# 10.5 POSITIVE TERM SERIES: COMPARISON TESTS

This section discusses how to determine whether some series converge or diverge by comparing them with other series which we already know converge or diverge. In the basic Comparison Test we compare the two series term by term. In the more powerful Limit Comparison Test, we compare limits of ratios of the terms of the two series. Finally, we focus on the parts of the terms of a series that determine whether the series converges or diverges.

## Comparison Test

Informally, if the individual terms of our series are smaller than the corresponding terms of a known convergent series, then our series converges. If our series is larger, term by term, than a known divergent series then our series diverges. If the individual terms of our series are larger than the corresponding terms of a convergent series or smaller than the corresponding terms of a divergent series, then our series may converge or diverge — the Comparison Test does not tell us.

**Comparison Test**

Suppose we want to determine whether $\sum_{k=1}^{\infty} a_k$ converges or diverges.

(a) If there is a convergent series $\sum_{k=1}^{\infty} c_k$ with $0 < a_k \le c_k$ for all k, then $\sum_{k=1}^{\infty} a_k$ converges.

(b) If there is a divergent series $\sum_{k=1}^{\infty} d_k$ with $a_k \ge d_k > 0$ for all k, then $\sum_{k=1}^{\infty} a_k$ diverges.

Proof: Since all of the terms of the $a_k$, $c_k$, and $d_k$ series are positive, their sequences of partial sums are all monotonic increasing. The proof compares the partial sums of the various series.

(a) Suppose that $0 < a_k \le c_k$ for all k and that $\sum_{k=1}^{\infty} c_k$ converges. Since $\sum_{k=1}^{\infty} c_k$ converges,

then the partial sums $t_n = \sum_{k=1}^{n} c_k$ approach a finite limit: $\lim_{n\to\infty} t_n = L$.

For each n, $s_n \le t_n$ (why?) so $\lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n = L$, and the sequence $\{ s_n \}$ is bounded by L. Finally, by the Monotone Convergence Theorem, we can conclude that $\{ s_n \}$ converges and that the series $\sum_{k=1}^{\infty} a_k$ converges.

(b) Suppose that $a_k \geq d_k > 0$ for all k and that $\sum_{k=1}^{\infty} d_k$ diverges. Since $\sum_{k=1}^{\infty} d_k$ diverges, then the partial sums $u_n = \sum_{k=1}^{n} d_k$ approach infinity: $\lim_{n\to\infty} u_n = \infty$. Then

$$\lim_{n\to\infty} s_n \geq \lim_{n\to\infty} u_n = \infty$$

so the sequence of partial sums $\{ s_n \}$ diverges and the series $\sum_{k=1}^{\infty} a_k$ diverges.

The Comparison Test requires that we select and compare our series against a series whose convergence or divergence is known, and that choice requires that we know a collection of series that converge and some that diverge. Typically, we pick a p–series or a geometric series to compare with our series, but this choice requires some experience and practice.

**Example 1:** Use the Comparison Test to determine the convergence or divergence of

(a) $\sum_{k=1}^{\infty} \frac{1}{k^2 + 3}$ and (b) $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ .

Solution: For these two series it is useful to compare with p–series for appropriate values of p.

(a) For all k, $\frac{1}{k^2+3} < \frac{1}{k^2}$ , and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (P–Test, p = 2)

so $\sum_{k=1}^{\infty} \frac{1}{k^2 + 3}$ converges.

(b) For all k, $\frac{k+1}{k^2} = \frac{1}{k} + \frac{1}{k^2} > \frac{1}{k}$ .

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (P–Test, p = 1), we conclude that $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ diverges.

**Practice 1:** Use the Comparison Test to determine the convergence or divergence of

(a) $\sum_{k=3}^{\infty} \frac{1}{\sqrt{k-2}}$ and (b) $\sum_{k=1}^{\infty} \frac{1}{2^k + 7}$ .

**Example 2:** A student has shown algebraically that $\frac{1}{k^2} < \frac{1}{k^2-1} < \frac{1}{k}$ for all $k \ge 2$. From this information and the Comparison Test, what can the student conclude about the convergence of the series $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ ?

Solution: Nothing. The Comparison Test only gives a definitive answer if our series is smaller than a convergent series or if our series is larger than a divergent series. In this example, our series is larger than a convergent series, $\sum \frac{1}{k^2}$ , and is smaller than a divergent series, $\sum \frac{1}{k}$ , so we can not conclude anything about the convergence of $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$.

However, we can show that if $k \ge 2$ then $\frac{1}{k^2-1} < \frac{2}{k^2}$ . Since $\sum_{k=2}^{\infty} \frac{2}{k^2}$ converges (P–Test), we can conclude that $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges. Next in this section we present a variation on the Comparison Test that allows us to quickly conclude that $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges.

## Limit Comparison Test

The exact value of the sum of a series depends on every part of the terms of the series, but if we are only asking about convergence or divergence, some parts of the terms can be safely ignored. For example, the three series with terms $1/k^2$ , $1/(k^2+1)$ , and $1/(k^2-1)$ converge to different values,

$$\sum_{k=2}^{\infty} \frac{1}{k^2} \approx 0.645 , \quad \sum_{k=2}^{\infty} \frac{1}{k^2+1} \approx 0.577 , \quad \sum_{k=2}^{\infty} \frac{1}{k^2-1} = 0.750 ,$$

but they all do converge. The "+ 1" and "– 1" in the denominators affect the value of the final sum, but they do not affect whether that sum is finite or infinite. When $k$ is a large number, the values of $1/(k^2+1)$ and $1/(k^2-1)$ are both very close to the value of $1/k^2$ , and the convergence or divergence of the series $\sum_{k=2}^{\infty} \frac{1}{k^2+1}$ and $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ can be predicted from the convergence or divergence of the series $\sum_{k=2}^{\infty} \frac{1}{k^2}$ .

The Limit Comparison Test states these ideas precisely.

**Limit Comparison Test**

Suppose $a_k > 0$ for all k, and we want to determine whether $\sum_{k=1}^{\infty} a_k$ converges or diverges.

If there is a series $\sum_{k=1}^{\infty} b_k$ so that $\lim_{k\to\infty} \frac{a_k}{b_k} = L$, a **positive, finite** value,

then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge or both diverge.

Idea for a proof: The key idea is that if $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ is a **positive, finite** value, then, when n is very large, $\frac{a_k}{b_k} \approx L$ so $a_k \approx L \cdot b_k$ and $\sum_{k=N}^{\infty} a_k \approx L \cdot \sum_{k=N}^{\infty} b_k$ . If one of these series converges, then so does the other. If one of these series diverges, then so does the other. When n is a relatively small number, the $a_k$ and $b_k$ values may not have a ratio close to L, but the first "few" values of a series do not affect the convergence or divergence of the series. A proof of the Limit Comparison Test is given in an Appendix after the Practice Answers.

**Example 3**: Put $a_k = \frac{1}{k^2}$ , $b_k = \frac{1}{k^2+1}$ , and $c_k = \frac{1}{k^2-1}$ and show that $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$ and $\lim_{k\to\infty} \frac{a_k}{c_k} = 1$. Since $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges (P–Test, p = 2) we can conclude that $\sum_{k=2}^{\infty} \frac{1}{k^2+1}$ and $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ both converge too.

Solution: $\frac{a_k}{b_k} = \frac{1/k^2}{1/(k^2+1)} = \frac{k^2+1}{k^2} = 1 + \frac{1}{k^2} \longrightarrow 1$ so $L = 1$ is positive and finite.

Similarly, $\frac{a_k}{c_k} = \frac{1/k^2}{1/(k^2-1)} = \frac{k^2-1}{k^2} = 1 - \frac{1}{k^2} \longrightarrow 1$ so $L = 1$ is positive and finite.

**Practice 2**: (a) Find a p–series to "limit–compare" with $\sum_{k=2}^{\infty} \frac{k^2 + 5k}{k^3 + k^2 + 7}$ .

(Suggestion: put $a_k = \frac{k^2 + 5k}{k^3 + k^2 + 7}$ and find a value of p so that $b_k = \frac{1}{k^p}$ and $\lim_{k\to\infty} \frac{a_k}{b_k}$ = L, a positive, finite value.) Does $\sum_{k=2}^{\infty} \frac{k^2 + 5k}{k^3 + k^2 + 7}$ converge?

(b) Find a p–series to compare with $\sum_{k=3}^{\infty} \frac{5}{\sqrt{k^4 - 11}}$ . Does $\sum_{k=3}^{\infty} \frac{5}{\sqrt{k^4 - 11}}$ converge ?

The Limit Comparison Test is particularly useful because it allows us to ignore some parts of the terms that cause algebraic difficulties but that have no effect on the convergence of the series.

## Using "Dominant Terms"

To use the Limit Comparison Test we need to pick a new series to compare with our given series. One effective way to pick the new series is to form the new series using the largest power of the variable (dominant term) from the numerator and the largest power of the variable (dominant term) from the denominator. The "dominant term" series consists of $\frac{\text{dominant term in the numerator}}{\text{dominant term in the denominator}}$ . Then the Limit Comparison Test allows us to conclude that the original series converges if and only if the "dominant term" series converges.

**Example 4**: For each of the given series, form a new series consisting of the dominant terms from the numerator and the denominator. Does the series of dominant terms converge?

(a) $\sum_{k=3}^{\infty} \frac{5k^2 - 3k + 2}{17 + 2k^4}$ (b) $\sum_{k=1}^{\infty} \frac{1 + 4k}{\sqrt{k^3 + 5k}}$ (c) $\sum_{k=1}^{\infty} \frac{k^{23} + 1}{5k^{10} + k^{26} + 3}$.

Solution: (a) The dominant terms of the numerator and denominator are $5k^2$ and $2k^4$, respectively, so the "dominant term" series is $\sum_{k=3}^{\infty} \frac{5k^2}{2k^4} = \frac{5}{2} \sum_{k=3}^{\infty} \frac{1}{k^2}$ which converges (P–Test, p = 2 ).

(b) The dominant terms of the numerator and denominator are $4k$ and $k^{3/2}$, respectively, so the "dominant term" series is $\sum_{k=1}^{\infty} \frac{4k}{k^{3/2}} = 4 \sum_{k=3}^{\infty} \frac{1}{k^{1/2}}$ which diverges (P–Test, p = 1/2 ).

(c) The dominant terms of the numerator and denominator are $k^{23}$ and $k^{26}$ , respectively, so the "dominant term" series is $\sum_{k=1}^{\infty} \frac{k^{23}}{k^{26}} = \sum_{k=1}^{\infty} \frac{1}{k^3}$ which converges (P–Test, p = 3 ).

Using the Limit Comparison Test to compare the given series with the "dominant term" series, we can conclude that the given series (a) and (c) converge and that the given series (b) diverges.

**Practice 3**: For each of the given series, form a new series consisting of the dominant terms from the numerator and the denominator. Does the series of dominant terms converge? Do the given series converge?

(a) $\sum_{k=1}^{\infty} \frac{3k^4 - 5k + 2}{1 + 17k^2 + 9k^5}$ (b) $\sum_{k=1}^{\infty} \frac{\sqrt{1+9k}}{k^2 + 5k - 2}$ (c) $\sum_{k=1}^{\infty} \frac{k^{25} + 1}{5k^{10} + k^{26} + 3}$.

Experienced users of series commonly use "dominant terms" to make quick and accurate judgments about the convergence or divergence of a series. With practice, so can you.

**PROBLEMS**

In problems 1 – 12 use the Comparison Test to determine whether the given series converge or diverge.

1. $\sum_{k=1}^{\infty} \frac{\cos^2(k)}{k^2}$
2. $\sum_{k=1}^{\infty} \frac{3}{k^3 + 7}$
3. $\sum_{n=3}^{\infty} \frac{5}{n - 1}$
4. $\sum_{n=1}^{\infty} \frac{2 + \sin(n)}{n^3}$
5. $\sum_{j=1}^{\infty} \frac{3 + \cos(j)}{j}$
6. $\sum_{j=1}^{\infty} \frac{\arctan(j)}{j^{3/2}}$
7. $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$
8. $\sum_{k=1}^{\infty} \frac{k - 1}{k \cdot 1.5^k}$
9. $\sum_{k=1}^{\infty} \frac{k + 9}{k \cdot 2^k}$
10. $\sum_{n=1}^{\infty} \frac{n^3 + 7}{n^4 - 1}$
11. $\sum_{n=1}^{\infty} \frac{1}{1 + 2 + 3 + \ldots + (n-1) + n}$
12. $\sum_{k=1}^{\infty} \frac{1}{k!} = \sum_{k=1}^{\infty} \frac{1}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (k-1) \cdot k}$

In problems 13 – 21 use the Limit Comparison Test (or the $N^{th}$ Term Test) to determine whether the given series converge or diverge.

13. $\sum_{k=3}^{\infty} \frac{k + 1}{k^2 + 4}$
14. $\sum_{j=1}^{\infty} \frac{7}{\sqrt{j^3 + 3}}$
15. $\sum_{w=1}^{\infty} \frac{5}{w + 1}$
16. $\sum_{n=1}^{\infty} \frac{7n^3 - 4n + 3}{3n^4 + 7n^3 + 9}$
17. $\sum_{k=1}^{\infty} \frac{k^3}{(1 + k^2)^3}$
18. $\sum_{k=1}^{\infty} \left( \frac{\arctan(k)}{k} \right)^2$
19. $\sum_{n=1}^{\infty} \left( \frac{5 - \frac{1}{n}}{n} \right)^3$
20. $\sum_{w=1}^{\infty} \left( 1 + \frac{1}{w} \right)^w$
21. $\sum_{j=1}^{\infty} \left( 1 - \frac{1}{j} \right)^j$

In problems 22 – 30 use "dominant term" series to determine whether the given series converge or diverge.

22. $\displaystyle\sum_{n=3}^{\infty} \frac{n+100}{n^2-4}$

23. $\displaystyle\sum_{k=1}^{\infty} \frac{7k}{\sqrt{k^3+5}}$

24. $\displaystyle\sum_{k=1}^{\infty} \frac{5}{k+1}$

25. $\displaystyle\sum_{j=1}^{\infty} \frac{j^3-4j+3}{2j^4+7j^6+9}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{5n^3+7n^2+9}{\left(1+n^3\right)^2}$

27. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\arctan(3n)}{2n}\right)^2$

28. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3-\frac{1}{k}}{k}\right)^2$

29. $\displaystyle\sum_{j=1}^{\infty} \frac{\sqrt{j^3+4j^2}}{j^2+3j-2}$

30. $\displaystyle\sum_{k=1}^{\infty} \frac{k+9}{k\cdot 2^k}$

**Putting it all together**

In problems 31 – 51 use any of the methods from this or previous sections to determine whether the given series converge or diverge. Give reasons for your answers.

31. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+10}{n^3-2}$

32. $\displaystyle\sum_{k=1}^{\infty} \frac{3k}{\sqrt{k^5+7}}$

33. $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2k+1}$

34. $\displaystyle\sum_{j=1}^{\infty} \frac{j^2-j+1}{3j^4+2j^2+1}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^3+n^2+6}{\left(3+n^2\right)^2}$

36. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\arctan(2n)}{3n}\right)^3$

37. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1-\frac{2}{k}}{k}\right)^3$

38. $\displaystyle\sum_{j=1}^{\infty} \frac{\sqrt{j^2+4j}}{j^3-2}$

39. $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{k\cdot 3^k}$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2+4}$

41. $\displaystyle\sum_{k=1}^{\infty} \frac{k+2}{\sqrt{k^2+1}}$

42. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k\pi)}{k+1}$

43. $\displaystyle\sum_{j=1}^{\infty} \frac{3}{e^j+j}$

44. $\displaystyle\sum_{n=1}^{\infty} \frac{(2+3n)^2+9}{\left(1+n^3\right)^2}$

45. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\tan(3)}{2+n}\right)^2$

46. $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$

47. $\displaystyle\sum_{n=1}^{\infty} \sin^3\left(\frac{1}{n}\right)$

48. $\displaystyle\sum_{n=1}^{\infty} \cos^2\left(\frac{1}{n}\right)$

49. $\displaystyle\sum_{j=1}^{\infty} \cos^3\left(\frac{1}{j}\right)$

50. $\displaystyle\sum_{n=1}^{\infty} \tan^2\left(\frac{1}{n}\right)$

51. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{2}{n}\right)^n$

## Review for Positive Term Series: Converge or Diverge

State whether the given series converge or diverge and give reasons for your answer. You may need any of the methods discussed so far as well as some ingenuity.

R1. $\sum_{n=3}^{\infty} \frac{5}{3^n}$

R2. $\sum_{j=3}^{\infty} \frac{5+\cos(j^3)}{j^2}$

R3. $\sum_{w=1}^{\infty} \frac{2}{3+\sin(w^3)}$

R4. $\sum_{n=1}^{\infty} \frac{5}{(1/3)^n}$

R5. $\sum_{k=1}^{\infty} e^{-k}$

R6. $\sum_{w=1}^{\infty} \sin^2\left(\frac{1}{w}\right)$ (Hint: for $0 \le x \le 1$, $\sin(x) \le x$)

R7. $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k^3}\right)$ (see the R6 hint)

R8. $\sum_{j=1}^{\infty} \cos^2\left(\frac{1}{j}\right)$

R9. $\sum_{n=3}^{\infty} \frac{5+\cos(n^2)}{n^3}$

R10. $\sum_{k=1}^{\infty} \frac{1}{k\cdot(3+\ln(k))}$

R11. $\sum_{j=1}^{\infty} \frac{1}{j\cdot(3+\ln(j))^2}$

R12. $\sum_{n=1}^{\infty} \frac{4}{n\cdot\arctan(n)}$

R13. $\sum_{n=3}^{\infty} \frac{4\cdot\arctan(n)}{n}$

R14. $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^3}$

R15. $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2}$

R16. $\sum_{j=1}^{\infty} \left(\frac{j}{2j+3}\right)^j$

R17. $\sum_{n=1}^{\infty} \frac{1+n}{1+n^2}$

R18. $\sum_{n=1}^{\infty} \left[\sin(n)-\sin(n+1)\right]$

R19. $\sum_{k=1}^{\infty} \sqrt{\frac{k^3+5}{k^5+3}}$

R20. $\sum_{k=1}^{\infty} \frac{1}{k^k}$

R21. $\sum_{n=1}^{\infty} n^{1/n}$

## Practice Answers

**Practice 1**: (a) For $k > 3$, $0 < k-2 < k$ so $0 < \sqrt{k-2} < \sqrt{k}$ and $\frac{1}{\sqrt{k-2}} > \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$.

$\sum_{k=3}^{\infty} \frac{1}{k^{1/2}}$ diverges (P–test, p = 1/2) so $\sum_{k=3}^{\infty} \frac{1}{\sqrt{k-2}}$ diverges.

(b) For $k > 1$, $2^k + 7 > 2^k > 0$ so $\frac{1}{2^k+7} < \frac{1}{2^k}$.

$\sum_{k=3}^{\infty} \frac{1}{2^k} = \sum_{k=3}^{\infty} \left(\frac{1}{2}\right)^k$ which is a convergent geometric series ( r = 1/2) so $\sum_{k=3}^{\infty} \frac{1}{2^k+7}$ converges.

**Practice 2**: (a) $a_k = \dfrac{k^2 + 5k}{k^3 + k^2 + 7}$ . Put $b_k = \dfrac{1}{k^1}$ . Then

$$\frac{a_k}{b_k} = \frac{\dfrac{k^2 + 5k}{k^3 + k^2 + 7}}{\dfrac{1}{k^1}} = \left(\frac{k^1}{1}\right)\frac{k^2 + 5k}{k^3 + k^2 + 7} = \frac{k^3 + 5k^2}{k^3 + k^2 + 7} = \frac{k^3\left(1 + \frac{5}{k}\right)}{k^3\left(1 + \frac{1}{k} + \frac{7}{k^3}\right)} \longrightarrow 1$$

so $L = 1$ is positive and finite, and $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge or both diverge.

Since we know $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (P–test, p=1 or as the Harmonic series),

we can conclude that $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2 + 5k}{k^3 + k^2 + 7}$ diverges

(b) $a_k = \dfrac{5}{\sqrt{k^4 - 11}}$ . Put $b_k = \dfrac{1}{\sqrt{k^4}} = \dfrac{1}{k^2}$ . Then

$$\frac{a_k}{b_k} = \frac{\dfrac{5}{\sqrt{k^4 - 11}}}{\dfrac{1}{k^2}} = \frac{k^2}{1} \cdot \frac{5}{\sqrt{k^4 - 11}} = \frac{5}{1}\sqrt{\frac{k^4}{k^4 - 11}} \longrightarrow \frac{5}{1}\sqrt{1} = 5$$

so $L = 5$ is positive and finite, and $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge or both diverge.

Since we know $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (P–test, p=2) , we can conclude that

$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5}{\sqrt{k^4 - 11}}$ converges.

**Practice 3**: (a) $\sum_{k=1}^{\infty} \frac{3k^4}{9k^5} = \frac{1}{3}\sum_{k=1}^{\infty} \frac{1}{k}$ which diverges (P–test, p = 1) so $\sum_{k=1}^{\infty} \frac{3k^4 - 5k + 2}{1 + 17k^2 + 9k^5}$ diverges.

(b) $\sum_{k=1}^{\infty} \frac{\sqrt{9k}}{k^2} = 3\sum_{k=1}^{\infty} \frac{k^{1/2}}{k^2} = 3\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ which converges (P–test, p = 3/2)

so $\sum_{k=1}^{\infty} \frac{\sqrt{1 + 9k}}{k^2 + 5k - 2}$ converges.

(c) $\sum_{k=1}^{\infty} \frac{k^{25}}{k^{26}} = \sum_{k=1}^{\infty} \frac{1}{k}$ which diverges (P–test, p = 1) so $\sum_{k=1}^{\infty} \frac{k^{25}+1}{5k^{10}+k^{26}+3}$ diverges.

**Appendix: Proof of the Limit Comparison Test**

(a) Suppose $\sum_{k=1}^{\infty} b_k$ converges and that $\lim_{k\to\infty} \frac{a_k}{b_k} = L$, a **positive, finite** value.

Since $\lim_{k\to\infty} \frac{a_k}{b_k} = L$, there is a value N so that $\frac{a_k}{b_k} < L + 1$ when $k \ge N$. Then $a_k < b_k \cdot (L + 1)$ when $k \ge N$, and $\sum_{k=N}^{\infty} a_k < (L+1) \cdot \sum_{k=N}^{\infty} b_k$. Since $\sum_{k=N}^{\infty} b_k$ converges, we can conclude that $\sum_{k=N}^{\infty} a_k$ converges so $\sum_{k=1}^{\infty} a_k$ also converges.

(b) Suppose $\sum_{k=1}^{\infty} b_k$ diverges and that $\lim_{k\to\infty} \frac{a_k}{b_k} = L$, a **positive, finite** value.

Since $\lim_{k\to\infty} \frac{a_k}{b_k} = L$, there is a value N so that $\frac{a_k}{b_k} > L/2 > 0$ when $k \ge N$. Then $a_k > \frac{L}{2} \cdot b_k$ when $k \ge N$, and $\sum_{k=N}^{\infty} a_k \ge \frac{L}{2} \cdot \sum_{k=N}^{\infty} b_k$. Since $\sum_{k=N}^{\infty} b_k$ diverges, we can conclude that $\sum_{k=N}^{\infty} a_k$ diverges so $\sum_{k=1}^{\infty} a_k$ also diverges.