10.3 GEOMETRIC AND HARMONIC SERIES

This section uses ideas from Section 10.2 about series and their convergence to investigate some special types of series. Geometric series are very important and appear in a variety of applications. Much of the early work in the 17th century with series focused on geometric series and generalized them. Many of the ideas used later in this chapter originated with geometric series. It is easy to determine whether a geometric series converges or diverges, and when one does converge, we can easily find its sum. The harmonic series is important as an example of a divergent series whose terms approach zero. A final type of series, called "telescoping," is discussed briefly. Telescoping series are relatively uncommon, but their partial sums exhibit a particularly nice pattern.

Geometric Series: $\sum_{k=0}^{\infty} C\cdot r^k = C + C\cdot r + C\cdot r^2 + C\cdot r^3 + \ldots$

Example 1: Bouncing Ball: A "super ball" is thrown 10 feet straight up into the air. On each bounce, it rebounds to four fifths of its previous height (Fig. 1) so the sequence of heights is 10 feet, 8 feet, 32/5 feet, 128/25 feet, etc. (a) How far does the ball travel (up and down) during its n^{th} bounce? (b) Use a sum to represent the total distance traveled by the ball.

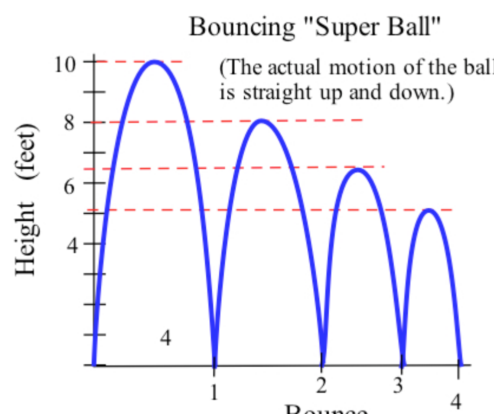

Fig. 1

Solution: Since the ball travels up and down on each bounce, the distance traveled during each bounce is twice the height of the ball on that bounce so $d_1 = 2(10 \text{ feet}) = 20$ feet, $d_2 = 16$ feet, $d_3 = 64/5$ feet, and, in general, $d_n = \frac{4}{5}\cdot d_{n-1}$. Looking at these values in another way,

$d_1 = 20$, $d_2 = \frac{4}{5}\cdot(20)$, $d_3 = \frac{4}{5}\, d_2 = \frac{4}{5}\cdot\frac{4}{5}\cdot 20 = (\frac{4}{5})^2 (20)$, $d_4 = \frac{4}{5}\cdot d_3 = \frac{4}{5}\cdot((\frac{4}{5})^2\cdot(20)) = (\frac{4}{5})^3\cdot(20)$,

and, in general, $d_n = (\frac{4}{5})^{n-1}\cdot(20)$.

In theory, the ball bounces up and down forever, and the total distance traveled by the ball is the sum of the distances traveled during each bounce (an up and down flight):

(first bounce) + (second bounce) + (third bounce) + (forth bounce) + . . .

$$= 20 + \frac{4}{5}(20) + (\frac{4}{5})^2 (20) + (\frac{4}{5})^3 (20) + \ldots$$

$$= 20\cdot(1 + \frac{4}{5} + (\frac{4}{5})^2 + (\frac{4}{5})^3 + \ldots) = 20\cdot\sum_{k=0}^{\infty} (\frac{4}{5})^k .$$

Practice 1: **Cake:** Three calculus students want to share a small square cake equally, but they go about it in a rather strange way. First they cut the cake into 4 equal square pieces, each person takes one square, and one square is left (Fig. 2). Then they cut the leftover piece into 4 equal square pieces, each person takes one square and one square is left. And they keep repeating this process. (a) What fraction of the total cake does each person "eventually" get? (b) Represent the amount of cake each person gets as a geometric series: (amount of first piece) + (amount of second piece) + . . .

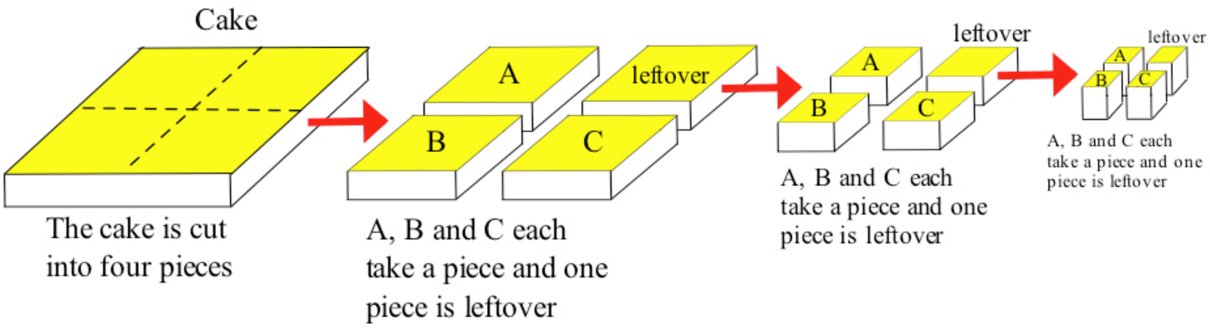

Fig. 2: Sharing a cake evenly among three students

Each series in the previous Example and Practice problems is a Geometric series, a series in which each term is a fixed multiple of the previous term. Geometric series have the form

$$\sum_{k=0}^{\infty} C\cdot r^k = C + C\cdot r + C\cdot r^2 + C\cdot r^3 + \ldots = C\cdot \sum_{k=0}^{\infty} r^k$$

with $C \neq 0$ and $r \neq 0$ representing fixed numbers. Each term in the series is r times the previous term. Geometric series are among the most common and easiest series we will encounter. A simple test determines whether a geometric series converges, and we can even determine the "sum" of the geometric series.

Geometric Series Theorem

The geometric series $\displaystyle\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \ldots \begin{cases} \textbf{converges} \text{ to } \dfrac{1}{1-r} & \text{if } |r| < 1 \\ \textbf{diverges} & \text{if } |r| \geq 1 \end{cases}$

Proof: **If $|r| \geq 1$**, then $|r^k|$ approaches 1 or $+\infty$ as k becomes arbitrarily large, so the terms $a_k = r^k$ of the geometric series do not approach 0. Therefore, by the n^{th} term test for divergence, the series diverges.

If $|r| < 1$, then the terms $a_k = r^k$ of the geometric series approach 0 so the series may or may not converge, and we need to examine the limit of the partial sums $s_n = 1 + r + r^2 + r^3 + \ldots + r^n$ of the series. For a geometric series, a clever insight allows us to calculate those partial sums:

$$(1 - r)\bullet s_n = (1 - r)\bullet(1 + r + r^2 + r^3 + \ldots + r^n)$$

$$= 1\bullet(1 + r + r^2 + r^3 + \ldots + r^n) - r\bullet(1 + r + r^2 + r^3 + \ldots + r^n)$$

$$= (1 + r + r^2 + r^3 + \ldots + r^n) - (r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1})$$

$$= 1 - r^{n+1} \quad .$$

Since $|r| < 1$ we know $r \neq 1$ so we can divide the previous result by $1 - r$ to get

$$s_n = 1 + r + r^2 + r^3 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r} - \frac{r^{n+1}}{1 - r} \quad .$$

This formula for the nth partial sum of a geometric series is sometimes useful, but now we are interested in the limit of s_n as n approaches infinity. Since $|r| < 1$, r^{n+1} approaches 0 as n approaches infinity, so we can conclude that the partial sums $s_n = \frac{1}{1 - r} - \frac{r^{n+1}}{1 - r}$ approach $\frac{1}{1-r}$ (as "$n \to \infty$").

The geometric series $\sum_{k=0}^{\infty} r^k$ converges to the value $\frac{1}{1 - r}$ when $-1 < r < 1$.

Finally, $\sum_{k=0}^{\infty} C\bullet r^k = C\bullet \sum_{k=0}^{\infty} r^k$ so we can easily determine whether or not $\sum_{k=0}^{\infty} C\bullet r^k$ converges and to what number.

Example 2: How far did the ball in Example 1 travel?

Solution: The distance traveled, $20(1 + \frac{4}{5} + (\frac{4}{5})^2 + (\frac{4}{5})^3 + \ldots)$, is a geometric series with $C = 20$ and $r = 4/5$. Since $|r| < 1$, the series $1 + \frac{4}{5} + (\frac{4}{5})^2 + (\frac{4}{5})^3 + \ldots$ converges to $\frac{1}{1 - r} = \frac{1}{1 - 4/5} = 5$, so the total distance traveled is

$$20(1 + \frac{4}{5} + (\frac{4}{5})^2 + (\frac{4}{5})^3 + \ldots) = 20(5) = \mathbf{100\ feet}.$$

Repeating decimal numbers are really geometric series in disguise, and we can use the Geometric Series Theorem to represent the exact value of the sum as a fraction.

Example 3: Represent the repeating decimals $0.\overline{4}$ and $0.\overline{13}$ as geometric series and find their sums.

Solution: $0.\overline{4} = 0.444\ldots = \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \ldots = \frac{4}{10} \bullet(1 + \frac{1}{10} + (\frac{1}{10})^2 + (\frac{1}{10})^3 + \ldots)$

which is a geometric series with $a = 4/10$ and $r = 1/10$. Since $|r| < 1$, the geometric series converges to $\frac{1}{1-r} = \frac{1}{1-1/10} = \frac{10}{9}$, and $0.\overline{4} = \frac{4}{10}\left(\frac{10}{9}\right) = \mathbf{\frac{4}{9}}$.

Similarly, $0.\overline{13} = 0.131313\ldots = \frac{13}{100} + \frac{13}{10000} + \frac{13}{1000000} + \ldots$

$$= \frac{13}{100} \cdot \left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \left(\frac{1}{100}\right)^3 + \ldots \right)$$

$$= \frac{13}{100} \cdot \left(\frac{1}{1-1/100} \right) = \frac{13}{100}\left(\frac{100}{99}\right) = \mathbf{\frac{13}{99}} .$$

Practice 2: Represent the repeating decimals $0.\overline{3}$ and $0.\overline{432}$ as geometric series and find their sums.

One reason geometric series are important for us is that some series involving powers of x are geometric series.

Example 4: $\sum_{k=0}^{\infty} 3x^k = 3 + 3x + 3x^2 + \ldots$ and $\sum_{k=0}^{\infty} (2x-5)^k = 1 + (2x-5) + (2x-5)^2 + \ldots$ are geometric series with $r = x$ and $r = 2x - 5$, respectively. Find the values of x for each series so that the series converges.

Solution: A geometric series converges if and only if $|r| < 1$, so the first series converges if and only if $|x| < 1$, or, equivalently, $-1 < x < 1$. The sum of the first series is $\frac{3}{1-x}$.

In the second series $r = 2x - 5$ so the series converges if and only if $|2x - 5| < 1$. Removing the absolute value and solving for x, we get $-1 < 2x - 5 < 1$, and (adding 5 to each side and then dividing by 2) $2 < x < 3$. The second series converges if and only if $2 < x < 3$. The sum of the second series is $\frac{1}{1-(2x-5)}$ or $\frac{1}{6-2x}$.

Practice 3: The series $\sum_{k=0}^{\infty} (2x)^k$ and $\sum_{k=0}^{\infty} (3x-4)^k$ are geometric series. Find the ratio r for each series, and find all values of x for each series so that the series converges.

The series in the previous Example and Practice are called "power series" because they involve powers of the variable x. Later in this chapter we will investigate other power series which are not geometric series (e.g., $1 + x + x^2/2 + x^3/3 + \ldots$), and we will try to find values of x which guarantee that the series converge.

Harmonic Series: $\sum_{k=1}^{\infty} \frac{1}{k}$

The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is one of the best known and most important **divergent** series. It is called the **harmonic series** because of its ties to music (Fig. 3).

A taut piece of string such as a guitar string or a piano wire can only vibrate so that an integer number of waves are formed. The fundamental mode determines the note being played, and the number and intensity of the harmonics (overtones) present determine the characteristic quality of the sound. Because of these different characteristic qualities, a middle C (264 vibrations per second) played on a piano can be distinguished from a C on a clarinet.

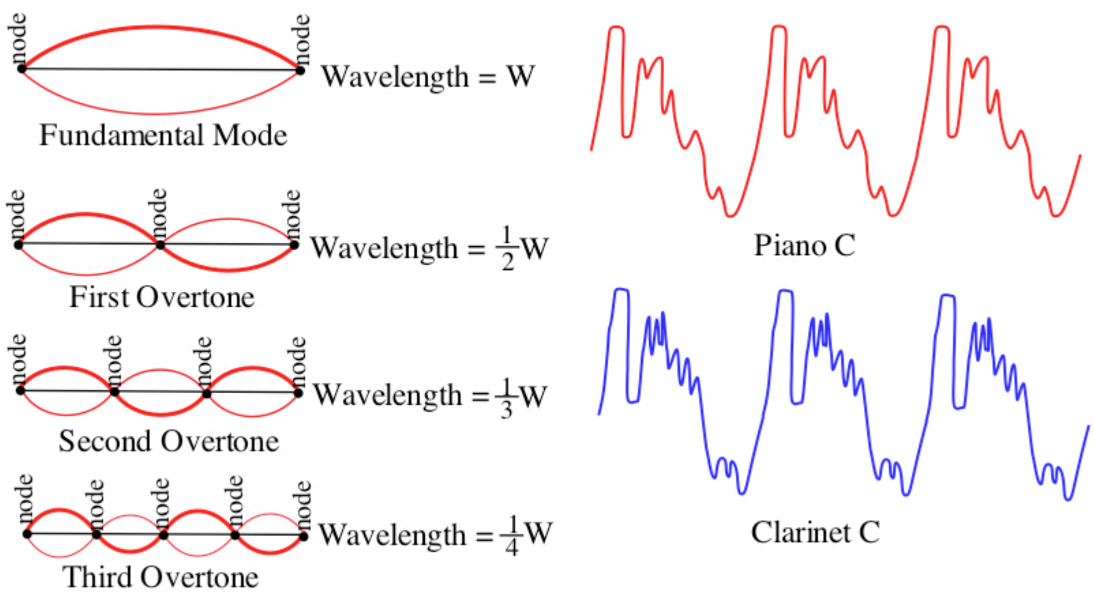

Fig. 3

If we simply calculate partial sums of the harmonic series, it is not clear that the series diverges — the partial sums s_n grow, but as n becomes large, the values of s_n grow very, very slowly. Fig. 4 shows the values of n needed for the partial sums s_n to finally exceed the integer values 4, 5, 6, 8, 10, and 15. To examine the divergence of the harmonic series, brain power is much more effective than a lot of computing power.

n	s_n
31	4.0224519544
83	5.00206827268
227	6.00436670835
1,674	8.00048557200
12,367	10.00004300827
1,835,421	15.00000378267

Fig. 4

We can show that the harmonic series is divergent by showing that the terms of the harmonic series can be grouped into an infinite number of disjoint "chunks" each of which has a sum larger than 1/2. The series $\sum_{k=1}^{\infty} \frac{1}{2}$ is clearly divergent because the partial sums grow arbitrarily large: by adding enough of the terms together we can make the partial sums, $s_n > n/2$, larger than any predetermined number. Then we can conclude that the partial sums of the harmonic series also approach infinity so the harmonic series diverges.

Theorem: The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ **diverges**.

Proof: (This proof is essentially due to Oresme in 1630, twelve years before Newton was born. In 1821 Cauchy included Oresme's proof in a "Course in Analysis" and it became known as Cauchy's argument.)

Let S represent the sum of the harmonic series, $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$, and group the terms of the series as indicated by the parentheses:

$$S = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\text{2 terms, each greater than or equal to 1/4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\text{4 terms, each greater than or equal to 1/8}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16}\right)}_{\text{8 terms, each greater than or equal to 1/16}} + \underbrace{\left(\frac{1}{17} + \ldots + \frac{1}{32}\right)}_{\text{16 terms, each greater than or equal to 1/32}} + \ldots$$

Each group in parentheses has a sum greater than 1/2, so

$S > 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \ldots$ and

the sequence of partial sums $\{ s_n \}$ does not converge to a finite number. Therefore, the harmonic series diverges.

The harmonic series is an example of a **divergent** series whose terms, $a_k = 1/k$, approach 0. If the terms of a series approach 0, the series may or may not converge — we need to investigate further.

Telescoping Series

Sailors in the seventeenth and eighteenth centuries used telescopes (Fig. 5) which could be extended for viewing and collapsed for storing. Telescoping series get their name because they exhibit a similar "collapsing" property. Telescoping series are rather uncommon. But they are easy to analyze, and it can be useful to recognize them.

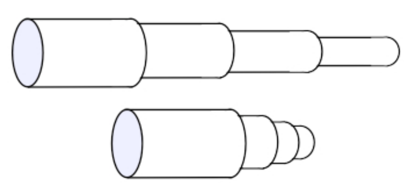

Fig. 5

Example 5: Determine a formula for the partial sum s_n of the series $\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]$.

Then find $\lim_{n\to\infty} s_n$. (Suggestion: It is tempting to algebraically consolidate terms, but the pattern is clearer in this case if you first write out all of the terms.)

Solution: $s_1 = a_1 = 1-\frac{1}{2}$. In later values of s_n, part of each term cancels part of the next term:

$$s_2 = a_1 + a_2 = \left(1-\frac{\mathbf{1}}{\mathbf{2}}\right) + \left(\frac{\mathbf{1}}{\mathbf{2}}-\frac{1}{3}\right) = 1-\frac{1}{3}$$

$$s_3 = a_1 + a_2 + a_3 = \left(1-\frac{\mathbf{1}}{\mathbf{2}}\right) + \left(\frac{\mathbf{1}}{\mathbf{2}}-\frac{\mathbf{1}}{\mathbf{3}}\right) + \left(\frac{\mathbf{1}}{\mathbf{3}}-\frac{1}{4}\right) = 1-\frac{1}{4}$$

In general, many of the pieces in each partial sum "collapse" and we are left with a simple form of s_n:

$$s_n = a_1 + a_2 + \ldots + a_{n-1} + a_n = \left(1-\frac{\mathbf{1}}{\mathbf{2}}\right) + \left(\frac{\mathbf{1}}{\mathbf{2}}-\frac{\mathbf{1}}{\mathbf{3}}\right) + \ldots + \left(\frac{\mathbf{1}}{\mathbf{n-1}}-\frac{\mathbf{1}}{\mathbf{n}}\right) + \left(\frac{\mathbf{1}}{\mathbf{n}}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1}$$

Finally, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$ so the series converges to 1: $\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right] = 1$.

Practice 4: Find the sum of the series $\sum_{k=3}^{\infty} \left[\sin\left(\frac{1}{k}\right) - \sin\left(\frac{1}{k+1}\right) \right]$.

PROBLEMS

In problems 1 – 6, rewrite each geometric series using the sigma notation and calculate the value of the sum.

1. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

2. $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots$

3. $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots$

4. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots$

5. $-\frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$

6. $1 + \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \ldots$

7. Rewrite each series in the form of a sum of r^k, and then show that

   (a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$, $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots = \frac{1}{2}$, and (b) for $a > 1$, $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \ldots = \frac{1}{a-1}$.

8. A ball is thrown 10 feet straight up into the air, and on each bounce, it rebounds to 60% of its previous height. (a) How far does the ball travel (up and down) during its n^{th} bounce. (b) Use a sum to represent the total distance traveled by the ball. (c) Find the total distance traveled by the ball.

9. An old tennis ball is thrown 20 feet straight up into the air, and on each bounce, it rebounds to 40% of its previous height. (a) How far does the ball travel (up and down) during its n^{th} bounce? (b) Use a sum to represent the total distance traveled by the ball. (c) Find the total distance traveled by the ball.

10. Eighty people are going on an expedition by horseback through desolate country. The people and gear require 90 horses, and additional horses are needed to carry food for the original 90 horses. Each additional horse can carry enough food to feed 3 horses for the trip. How many additional horses are needed? (The original 90 horses will require 30 extra horses to carry their food. The 30 extra horses require 10 more horses to carry their food. etc.)

11. The mathematical diet you are following says you can eat "half of whatever is on the plate," so first you bite off one half of the cake and put the other half back on the plate. Then you pick up the remaining half from the plate (it's "on the plate"), bite off half of that, and return the rest to the plate. And you continue this silly process of picking up the piece from the plate, biting off half, and returning the rest to the plate. (a) Represent the total amount you eat as a series. (b) How much of the cake is **left** after 1 bite, 2 bites, n bites? (c) "Eventually," how much of the cake do you eat?

12. Suppose in Fig. 6 we begin with a square with sides of length 1 (area = 1) and construct another square inside by connecting the midpoints of the sides. Then the new square has area 1/2. If we continue the process of constructing each new square by connecting the midpoints of the sides of the previous square, we get a sequence of squares each of which has 1/2 half the area of the previous square. Find the total area of all of the squares.

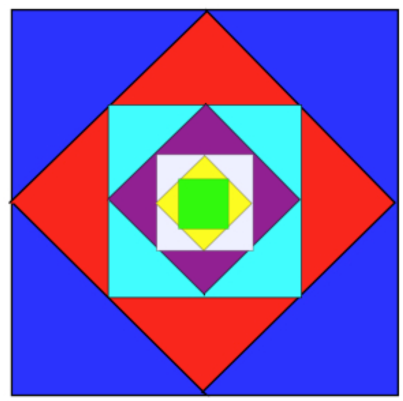

Fig. 6

13. Suppose in Fig. 7 we begin with a triangle with area 1 and construct another triangle inside by connecting the midpoints of the sides. Then the new triangle has area 1/4. Imagine that this construction process is continued and find the total area of all of the triangles.

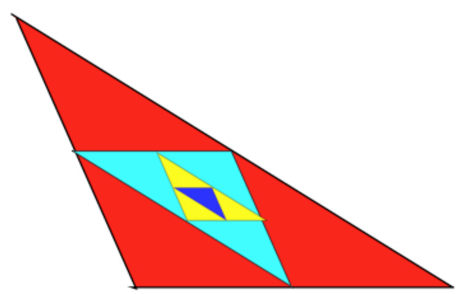

Fig. 7

14. Suppose in Fig. 8 we begin with a circle of radius 1 and construct 2 more circles inside, each with radius 1/2. Continue the process of constructing two new circles inside each circle from the previous step and find the total area of the circles.

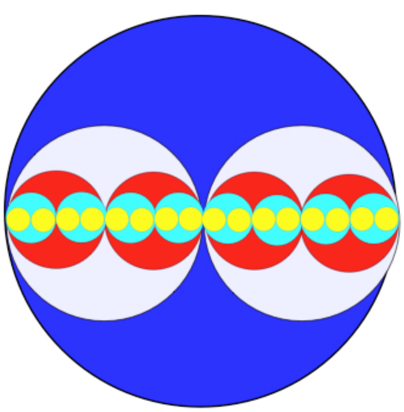

Fig. 8

15. The construction of the Helga von Koch snowflake begins with an equilateral triangle of area 1 (Fig. 9). Then each edge is subdivided into three equal lengths, and three equilateral triangles, each with area 1/9, are built on these "middle thirds" adding a total of 3(1/9) to the original area. The process is repeated: at the next stage, 3•4 equilateral triangles, each with area 1/81, are built on the new "middle thirds" adding 3•4•(1/81) more area. (a) Find the total area that results when this process is repeated forever.

number of edges	# triangles added	area of each triangle	total area added
3	0	0	0
$3\cdot 4$	3	1/9	3/9
$3\cdot 4^2$	$3\cdot 4$	$1/9^2$	$3\cdot 4/9^2 = (3/9)(4/9)$
$3\cdot 4^3$	$3\cdot 4^2$	$1/9^3$	$3\cdot 4^2/9^3 = (3/9)(4^2/9^2)$
$3\cdot 4^4$	$3\cdot 4^3$	$1/9^4$	$3\cdot 4^3/9^4 = (3/9)(4^3/9^3)$

so the total added area of the snowflake is

$$1 + \frac{3}{9} + \left(\frac{3}{9}\right)\left(\frac{4}{9}\right) + \left(\frac{3}{9}\right)\left(\frac{4^2}{9^2}\right) + \left(\frac{3}{9}\right)\left(\frac{4^3}{9^3}\right) + \ldots$$

(b) Express the perimeter of the Koch Snowflake as a geometric series and find its sum.

(The area is finite, but the perimeter is infinite.)

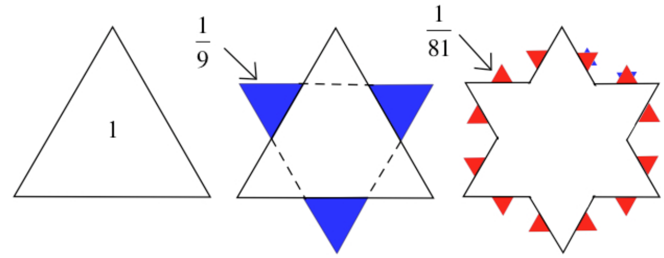

Fig. 9: Helga von Koch Snowflake

16. Harmonic Tower: The base of a tower is a cube whose edges are each one foot long. On top of it are cubes with edges of length 1/2, 1/3, 1/4, ... (Fig. 10).

(a) Represent the total height of the tower as a series. Is the height finite?

(b) Represent the total surface area of the cubes as a series.

(c) Represent the total volume of the cubes as a series.

(In the next section we will be able to determine if this surface area and volume are finite or infinite.)

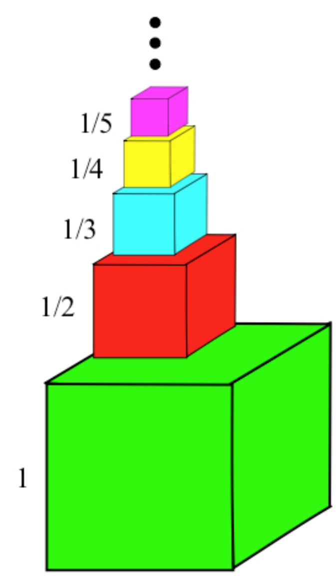

Fig. 10: Harmonic Tower

17. The base of a tower is a sphere whose radius is each one foot long. On top of each sphere is another sphere with radius one half the radius of the sphere immediately beneath it (Fig 11). (a) Represent the total height of the tower as a series and find its sum. (b) Represent the total surface area of the spheres as a series and find its sum. (c) Represent the total volume of the spheres as a series and find its sum.

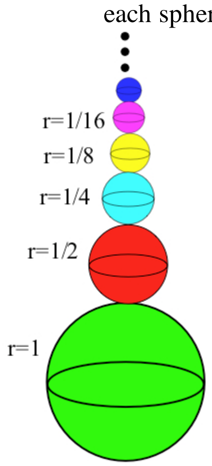

Fig. 11

18. Represent the repeating decimals $0.\overline{6}$ and $0.\overline{63}$ as geometric series and find the value of each series as a simple fraction.

19. Represent the repeating decimals $0.\overline{8}$, $0.\overline{9}$, and $0.\overline{285714}$ as geometric series and find the value of each series as a simple fraction.

20. Represent the repeating decimals $0.\overline{a}$, $0.\overline{ab}$, and $0.\overline{abc}$ as geometric series and find the value of each series as a simple fraction. What do you think the simple fraction representation is for $0.\overline{abcd}$?

In problems 21 – 32, find all values of x for which each geometric series converges.

21. $\sum_{k=1}^{\infty} (2x+1)^k$ 22. $\sum_{k=1}^{\infty} (3-x)^k$ 23. $\sum_{k=1}^{\infty} (1-2x)^k$

24. $\sum_{k=1}^{\infty} 5x^k$ 25. $\sum_{k=1}^{\infty} (7x)^k$ 26. $\sum_{k=1}^{\infty} (x/3)^k$

27. $1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots$ 28. $1 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3} + \ldots$ 29. $1 + 2x + 4x^2 + 8x^3 + \ldots$

30. $\sum_{k=1}^{\infty} (2x/3)^k$ 31. $\sum_{k=1}^{\infty} \sin^k(x)$ 32. $\sum_{k=1}^{\infty} e^{kx} = \sum_{k=1}^{\infty} (e^x)^k$

33. One student thought the formula was $1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$. The second student said "That can't be right. If we replace x with 2, then the formula says the sum of the positive numbers $1 + 2 + 4 + 8 + \ldots$ is a negative number $\frac{1}{1-2} = -1$." Who is right? Why?

34. The Classic Board Problem: If you have identical 1 foot long boards, they can be arranged to hang over the edge of a table. One board can extend 1/2 foot beyond the edge (Fig. 12), two boards can extend 1/2 + 1/4 feet, and, in general, n boards can extend 1/2 + 1/4 + 1/6 + . . . + 1/(2n) feet beyond the edge.

(a) How many boards are needed for an arrangement in which the entire top board is beyond the edge of the table?

(b) How many boards are needed for an arrangement in which the entire top two boards are beyond the edge of the table?

(c) How far can an arrangement extend beyond the edge of the table?

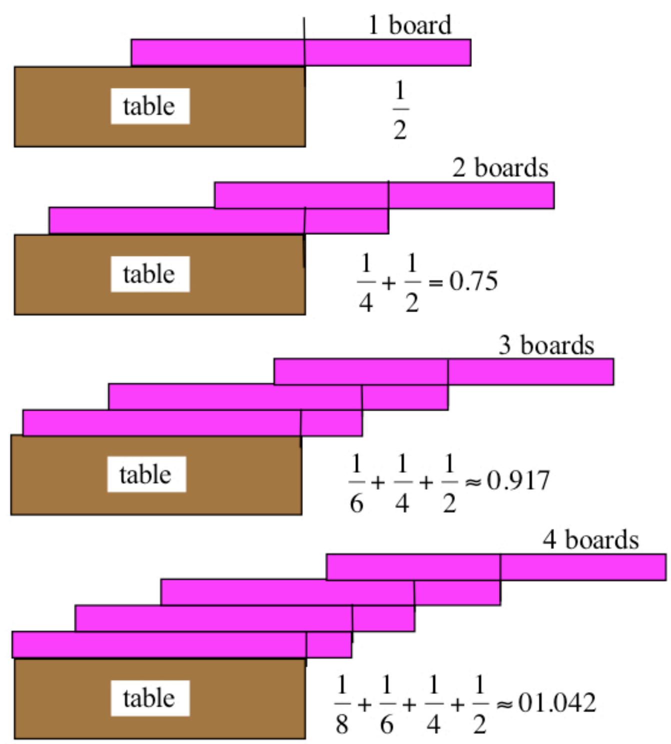

Fig. 12

In problems 35 – 40, calculate the value of the partial sum for n = 4 and n = 5 and find a formula for s_n .

(The patterns may be more obvious if you do not simplify each term.)

35. $\sum_{k=3}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}\right]$ 36. $\sum_{k=1}^{\infty}\left[\frac{1}{k}-\frac{1}{k+2}\right]$ 37. $\sum_{k=1}^{\infty}[k^3-(k+1)^3]$

38. $\sum_{k=1}^{\infty}\ln\left(\frac{k}{k+1}\right)$ 39. $\sum_{k=3}^{\infty}[f(k)-f(k+1)]$ 40. $\sum_{k=1}^{\infty}[g(k)-g(k+2)]$

In problems 41 – 44, calculate s_4 and s_5 for each series and find the limit of s_n as n approaches infinity. If the limit is a finite value, it represents the value of the infinite series.

41. $\sum_{k=1}^{\infty}\sin(\frac{1}{k})-\sin(\frac{1}{k+1})$ 42. $\sum_{k=2}^{\infty}\cos(\frac{1}{k})-\cos(\frac{1}{k+1})$ 43. $\sum_{k=2}^{\infty}\frac{1}{k^2}-\frac{1}{(k+1)^2}$

44. $\sum_{k=3}^{\infty}\ln(1-\frac{1}{k^2})$ (Suggestion: Rewrite $1-\frac{1}{k^2}$ as $\dfrac{1-\frac{1}{k}}{1-\frac{1}{k+1}}$)

Problems 45 and 46 are outlines of two "proofs by contradiction" that the harmonic series is divergent. Each proof starts with the assumption that the "sum" of the harmonic series is a finite number, and then an obviously false conclusion is derived from the assumption. Verify that each step follows from the assumption and previous steps, and explain why the conclusion is false.

45. Assume that $H = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\ldots$ is a finite number, and let $O = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$ be the sum of the "odd reciprocals," and $E = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots$ be the sum of the "even reciprocals." Then

(i) $H = O + E$, (ii) each term of O is larger than the corresponding term of E so $O > E$,

and (iii) $E = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots = \frac{1}{2}\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\} = \frac{1}{2}H$.

Therefore $H = O + E > \frac{1}{2}H + \frac{1}{2}H = H$ (so "H is strictly bigger than H," a contradiction).

II. Assume that $H = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\ldots$ is a finite number, and, starting with the second term, group the terms into groups of three. Then, using $\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1} > 3\cdot\frac{1}{n}$, we have

$$
\begin{aligned}
H &= 1+(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7})+(\tfrac{1}{8}+\tfrac{1}{9}+\tfrac{1}{10})+\ldots \\
&> 1+(\quad 1 \quad)+(\quad \tfrac{1}{2} \quad)+(\quad \tfrac{1}{3} \quad)+\ldots \\
&= 1+(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots) = 1 + H
\end{aligned}
$$

Therefore, $H > 1 + H$ (so "H is bigger than 1 + H").

46. Jacob Bernoulli (1654–1705) was a master of understanding and manipulating series by breaking a difficult series into a sum of easier series. He used that technique to find the sum of the non–geometric series

$\sum_{k=1}^{\infty} \frac{k}{2^k}$ = 1/2 + 2/4 + 3/8 + 4/16 + 5/32 + . . . + $k/2^k$ + . . . in his book Ars Conjectandi , 1713.

Show that	1/2 +	2/4 +	3/8 +	4/16 +	5/32 + . . . +	$n/2^n$ + . . .	can be written as
	1/2 +	1/4 +	1/8 +	1/16 +	1/32 + . . . +	$1/2^n$ + . . .	(a)
plus		1/4+	1/8 +	1/16 +	1/32 + . . . +	$1/2^n$ + . . .	(b)
plus			1/8 +	1/16 +	1/32 + . . . +	$1/2^n$ + . . .	(c)
plus . . . etc.							

Find the values of the geometric series (a), (b), (c), etc. and then find the sum of these values (another geometric series).

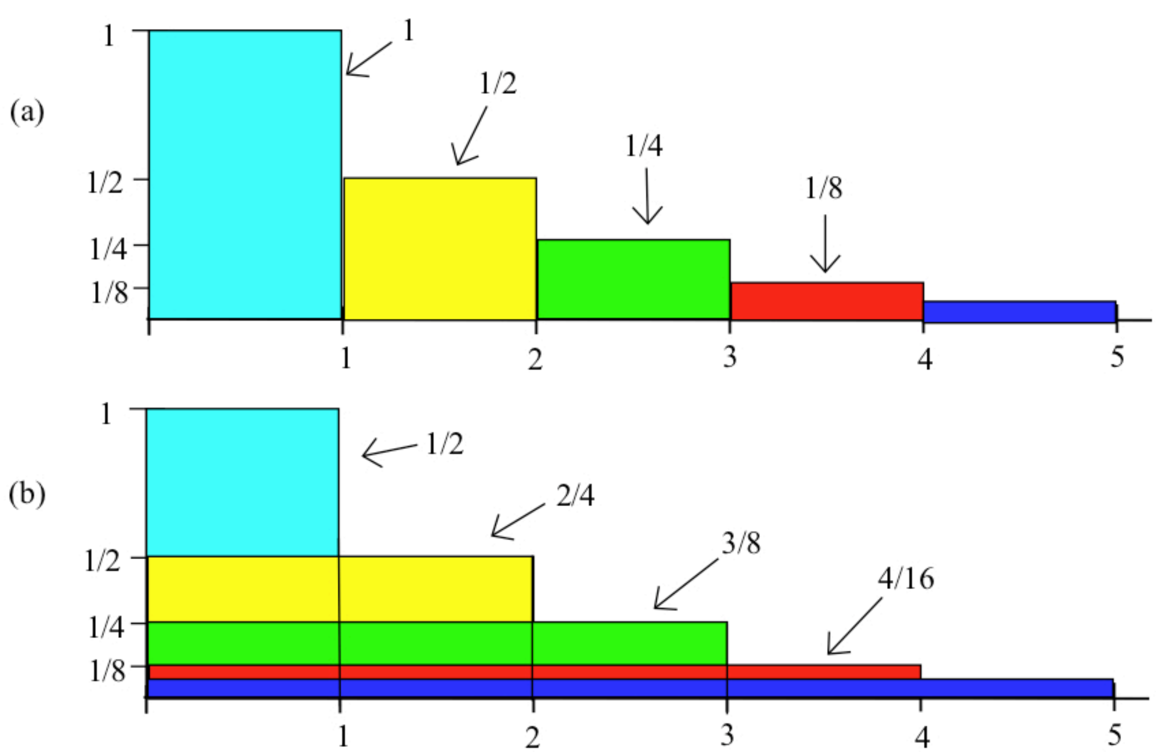

Fig. 13

47. Bernoulli's approach in problem 46 can also be interpreted as a geometric argument for representing the area in Fig. 13 in two different ways. (a) Represent the total area in Fig. 13a as a (geometric) sum of the areas of the side–by–side rectangles, and find the sum of the series. (b) Represent the total area of the stacked rectangles in Fig. 13b as a sum of the areas of the horizontal slices.

Since both series represent the same total area, the values of the series are equal.

48. Use the approach of problem 46 to find a formula for the sum of

(a) the value of $\sum_{k=1}^{\infty} \frac{k}{3^k} = 1/3 + 2/9 + 3/27 + 4/81 + \ldots + k/3^k + \ldots$ and

(b) a formula for the value of $\sum_{k=1}^{\infty} \frac{k}{c^k} = 1/c + 2/c^2 + 3/c^3 + 4/c^4 + \ldots + k/c^k + \ldots$ for $c > 1$.

(answers: (a) 3/4, (b) $c/(c-1)^2$)

Practice Answers

Practice 1: (a) Since they each get equal shares, and the whole cake is distributed, they each get 1/3 of the cake. More precisely, after step 1, 1/4 of the cake remains and 3/4 was shared. After step 2, $(1/4)^2$ of the cake remains and $1 - (1/4)^2$ was shared. After step n, $(1/4)^n$ of the cake remains and $1 - (1/4)^n$ was shared. So after step n, each student has $(\frac{1}{3})(1 - (1/4)^n)$ of the cake. "Eventually," each student gets (almost) $\frac{1}{3}$ of the cake.

(b) $(\frac{1}{4}) + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \ldots = (\frac{1}{4})\{1 + (\frac{1}{4}) + (\frac{1}{4})^2 + \ldots\}$

Practice 2: $0.\overline{3} = 0.333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots = \frac{3}{10} \cdot (1 + \frac{1}{10} + (\frac{1}{10})^2 + (\frac{1}{10})^3 + \ldots)$

which is a geometric series with $a = 3/10$ and $r = 1/10$. Since $|r| < 1$, the geometric series

converges to $\frac{1}{1-r} = \frac{1}{1-1/10} = \frac{10}{9}$, and $0.\overline{3} = \frac{3}{10}(\frac{10}{9}) = \mathbf{\frac{3}{9} = \frac{1}{3}}$.

Similarly, $0.\overline{432} = 0.432432432\ldots = \frac{432}{1000} + \frac{432}{1000000} + \frac{432}{1000000000} + \ldots$

$$= \frac{432}{1000} \cdot (1 + \frac{1}{1000} + (\frac{1}{1000})^2 + (\frac{1}{100})^3 + \ldots)$$

$$= \frac{432}{1000} \cdot (\frac{1}{1-1/1000}) = \frac{432}{1000}(\frac{1000}{999}) = \mathbf{\frac{432}{999} = \frac{16}{37}}.$$

Practice 3: $r = 2x$: If $|2x| < 1$, then $-1 < 2x < 1$ so $-1/2 < x < 1/2$.

$\sum_{k=0}^{\infty} (2x)^k$ converges (to $\frac{1}{1-2x}$) when $-1/2 < x < 1/2$.

$r = 3x - 4$: If $|3x - 4| < 1$, then $-1 < 3x - 4 < 1$ so $3 < 3x < 5$ and $1 < x < 5/3$.

$\sum_{k=0}^{\infty} (3x-4)^k$ converges (to $\frac{1}{1-(3x-4)} = \frac{1}{5-3x}$) when $1 < x < 5/3$.

Practice 4: Let $s_n = \sum_{k=3}^{n} \sin(\frac{1}{k}) - \sin(\frac{1}{k+1})$

$= \{\sin(\frac{1}{3}) - \sin(\frac{1}{4})\} + \{\sin(\frac{1}{4}) - \sin(\frac{1}{5})\} + \{\sin(\frac{1}{5}) - \sin(\frac{1}{6})\} + ... + \{\sin(\frac{1}{n}) - \sin(\frac{1}{n+1})\}$

$= \sin(\frac{1}{3}) - \sin(\frac{1}{n+1})$.

Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \{\sin(\frac{1}{3}) - \sin(\frac{1}{n+1})\} = \sin(\frac{1}{3})$ so the series converges to $\sin(\frac{1}{3})$:

$\sum_{k=1}^{\infty} \sin(\frac{1}{k}) - \sin(\frac{1}{k+1}) = \sin(\frac{1}{3}) \approx 0.327$.

Appendix: MAPLE and WolframAlpha for Partial Sums of Geometric Series

MAPLE command for

$\sum_{n=0}^{100} \frac{3}{2^n}$: sum(3*(1/2)^n , n=0..100) ; (then press ENTER key)

$\sum_{n=0}^{\infty} \frac{3}{2^n}$: sum(3*(1/2)^n , n=0..infinity) ; (then press ENTER key)

WolframAlpha (free at http://www.wolframalpha.com)

Asking sum 3/2^k , k = 1 to infinity

gives the sum of the infinite series and a graph of the partial sums.

Asking sum 3/2^k , k = 1 to 100

gives the sum as an exact fraction (strange) and as a decimal.