10.12 INTRODUCTION TO FOURIER SERIES

When we discussed Maclaurin series in earlier sections we used information about the derivatives of a function f(x) to create an infinite series of the form

$$\sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 \cdot x^1 + a_2 \cdot x^2 + a_3 \cdot x^3 + a_4 \cdot x^4 + \ldots$$

In this case the "building blocks" of the series were powers of x $(x^0, x^1, x^2, x^3, \ldots)$. and this is why the series is called a "power series." The coefficients $a_n = \frac{f^{(n)}(0)}{n!}$ of the Maclaurin series told us how much of each power to include in our series, and we found the formula involving derivatives that enabled us to calculate the values of the coefficients. For general Taylor series the "building blocks" were powers of (x-c) for some fixed center c, and we had a similar formula to calculate the values of the coefficients for those series. We then used the Taylor and Maclaurin series to approximate functions "near c" and sometimes even for "every value of x."

Fourier series have a number of similarities with power series:

we will approximate functions using "building blocks"

we will need to calculate the values of the coefficients of the building blocks, and

we will need to be concerned about where the approximations are "good."

But our building blocks for the Fourier series will be the trigonometric functions sin(x), sin(2x), sin(3x), ... and cos(x), cos(2x), cos(3x), ... and the result will be a "trigonometric series." Our formula for calculating the values of the coefficients will involve integrals rather than derivatives. Finally, since each building block repeats its values every 2π units, we will only (at first) use Fourier series to approximate 2π-periodic functions. (This is not as serious a restriction as it might seem.)

First goal: Finding the coefficients

Our first goal is to find a relationship between the function f(x) and the coefficients a_n and b_n in the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cdot \cos(nx) + b_n \cdot \sin(nx)\}$$

$$= \frac{a_0}{2} + \{a_1 \cdot \cos(x) + b_1 \cdot \sin(x)\} + \{a_2 \cdot \cos(2x) + b_2 \cdot \sin(2x)\} + \{a_3 \cdot \cos(3x) + b_3 \cdot \sin(3x)\} + \ \ldots$$

The key to finding the values of the coefficients depends on the following results about the integrals of the products of any two of our building blocks, sin(mx) and cos(nx), for positive integer values of m and n.

$$\int_{x=0}^{2\pi} \sin(nx)\cdot\sin(mx)\,dx = \begin{cases} \pi & \text{if } m=n \quad \text{(using integral formula \#13)} \\ 0 & \text{if } m\neq n \quad \text{(using integral formula \#25)} \end{cases}$$

$$\int_{x=0}^{2\pi} \cos(nx)\cdot\cos(mx)\,dx = \begin{cases} \pi & \text{if } m=n \quad \text{(using integral formula \#14)} \\ 0 & \text{if } m\neq n \quad \text{(using integral formula \#26)} \end{cases}$$

$$\int_{x=0}^{2\pi} \sin(nx)\cdot\cos(mx)\,dx = 0 \quad \text{for all integer values of m and n} \quad \text{(using integral formula \#27)}$$

The technical term is that the set of functions { sin(x), cos(x), sin(2x), cos(2x), sin(3x), cos(3x), ...} forms an "orthogonal" family on the interval $[0,2\pi]$: for any two functions from this set

$$\int_{x=0}^{2\pi} f(x)\cdot g(x)\,dx = \begin{cases} \pi & \text{if f and g are the same member of the set} \\ 0 & \text{if f and g are different members of the set} \end{cases}$$

Example 1: Suppose we have the function f(x) = 3 + 2cos(3x)-7sin(4x) that is already a finite trigonometric series with $a_0 = 6$, $a_3 = 2$, $b_4 = -7$, and all of the other coefficients are 0. Evaluate the integral of the product of f(x) with a general building block, sin(nx) (for every value of n).

Solution:

$$\int_{x=0}^{2\pi} \sin(nx)\cdot f(x)\,dx = \int_{x=0}^{2\pi} \sin(nx)\cdot(\,3+2\cos(3x)-7\sin(4x)\,)\,dx$$

$$= \int_{x=0}^{2\pi} \sin(nx)\cdot\ 3\,dx + \int_{x=0}^{2\pi} \sin(nx)\cdot 2\cos(3x) + \int_{x=0}^{2\pi} \sin(nx)\cdot(-7\sin(4x)\,)\,dx$$

$$= 0 + 0 + \begin{cases} -7\pi & \text{if } n=4 \\ 0 & \text{if } n\neq 4 \end{cases}\ .$$

It appears that we can find the value of b_n by dividing this last value by π :

$$b_n = \frac{1}{\pi}\int_{x=0}^{2\pi} \sin(nx)\cdot f(x)\,dx$$

Practice 1: Using the same function f(x) = 3 + 2cos(3x)-7sin(4x), evaluate the integral of the product with the cosine blocks, cos(nx), and show that

$$a_3 = \frac{1}{\pi}\int_{x=0}^{2p} \cos(3x)\cdot f(x)\,dx \quad \text{and} \quad a_n = \frac{1}{\pi}\int_{x=0}^{2p} \cos(nx)\cdot f(x)\,dx \quad \text{for } n=1,2,3,\ \dots$$

The general cases for the Example and Practice problems say that if f(x) is already a trigonometric polynomial with

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cdot \cos(nx) + b_n \cdot \sin(nx) \}$$

$$= \frac{a_0}{2} + \{ a_1 \cdot \cos(x) + b_1 \cdot \sin(x) \} + \{ a_2 \cdot \cos(2x) + b_2 \cdot \sin(2x) \} + \{ a_3 \cdot \cos(3x) + b_3 \cdot \sin(3x) \} + \ ...$$

then we can find the coefficients of the terms of f(x) using

$$a_n = \frac{1}{\pi} \int_{x=0}^{2\pi} \cos(nx) \cdot f(x) \, dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{x=0}^{2\pi} \sin(nx) \cdot f(x) \, dx \ .$$

What we have just done is similar to our beginning work with Maclaurin series in Section 10.10. There we started with a polynomial P(x) and found that the coefficients were given by the formula $a_n = \frac{f^{(n)}(0)}{n!}$ using derivatives of P(x). Here we started with a trigonometric polynomial f(x) and found that the coefficients a_n and b_n were given by formulas using integrals of products with f(x). In Section 10.10 we then extended the Maclaurin series to functions that were not polynomials. Here we will make a similar extension of trigonometric series to functions that are not trigonometric polynomials.

Definition: If f(x) is integrable on the interval $[0, 2\pi]$, then the

Fourier Series of f(x) is $\frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cdot \cos(nx) + b_n \cdot \sin(nx) \}$ with

Fourier coefficients

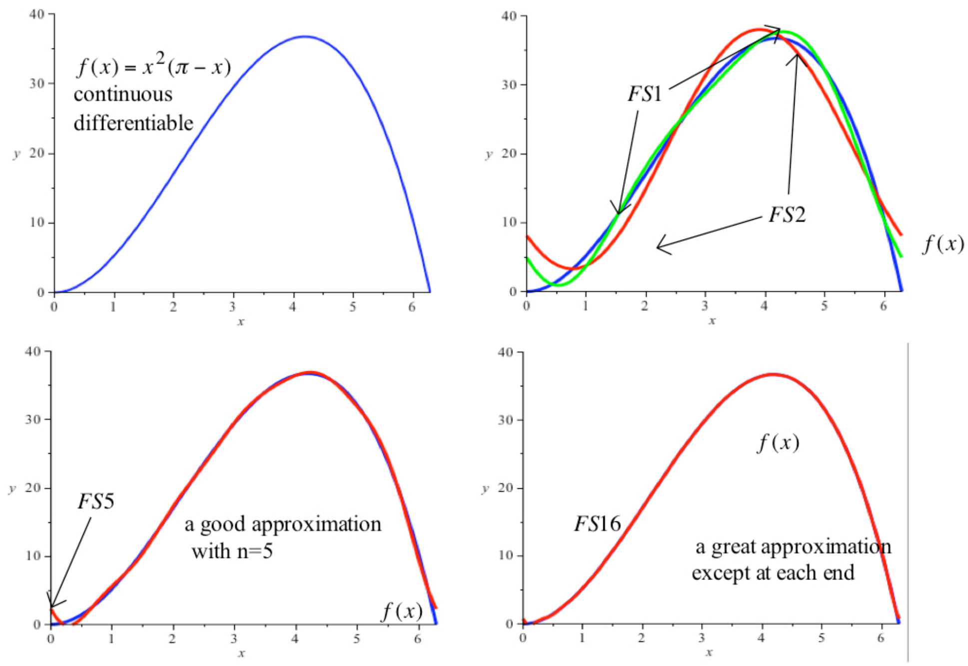
$$a_n = \frac{1}{\pi} \int_{x=0}^{2\pi} \cos(nx) \cdot f(x) \, dx \quad \text{and}$$

$$b_n = \frac{1}{\pi} \int_{x=0}^{2\pi} \sin(nx) \cdot f(x) \, dx \quad \text{for } n=0, 1, 2, 3, \ldots$$

Before going further with the development and discussion, let’s take a look at how this actually works with some different types of functions – a differentiable function, a function that is continuous but not differentiable, and a function that is not even continuous. At the end of this section is a MAPLE program that automatically calculates the Fourier coefficients and builds the Fourier series.

Fourier Series approximation of a differentiable function $f(x) = x^2 \cdot (2\pi - x)$

The series of graphs below show how the higher degree Fourier series of f become better and better approximations of the graph of f on the interval $[0,2\pi]$.



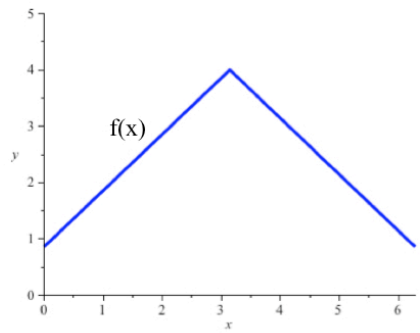
FS5(x) = 20.67085112-12.56637062*cos(x)-12.*sin(x)-3.141592654*cos(2*x)-1.5*sin(2*x)
-1.396263402*cos(3*x)-.4444444444*sin(3*x)-.7853981635*cos(4*x)-.1875*sin(4*x)
-.5026548246*cos(5*x)-0.096*sin(5*x)

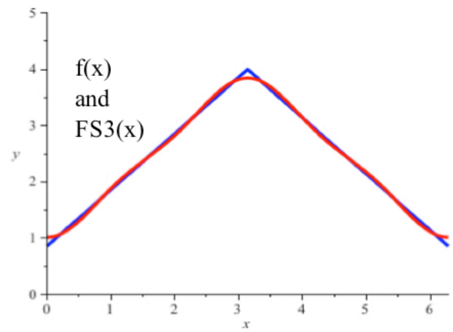
Taylor and Maclaurin series require that we have a differentiable function of f or we could not calculate the coefficients for the series. For Fourier series, however, we only need that the function of f be integrable, a much less demanding condition. The next example illustrates the convergence of the Fourier series of a function that has a "corner" so it is not differentiable (at that point).

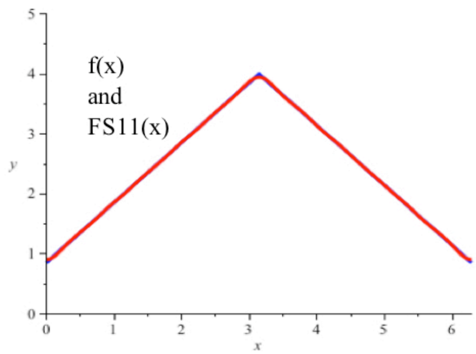
Note: We only need f to be an integrable function in order to be able to calculate the coefficients of the Fourier series, but that is not enough to guarantee that the Fourier series we get converges to f(x) for every value of x. In fact, there are continuous functions for which the Fourier series does not converge to f(x) for an infinite number of vales of x. The whole study of conditions that do and do not guarantee the convergence of Fourier series led to some very interesting, very beautiful and very deep results in mathematics.

Fourier Series approximation of a continuous function with a corner: $f(x) = 4 - |\pi - x|$

These figures illustrate that the Fourier series can converge to a function that is not differentiable at a point.







The formula for the 9th degree Fourier series is

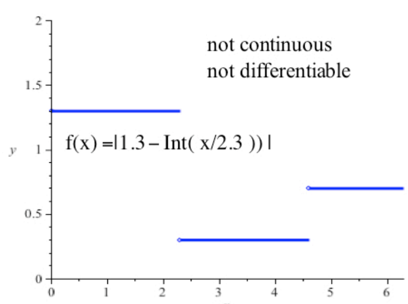
f(x) = 2.429203673-1.273239544*cos(x)
-.1414710605*cos(3*x)
-0.05092958178*cos(5*x)
-0.02598448805*cos(7*x)
-0.01571900067*cos(9*x)

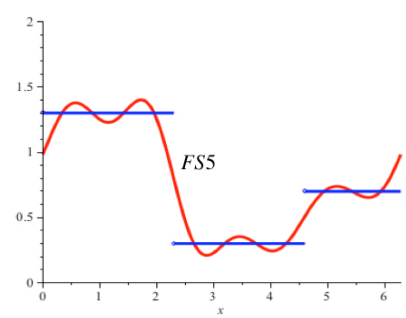
Because of the symmetry of f(x) around π, all of the sine terms, the b_n, are 0.

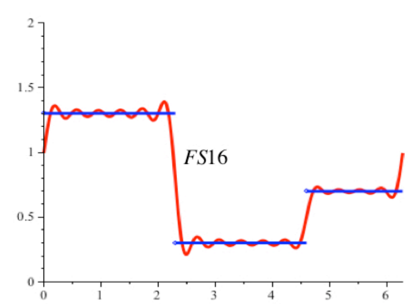
The next example illustrates that the Fourier coefficients can even be found for functions which have some discontinuities (but they can still be integrated), and that the resulting Fourier series can still do a good job of approximating the function between the breaks.

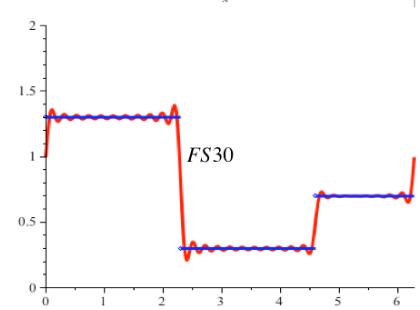
Fourier series approximation of a discontinuous function:

$f(x) = |1.3 - \text{Int}(x/2.3)|$





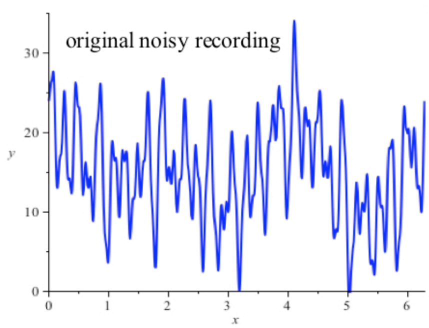


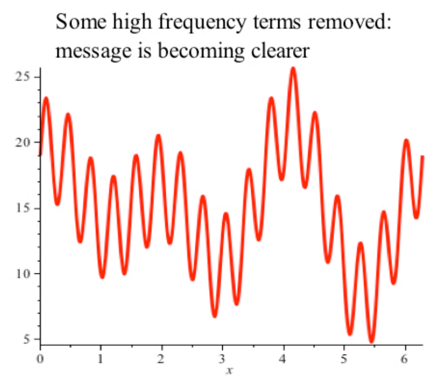


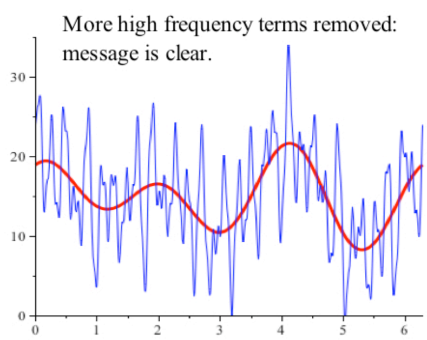
What good are Fourier Series?

Historically Fourier series were important because they provided a method of analyzing data that arose from the interaction of various periodic influences such as the orbits of the planets (influenced by the gravitational attractions of the sun and other planets whose orbits are almost periodic) or the tides (influenced by the attraction of the moon and also by local conditions). More recently, Fourier series provide a way to efficiently store and regenerate the musical tones of various instruments. Each tone for an instrument can be efficiently stored by saving only the coefficients, and we only need to save the first "several" terms of the series since the higher order coefficients correspond to high frequency sounds that are beyond the human hearing range.

Fourier series, and variations on that idea, have also been used extensively in signal processing and to clean up noisy signals. Since "random noise" is usually high frequency, the Fourier series of each piece of the signal can be calculated (automatically) and only the lower degree terms kept in order to reproduce a "clean" result. The following figures illustrate a noisy signal that has been cleaned up in this way. This might represent an attempt to reclaim voice information from a recording that has been damaged or originally contained a lot of background noise.







And Fourier series still enable us to solve problems of the type that led Fourier to develop them in the first place: if we know the temperature at each point on the boundary of a region (such as a solid circular disk) what will the temperature become at each point on the interior of the disk? And they are very useful for solving certain types of differential equations.

What if the function is not 2π-periodic?

If the function f is periodic with period P , we can "squeeze" f to create a new function g that does repeat every 2π units. Then after calculating the Fourier series for g, we can "unsqueeze" g to get a series for f.

Problems

The point of this section was only to use the Taylor series approach to create a new kind of infinite series, one whose building blocks were trigonometric functions, and the to show that this approach actually worked to approximate functions that were not even differentiable or continuous. There are still other infinite series that use different building blocks, but the point here was to illustrate that not all useful infinite series are power series.

Main Reference

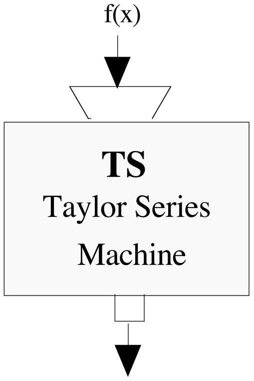
Fourier Analysis, T.W. Korner, Cambridge University Press, 1988 (available in paperback)

This is a beautifully written book that is willing to forego some of the completeness and generalities of the results in order to present an understandable development of the main ideas and their applications and the personalities involved in this development. The level is aimed at a university student in their 3rd or 4th year with a strong background in mathematics (physics helps too), but the stories and descriptions of the applications and the flow of ideas are mostly accessible to very good and motivated students with a year of calculus.

Comparing Taylor and Fourier Series

Taylor (Maclauren) Series

Think of the **TS** below as a machine that gives the **Taylor series** of any function dropped into it.



TS[f(x)]

the Taylor series for f(x)

We know this TS machine operates by creating the Taylor series

$$TS[\ f(x)\ ] = a_0 + a_1 * x + a_2 * x \ \ + a_3 * x \ \ + ...$$

where each $a_n = \dfrac{f^{(n)}(x)}{n!}$

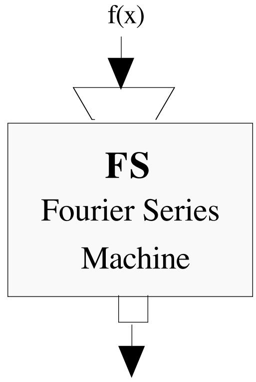
If we drop a **polynomial** P(x) into the TS machine, the output Taylor series is *exactly* the polynomial P(x) we put in.

If we drop a **general function** f(x) (e.g., sin(x) or e^x) into the TS machine, the output Taylor series is a polynomial which *approximates* the original f(x) we put in.

Why bother? Because polynomials are "really easy" to work with and are very well understood.

Fourier Series

Think of the **FS** below as a machine that gives the **Fourier series** of any function dropped into it.



FS[f(x)]

the Fourier series for f(x)

We know this FS machine operates by creating the Fourier series

$$FS[\ f(x)\ ] = \frac{a_0}{2} + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \ ...$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos(\ nx\ ) * f(x)\,dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} \sin(\ nx\ ) * f(x)\,dx$$

If we drop a **trigonometric polynomial** TP(x) into the FS machine, the output Fourier series is *exactly* the trigonometric polynomial TP(x) we put in.

If we drop a **general function** f(x) (e.g., 1 + x or |x| or a step function or a function given by data) into the FS machine, the output Fourier series is a trigonometric polynomial which *approximates* the original f(x) we put in.

Why bother? Because trigonometric polynomials are "really easy" to work with and are very well understood (+ other reasons).

MAPLE

You don't need a computer to calculate and graph the Fourier series of a function (your calculator can do the work), but Maple does a really nice job. The following program will automatically calculate the coefficients, build the Fourier series and graph both the original function and its Fourier series. One variation will even animate the approximation degree by degree. To use the program you need to enter the formula for the function (f:=x-> ...) and the degree of the approximation you want (N:= ...). The comments in italics are not part of the program.

```
with(plots):
f:=x->abs(1.3-floor(x/2.3));                    the original function – pick your own
a0:=evalf((1/Pi)*int(f(x),x=0..2*Pi)):
PF:=plot(f(x), x=0..2*Pi, y=0..2, color=blue, thickness=3, discont=true):
a0:=evalf((1/Pi)*int(f(x),x=0..2*Pi)):
```

The next commands automatically construct the Fourier series

```
N:=15:                    degree of the approximating Fourier series – pick your own
fs[0]:=a0/2;
FSplot[0]:=plot(fs[0], x=0..2*Pi, color=red, thickness=2, title="degree = "||i):
R[0]:=display(PF,FSplot[0], title="degree = 0"):
for i from 1 to N do
a:=evalf((1/Pi)*int(cos(i*x)*f(x), x=0..2*Pi)):
b:=evalf((1/Pi)*int(sin(i*x)*f(x), x=0..2*Pi)):
fs[i]:=fs[i-1]+a*cos(i*x)+b*sin(i*x);od:
fs[N];                    prints the approximating formula of the Fourier series of degree N
plot({f(x),fs[N]}, x=0..2*Pi, color=[red,blue], thickness=2, title="degree = "||N);   plots f and Nth degree Fourier series
```

This set of commands will animate the approximation degree by degree

```
for i from 1 to N do
FSplot[i]:=plot(fs[i], x=0..2*Pi, color=red, thickness=2, title="degree = "||i):
R[i]:=display(PF, FSplot[i]):
od:
M:=[seq(R[i], i=0..N)]:
display(M, axes=normal, insequence=true);
```