 1]$, $|f^{(n+1)}(x)| = |e^x| \le e^1$. A "crude" but "easy to use" bound for e^1 is $e^1 < (3)^1 = 3 = M$. (A more precise bound is $e^1 < (2.72)^1 < 2.72$.)

Then $|R_3(x)| < M\cdot\frac{|x-c|^{n+1}}{(n+1)!} < 3\cdot\frac{1}{24} = 0.125$.

For all $-1 < x < 1$, $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ is within 0.125 of e^x.

(b) For x in the interval $[-3, 2]$: $|(x-c)^{n+1}| = |x^4| \le |(-3)^4| = 81$ and $(n+1)! = 4! = 24$.

For x in $[-3, 2]$, $|f^{(n+1)}(x)| = |e^x| \le e^2$. A "crude" but "easy to use" bound for e^2 is $e^2 < (3)^2 = 9 = M$. (A more precise bound is $e^2 < (2.72)^2 < 7.4$.)

Then $| R_3(x) | < M \cdot \frac{|x-c|^{n+1}}{(n+1)!} < 9 \cdot \frac{81}{24} = 30.375$. Obviously we cannot have much confidence in our use of $P_3(x)$ to approximate e^x on the interval $[-3, 2]$.

(c) For x in the interval $[-0.2, 0.3]$: $| (x-c)^{n+1} | = | x^4 | \le | 0.3^4 | = 0.0081$.

$(n+1)! = 4! = 24$.

For x in $[-0.2, 0.3]$, $| f^{(n+1)}(x) | = | e^x | \le e^{0.3}$. A bound for $e^{0.3}$ is

$e^{0.3} < (2.72)^{0.3} < 1.4 = M$ — obtained using a calculator .

Then $| R_3(x) | < M \cdot \frac{|x-c|^{n+1}}{(n+1)!} < 1.4 \cdot \frac{0.0081}{24} = 0.0004725$.

For all $-0.2 < x < 0.3$, $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ is within 0.0004725 of e^x .

When the interval is small, we can be confident that $P_3(x)$ provides a good approximation of e^x , but as the interval grows, so does our bound on the remainder. To guarantee a good approximation on a larger interval, we typically need $(n+1)!$ to be larger so we need to use a higher degree Taylor Polynomial $P_n(x)$.

Practice 3: Find a value of n to guarantee that $P_n(x)$ is within 0.001 of e^x for x in the interval $[-3, 2]$.

Example 4: We want to approximate the values of $f(x) = \sin(x)$ on the interval $[-\pi/2, \pi/2]$ with an error less that 10^{-10}. How many terms of the Maclaurin series for $\sin(x)$ are needed?

Solution: For every value of n, $| f^{(n+1)}(x) |$ is $| \sin(x) |$ or $| \cos(x) |$ so $M = 1$ in the Bound for the Remainder. Then "error" $= | R_n(x) | < 1 \cdot \frac{|x-0|^{n+1}}{(n+1)!} \le \frac{(\pi/2)^{n+1}}{(n+1)!}$, and we need to find a value of n so that $\frac{(\pi/2)^{n+1}}{(n+1)!}$ is less than 10^{-10}. A bit of numerical experimentation on a calculator shows that

$$\frac{(\pi/2)^{14}}{14!} \approx 6.39 \text{ x } 10^{-9} , \quad \frac{(\pi/2)^{15}}{15!} \approx 6.69 \text{ x } 10^{-10} , \text{ and } \frac{(\pi/2)^{16}}{16!} \approx 6.57 \text{ x } 10^{-11}$$

so we can take $n = 15$: $P_{15}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!}$.

If $-\pi/2 \le x \le \pi/2$, then $| P_{15}(x) - \sin(x) | < 10^{-10}$.

Practice 4: How many terms of the Maclaurin series for e^x are needed to approximate e^x to within 10^{-10} for $0 \le x \le 1$?

Calculator Notes

Imagine that you are in charge of designing or selecting an algorithm for a calculator to use when the SIN button is pushed. (Smartest move: find a mathematician who knows about "numerical analysis" and the design and implementation of algorithms.) You know that if the value of x is relatively close to 0, then SIN(x) can be quickly approximated to 10 digits (the size of the display of the calculator) by using a "few" terms of the Taylor series for sin(x): if $-1.57 \le x \le 1.57$, then

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!}$$

$$= x\left(1 - \frac{x^2}{2\cdot 3}\left(1 - \frac{x^2}{4\cdot 5}\left(1 - \frac{x^2}{6\cdot 7}\left(1 - \frac{x^2}{8\cdot 9}\left(1 - \frac{x^2}{10\cdot 11}\left(1 - \frac{x^2}{12\cdot 13}\left(1 - \frac{x^2}{14\cdot 15}\right)\right)\right)\right)\right)\right)\right)$$

gives the value of sin(x) with 10 digits of accuracy.

(The second pattern looks more complicated, but is usually preferred because it uses fewer multiplications and avoids very large values such as x^{15} and 15!) But you also want your algorithm to give 10 digits of accuracy even when x is larger, say 10 or 101.7. Rather than computing <u>many</u> more terms of the Maclaurin series for sine, some algorithms simply shift the problem closer to 0. First they use the fact that sin(x) = sin(x – 2π) to keep shifting the problem until the argument is in the interval [0, 2π]:

$\sin(10) = \sin(10 - 2\pi) = \sin(3.71681469)$

$\sin(101.7) = \sin(101.7 - 2\pi) = \sin(101.7 - 4\pi) = \ldots = \sin(101.7 - 32\pi) = \sin(1.169035085)$.

Once the argument is between 0 and 2π, additional trigonometric facts are used:

if the new value of x is larger than π, use $\sin(x) = -\sin(x - \pi)$ to replace "x" with "x – π" (and keep track of the change in sign of the answer). The new x value is in the interval [0, π].

Finally, we can shift the problem into the interval [0, π/2]:

if the new value of x is larger than π/2, use $\sin(x) = \sin(\pi - x)$ to replace "x" with "π – x." This new x value is in the interval $[0, \pi/2] \approx [0, 1.57]$ and the 7 terms of the sine series shown above are sufficient to approximate sin(x) with 10 digits of accuracy.

There are, however, major problems when calculators encounter the sine or exponential of a very large number. Since calculators only store the leading finite number of digits of a number (usually 10 or 12 digits), the calculator can not distinguish large numbers that differ past that leading number of stored digits: one calculator correctly says that (10^12+1) – 10^12 = 1, but it incorrectly reports that (10^13+1) – 10^13 = 0. Since it calculates "10^13+1 = 10^13", it also would falsely report the same values for sin(10^13+1) and sin(10^13). In fact, the people who programmed this particular type of calculator recognized the problem, and the calculator gives an error message if it is asked to calculate sin(10^11). This particular calculator reports e^230 ≈ 7.7×10^{99}. It reports an error for e^231 since the largest number it can display is 9.9×10^{99} and e^231 exceeds that value. What happens on your calculator?

PROBLEMS

In problems 1 – 10, calculate the Taylor polynomials P_0, P_1, P_2, P_3, and P_4 for the given function centered at the given value of c. Then graph the function and the Taylor polynomials on the given interval.

1. $f(x) = \sin(x)$, $c = 0$, $[-2, 4]$
2. $f(x) = \cos(x)$, $c = 0$, $[-2, 4]$
3. $f(x) = \ln(x)$, $c = 1$, $[0.1, 3]$
4. $f(x) = \arctan(x)$, $c = 0$, $[-3, 3]$
5. $f(x) = \sqrt{x}$, $c = 1$, $[0, 3]$
6. $f(x) = \sqrt{x}$, $c = 9$, $[0, 20]$
7. $f(x) = (1 + x)^{-1/2}$, $c = 0$, $[-2, 3]$
8. $f(x) = e^{2x}$, $c = 0$, $[-2, 4]$
9. $f(x) = \sin(x)$, $c = \pi/2$, $[-1, 5]$
10. $f(x) = \sin(x)$, $c = \pi$, $[-1, 5]$

In problems 11 – 18, a function f(x) and a value of n are given. Determine a formula for $R_n(x)$ and find a bound for $|R_n(x)|$ on the given interval. This bound for $|R_n(x)|$ is our "guaranteed accuracy" for P_n to approximate f(x) on the given interval. (Use $c = 0$.)

11. $f(x) = \sin(x)$, $n = 5$, $[-\pi/2, \pi/2]$
12. $f(x) = \sin(x)$, $n = 9$, $[-\pi/2, \pi/2]$
13. $f(x) = \sin(x)$, $n = 5$, $[-\pi, \pi]$
14. $f(x) = \sin(x)$, $n = 9$, $[-\pi, \pi]$
15. $f(x) = \cos(x)$, $n = 10$, $[-1, 2]$
16. $f(x) = \cos(x)$, $n = 10$, $[-1, 5]$
17. $f(x) = e^x$, $n = 6$, $[-1, 2]$
18. $f(x) = e^x$, $n = 10$, $[-1, 3]$

In problems 19 – 24, determine how many terms of the Taylor series for f(x) are needed to approximate f to within the specified error on the given interval. (For each function use the center $c = 0$.)

19. $f(x) = \sin(x)$ within 0.001 on $[-1, 1]$
20. $f(x) = \sin(x)$ within 0.001 on $[-3, 3]$
21. $f(x) = \sin(x)$ within 0.00001 on $[-1.6, 1.6]$
22. $f(x) = \cos(x)$ within 0.001 on $[-2, 2]$
23. $f(x) = e^x$ within 0.001 on $[0, 2]$
24. $f(x) = e^x$ within 0.001 on $[-1, 4]$

Series Approximations of π

The following problems illustrate some of the ways series have been used to obtain very precise approximations of π. Several of these methods use the series for arctan(x),

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},$$

which converges rapidly if $|x|$ is close to zero.

Method I: $\tan(\frac{\pi}{4}) = 1$ so $\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and

$$\pi = 4\left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right\}.$$

25. (a) Approximate π as $4\left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \right\}$ and compare this value with the value your calculator gives for π.

(b) The series for arctan(1) is an alternating series so we have an "easy" error bound. Use the error bound for an alternating series to find a bound for the error if 50 terms of the arctan(1) series are used.

(c) Using the error bound for an alternating series, how many terms of the arctan(1) series are needed to guarantee that the series approximation of π is within 0.0001 of the exact value of π?

(The arctan(1) series converges so slowly that it is not used to approximate π.)

Method II: $\tan(a + b) = \dfrac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$ so $\tan\left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) \right) = \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = 1$. Then

$\frac{\pi}{4} = \arctan(1) = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$, and the series for $\arctan\left(\frac{1}{2}\right)$ and $\arctan\left(\frac{1}{3}\right)$ converge much more rapidly than the series for arctan(1).

26. (a) Approximate π as

$4\left\{ \text{(sum of the first 4 terms of the } \arctan\left(\tfrac{1}{2}\right) \text{ series)} + \text{(sum of the first 4 terms of the } \arctan\left(\tfrac{1}{3}\right) \text{ series)} \right\}$.

Then compare this value with the value your calculator gives for π.

(b) The series for $\arctan\left(\frac{1}{2}\right)$ and $\arctan\left(\frac{1}{3}\right)$ are each alternating series. Use the error bound for an alternating series to find a bound for the error if 10 terms of each series are used.

(c) How many terms of each series are needed to guarantee that the series approximation of π is within 0.0001 of the exact value of π?

Other Methods: We will not justify these methods, but they converge to π more rapidly than the first two methods.

A: $\frac{\pi}{4} = 4 \cdot \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$ (due to Machin in 1706)

B: $\pi = 48 \cdot \arctan\left(\frac{1}{18}\right) + 32 \cdot \arctan\left(\frac{1}{57}\right) - 20 \cdot \arctan\left(\frac{1}{239}\right)$

27. (a) Use the first 3 terms of each series in formula A to approximate π. How much does it differ from the value your calculator gives you?

(b) Why does formula A converge more rapidly (using fewer terms) than methods I and II?

28. (a) Use the first 3 terms of each series in formula B to approximate π. How much does it differ from the value your calculator gives you?

(b) Why does formula B converge more rapidly (using fewer terms) than Methods I and II and formula A?

Practice Answers

Practice 1: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ so

$P_0(x) = 1$, $P_2(x) = 1 - \frac{x^2}{2}$, and $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$. Their graphs are shown in Fig. 3.

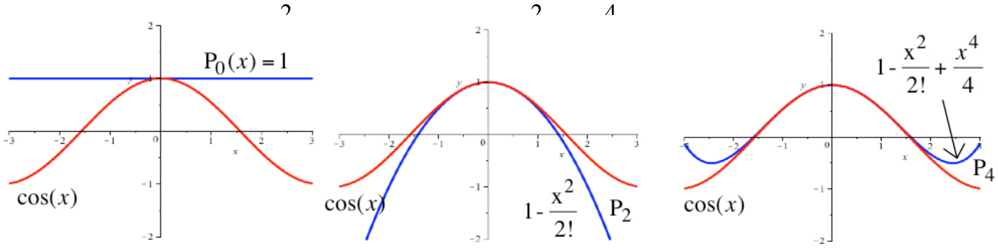

Fig. 3

Practice 2:

$$\cos(x) = -(x - \pi/2) + \frac{1}{3!}(x - \pi/2)^3 - \frac{1}{5!}(x - \pi/2)^5 + \frac{1}{7!}(x - \pi/2)^7 - \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)!} (x - \pi/2)^{2n+1} .$$

Then $P_0(x) = 0$, $P_1(x) = -(x - \pi/2)$, and $P_3(x) = -(x - \pi/2) + \frac{1}{6}(x - \pi/2)^3$. The graphs of $\cos(x)$, $P_1(x)$, and $P_3(x)$ are shown in Fig. 4.

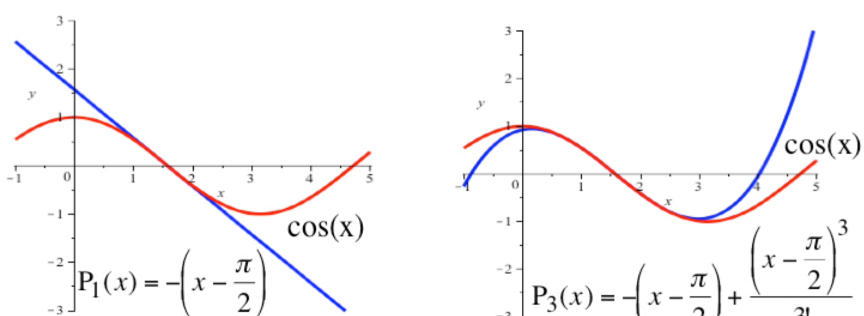

Fig. 4

Practice 3: For x in the interval $[-3, 2]$, $| (x-c)^{n+1} | = | x^3 | \le | (-3)^3 | = 27$.

For x in $[-3, 2]$, $| f^{(n+1)}(x) | = | e^x | \le e^2$. A "crude" bound for e^2 is $e^2 < (3)^2 = 9 = M$.

Then $| R_n(x) | < M \cdot \frac{(x-c)^{n+1}}{(n+1)!} < 9 \cdot \frac{27}{n!}$, and we want a value of n so $9 \cdot \frac{27}{n!} \le 0.001$:

we want $n! \ge \frac{(9)(27)}{0.001} = 243{,}000$. Using a calculator, we see that $8! = 40{,}320$ is not large enough, but $9! = 362{,}880 > 243{,}000$ so we can use $n = 9$.

For x in the interval $[-3, 2]$, $P_9(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!}$ is within 0.001 of e^x .

Practice 4: $R_n(x) \le 10^{-10}$. $f(x) = e^x$, $c = 0$, and for every n, $f^{(n+1)}(x) = e^x$. For $0 \le x \le 1$,

$| f^{(n+1)}(x) | = | e^x | \le e < 2.72 = M$. We want to find a value for n so

$M \cdot \frac{(x-c)^{n+1}}{(n+1)!} = 2.72 \cdot \frac{(1-0)^{n+1}}{(n+1)!} < 10^{-10}$. Some numerical experimentation on a calculator

shows that $2.72 \cdot \frac{1}{15!} \approx 1.58\ 10^{-9}$ and $2.72 \cdot \frac{1}{16!} \approx 9.9\ 10^{-11}$ so we can take $n = 15$.

For $0 \le x \le 1$, $P_{15}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^{15}}{15!}$ is within 10^{-10} of e^x .

Appendix: Idea and details of a Proof of Taylor's Formula with Remainder

Main idea of the proof:

We define a new differentiable function g(t) and show that $g(x) = 0$ and $g(c) = 0$.

Then, by Rolle's Theorem, we can conclude that there is a number z , between x and c, so that $g'(z) = 0$.

Finally, we set $g'(z) = 0$ and algebraically obtain the given formula for $R_n(x)$.

Let $R_n(x) = f(x) - P_n(x)$ be the difference between f(x) and the nth Taylor polynomial for $P_n(x)$.

Define a differentiable function g(t) to be

$$g(\mathbf{t}) = f(x) - \left\{ f(\mathbf{t}) + f'(\mathbf{t})(x-\mathbf{t}) + \frac{f''(\mathbf{t})}{2!}(x-\mathbf{t})^2 + \frac{f'''(\mathbf{t})}{3!}(x-\mathbf{t})^3 + \ldots + \frac{f^{(n)}(\mathbf{t})}{n!}(x-\mathbf{t})^n \right\} - R_n(x)\frac{(x-\mathbf{t})^{n+1}}{(x-c)^{n+1}} .$$

This may seem to be a strange way to define a function, but it turns out to have the properties we need:

$$g(\mathbf{x}) = f(x) - \left\{ f(\mathbf{x}) + 0 + 0 + 0 + \ldots + 0 \right\} - R_n(x)\frac{0}{(x-c)^{n+1}} = f(x) - f(x) = 0 \text{ , and}$$

$$g(\mathbf{c}) = f(x) - \left\{ f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \frac{f^{(n)}(c)}{n!}(x-c)^n \right\}$$
$$- R_n(x)\frac{(x-c)^{n+1}}{(x-c)^{n+1}}$$
$$= f(x) - P_n(x) - R_n(x) = 0 \quad \text{since } R_n(x) = f(x) - P_n(x) .$$

Then, by Rolle's Theorem, there is a number z, strictly between x and c, so $g'(z) = 0$.

Notice that g is defined to be a function of t so we treat x and c as constants and differentiate with respect to t. The key pattern is that when we differentiate a term such as $\frac{f'''(\mathbf{t})}{3!}\cdot(x-\mathbf{t})^3$ with respect to t, we need to use the product rule. The resulting derivative has two terms:

$$\frac{\mathbf{d}}{\mathbf{dt}}\left\{\frac{f'''(\mathbf{t})}{3!}(x-\mathbf{t})^3\right\} = \frac{f'''(\mathbf{t})}{3!}\frac{\mathbf{d}}{\mathbf{dt}}(x-\mathbf{t})^3 + (x-\mathbf{t})^3\cdot\frac{\mathbf{d}}{\mathbf{dt}}\frac{f'''(\mathbf{t})}{3!}$$

$$= \frac{f'''(\mathbf{t})}{3!}(3)(x-\mathbf{t})^2(-1) + \frac{f^{(4)}(\mathbf{t})}{3!}(x-\mathbf{t})^3 = -\frac{f'''(\mathbf{t})}{2!}(x-\mathbf{t})^2 + \frac{f^{(4)}(\mathbf{t})}{3!}(x-\mathbf{t})^3 .$$

When we differentiate g(t) with respect to t, we get a complicated pattern, but most of the terms cancel:

$$\begin{aligned} g'(t) = \frac{d}{dt}g(t) = 0 - \Big\{ & f'(t) \\ & - f'(t) + f''(t)(x-t) \\ & - f''(t)(x-t) + \frac{f'''(t)}{2!}(x-t)^2 \\ & - \frac{f'''(t)}{2!}(x-t)^2 + \frac{f^{(4)}(t)}{3!}(x-t)^3 \\ & - \ldots - \frac{f^{(n)}(t)}{(n-1!)}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n \Big\} - R_n(x)\cdot(n+1)\cdot\frac{(x-t)^n(-1)}{(x-c)^{n+1}} \end{aligned}$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + R_n(x)\cdot(n+1)\cdot\frac{(x-t)^n}{(x-c)^{n+1}}$$

$$= (x-t)^n\left\{R_n(x)\cdot(n+1)\cdot\frac{1}{(x-c)^{n+1}} - \frac{f^{(n+1)}(t)}{n!}\right\}.$$

By Rolle's Theorem, there is a value $\mathbf{z}$, between x and c, for the variable t so $g'(\mathbf{z}) = 0$. Then

$$(x-\mathbf{z})^n\left\{R_n(x)\cdot(n+1)\cdot\frac{1}{(x-c)^{n+1}} - \frac{f^{(n+1)}(\mathbf{z})}{n!}\right\} = 0 .$$

$\mathbf{z}$ is strictly between x and c so $\mathbf{z} \neq x$ and we can divide each side by $(x-\mathbf{z})^n$ to get

$$R_n(x)\cdot(n+1)\cdot\frac{1}{(x-c)^{n+1}} - \frac{f^{(n+1)}(\mathbf{z})}{n!} = 0 .$$

Finally, $R_n(x)\cdot(n+1)\cdot\frac{1}{(x-c)^{n+1}} = \frac{f^{(n+1)}(\mathbf{z})}{n!}$ so $R_n(x) = \frac{f^{(n+1)}(\mathbf{z})}{n!(n+1)}\cdot(x-c)^{n+1} = \frac{f^{(n+1)}(\mathbf{z})}{(n+1)!}(x-c)^{n+1}$,

the result we wanted to prove.

MAPLE command to plot cos(x) and $P_3(x)$.

```
plot({1-x^2/2+x^4/24,cos(x)},x=-3..3,y=-2..2,color=[blue,red],thickness=3);
```