10.6 ALTERNATING SERIES

In the last two sections we considered tests for the convergence of series whose terms were all positive. In this section we examine series whose terms change signs in a special way: they alternate between positive and negative. And we present a very easy–to–use test to determine if these alternating series converge.

An **alternating series** is a series whose terms alternate between positive and negative. For example, the following are alternating series:

(1) $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots + (-1)^{k+1}\frac{1}{k} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k}$$ (alternating harmonic series)

(2) $$-\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \ldots + (-1)^{k}\frac{k}{k+2} + \ldots = \sum_{k=1}^{\infty} (-1)^{k}\frac{k}{k+2}$$

(3) $$-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{11}} - \ldots + (-1)^{k}\frac{1}{\sqrt{2k+1}} + \ldots = \sum_{k=1}^{\infty} (-1)^{k}\frac{1}{\sqrt{2k+1}} \; .$$

Figures 1, 2 and 3 show graphs and tables of values of several partial sums s_n for each of these series. As we move from left to right in each graph (as n increases), the partial sums alternately get larger and smaller, a common pattern for the partial sums of alternating series. The same pattern appears in the tables.

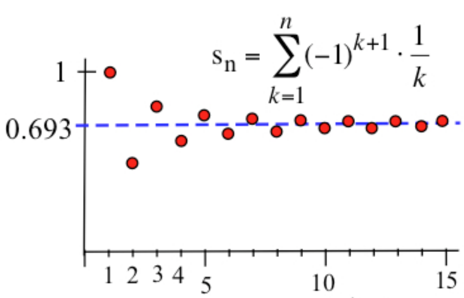

Fig. 1

n	s_n
1	1.0
2	0.5
3	0.8333
4	0.5833
5	0.7833
6	0.6166
7	0.7595
8	0.6345
9	0.7456
10	0.6456

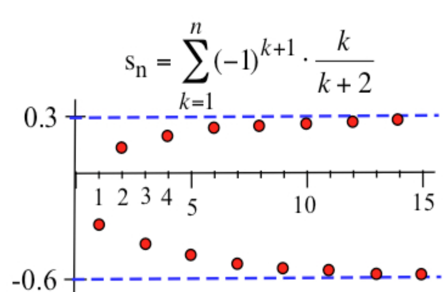

Fig. 2

n	s_n
1	–0.3333
2	0.1667
3	– 0.4333
4	0.2333
5	–0.4810
6	0.2690
7	– 0.5087
8	0.2913
9	– 0.5269
10	0.3064

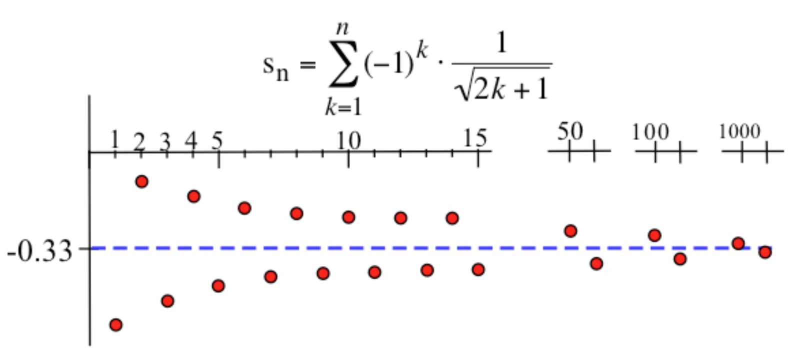

Fig. 3

n	s_n
1	–0.5774
2	– 0.1301
3	– 0.5081
4	– 0.1748
50	–0.2828
51	– 0.3813
1000	– 0.3211
1001	– 0.3435

Alternating Series Test

The following result provides a very easy way to determine that some alternating series converge. It says that if the absolute values of the terms decrease monotonically to 0 then the series converges. This is the main result for alternating series.

Alternating Series Test

If the numbers a_n satisfy the three conditions

(i) $a_n > 0$ for all n (each a_n is positive)

(ii) $a_n > a_{n+1}$ (the terms a_n are monotonically decreasing)

(iii) $\lim_{n\to\infty} a_n = 0$

then the **alternating series** $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof: In order to show that the alternating series converges, we need to show that the sequence of partial sums approaches a finite limit, and we do so in this case by showing that the sequence of even partial sums $\{ s_2, s_4, s_6, \ldots \}$ and the sequence of odd partial sums $\{s_1, s_3, s_5, \ldots\}$ each approach the same value.

Even partial sums:

$s_2 = a_1 - a_2 > 0$ since $a_1 > a_2$

$s_4 = a_1 - a_2 + a_3 - a_4 = s_2 + (a_3 - a_4) > s_2$ since $a_3 > a_4$

$s_6 = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = s_4 + (a_5 - a_6) > s_4$ since $a_5 > a_6$.

In general, the sequence of even partial sums is positive and increasing

$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2}) > s_{2n} > 0$ since $a_{2n+1} > a_{2n+2}$.

Also,

$$s_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots - a_{2n-2} + a_{2n-1} - a_{2n}$$
$$= a_1 - (a_2 - a_3) - (a_4 - a_5) \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n} < a_1 .$$

so the sequence of even partial sums is bounded above by a_1.

Since the sequence $\{ s_2, s_4, s_6, \ldots \}$ of even partial sums is increasing and bounded, the Monotone Convergence Theorem of Section 10.1 tells us that sequence of even partial sums converges to some finite limit: $\lim_{n\to\infty} s_{2n} = L$.

Odd partial sums:

$$s_{2n+1} = s_{2n} + a_{2n+1} \quad \text{so} \quad \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1} = L + 0 = L .$$

Since the sequence of even partial sums and the sequence of odd partial sums both approach the same limit L, we can conclude that the limit of the sequence of partial sums is L and that the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to L: $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = L$.

Example 1: Show that each of the three alternating series satisfies the three conditions in the hypothesis of the Alternating Series Test. Then we can conclude that each of them converges.

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

(b) $\frac{3}{1} - \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{3}} - \frac{3}{\sqrt{4}} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{\sqrt{n}}$

(c) $\frac{7}{2\cdot\ln(2)} - \frac{7}{3\cdot\ln(3)} + \frac{7}{4\cdot\ln(4)} - \ldots = \sum_{n=2}^{\infty} (-1)^{n} \frac{7}{n\cdot\ln(n)}$.

Solution: (a) This series is called the **alternating harmonic series**. (i) $a_n = \frac{1}{n} > 0$ for all positive n. (ii) Since $n < n+1$, then $\frac{1}{n} > \frac{1}{n+1}$ and $a_n > a_{n+1}$.

(iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, the alternating harmonic series converges.

Fig. 1 shows some partial sums for the alternating harmonic series.

(b) $a_n = \frac{3}{\sqrt{n}} > 0$. Since $n < n+1$, we have $\sqrt{n} < \sqrt{n+1}$ and $\frac{3}{\sqrt{n}} > \frac{3}{\sqrt{n+1}}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{\sqrt{n}} = 0$. The series $\frac{3}{1} + \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{3}} + \frac{3}{\sqrt{4}} + \ldots$ converges.

(c) $a_n = \frac{7}{n\cdot\ln(n)} > 0$ for $n \geq 2$. Since $n < n+1$ and $\ln(n) < \ln(n+1)$, we have

$n\cdot\ln(n) < (n+1)\cdot\ln(n+1)$ and $\frac{7}{n\cdot\ln(n)} > \frac{7}{(n+1)\cdot\ln(n+1)}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7}{n\cdot\ln(n)} = 0$. The series $\sum_{n=2}^{\infty} (-1)^{n} \frac{7}{n\cdot\ln(n)}$ converges.

Practice 1: Show that these two alternating series satisfy the three conditions in the hypothesis of the Alternating Series Test. Then we can conclude that each of them converges.

(a) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

(b) $\frac{3}{\ln(2)} - \frac{3}{\ln(3)} + \frac{3}{\ln(4)} - \frac{3}{\ln(5)} + \ldots = \sum_{n=2}^{\infty} (-1)^{n} \frac{3}{\ln(n)}$

Example 2: Does $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+2}$ converge?

Solution: $a_n = \frac{n}{n+2} > 0$, but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+2} = 1 \neq 0$. Since the terms do not approach 0, we can conclude from the N^{th} Term Test For Divergence (Section 10.2) that the series diverges. Fig. 2 shows some of the partial sums for this series. You should notice that the even and the odd partial sums in Fig. 2 are approaching two different values.

Practice 2: (a) Does $\sum_{n=1}^{\infty} (-1)^{n+1} n$ converge? (b) Does $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{2n+1}}$ converge?

Example Of A Divergent Alternating Series

If the terms of a series, any series, do not approach 0, then the series must diverge (N^{th} Term Test For Divergence). If the terms do approach 0 the series may converge or may diverge. There are divergent alternating series whose terms approach 0 (but the approach to 0 is not monotonic). For example, $\frac{3}{2} - \frac{1}{2} + \frac{3}{4} - \frac{1}{4} + \frac{3}{6} - \frac{1}{6} + \frac{3}{8} - \frac{1}{8} + \ldots$ is an alternating series whose terms approach 0, but the series diverges.

The even partial sums of our new series are

$s_2 = \frac{3}{2} - \frac{1}{2} = 1$,

$s_4 = \frac{3}{2} - \frac{1}{2} + \frac{3}{4} - \frac{1}{4} = (\frac{3}{2} - \frac{1}{2}) + (\frac{3}{4} - \frac{1}{4}) = 1 + \frac{1}{2}$,

$s_6 = \frac{3}{2} - \frac{1}{2} + \frac{3}{4} - \frac{1}{4} + \frac{3}{6} - \frac{1}{6} = (\frac{3}{2} - \frac{1}{2}) + (\frac{3}{4} - \frac{1}{4}) + (\frac{3}{6} - \frac{1}{6}) = 1 + \frac{1}{2} + \frac{1}{3}$,

and $s_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$.

You should recognize that these partial sums are the partial sums of the harmonic series, a divergent series, so the partial sums of our new series diverge and our new series is divergent. If the terms of an alternating series approach 0, but not monotonically, then the Alternating Series Test does not apply, and the series may converge or it may diverge.

Approximating the Sum of an Alternating Series

If we know that a series converges and if we add the first "many" terms together, then we expect that the resulting partial sum is close to the value S obtained by adding all of the terms together. Generally, however, we do not know how close the partial sum is to S. The situation with many alternating series is much nicer. The next result says that for some alternating series (those that satisfy the three conditions in the next box), the difference between S and the n^{th} partial sum of the alternating series, $| S - s_n |$, is less than the magnitude of the next term in the series, a_{n+1}.

Estimation Bound for Alternating Series

If S is the sum of an alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$

that satisfies the three conditions

(i) $a_n > 0$ for all n (each a_n is positive),

(ii) $a_n > a_{n+1}$ (the terms are monotonically decreasing),

and (iii) $\lim_{n\to\infty} a_n = 0$ (the terms approach 0),

then the n^{th} partial sum s_n is within a_{n+1} of the sum S: $s_n - a_{n+1} < S < s_n + a_{n+1}$

and $|$ approximation "error" using s_n as an estimate of S $| = | S - s_n | < a_{n+1}$.

Note: This Estimation Bound only applies to alternating series. It is tempting, but wrong, to use it with other types of series.

Geometric idea behind the Estimation Bound:

If we have an alternating series that satisfies the hypothesis of the Estimation Bound, then the graph of the sequence $\{s_n\}$ of partial sums is "trumpet–shaped" or "funnel–shaped" (Fig. 4). The partial sums are alternately above and below the value S, and they "squeeze" in on the value S. Since the distance from s_n to S is less than the distance between the successive terms s_n and s_{n+1} (Fig. 5), then $| S - s_n | < | s_n - s_{n+1} | = a_{n+1}$.

Proof of the Estimation Bound for Alternating Series:

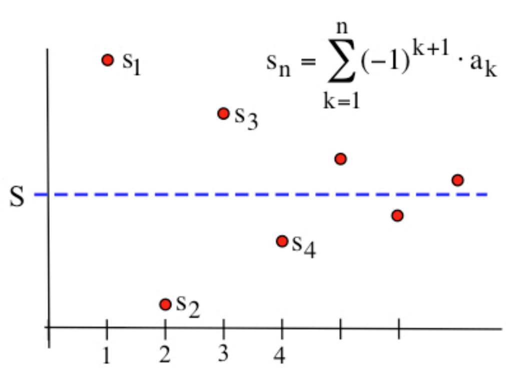

Fig. 4

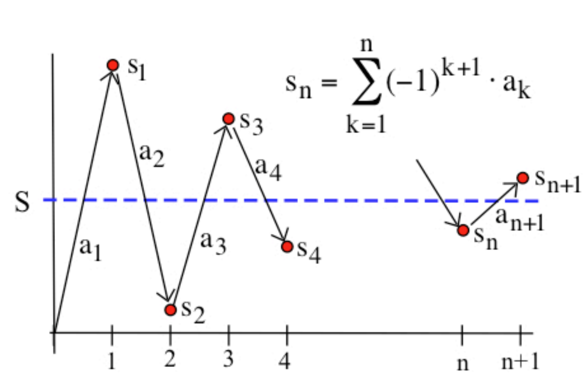

Fig. 5

$$S - s_n = (a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1}a_n + (-1)^{n+2}a_{n+1} \ldots) - (a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1}a_n)$$
$$= (-1)^{n+2}a_{n+1} + (-1)^{n+3}a_{n+2} + (-1)^{n+4}a_{n+3} + \ldots$$
$$= (-1)^{n+2}(a_{n+1} - a_{n+2} + a_{n+3} - \ldots).$$

Then $0 \le | S - s_n | = | (-1)^{n+2}(a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + a_{n+5} \ldots) |$

$= a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + a_{n+5} \ldots$ since $|(-1)^{n+2}| = 1$ and the rest is positive

$= a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \ldots < a_{n+1}$.

The Estimation Bound is typically used in two different ways. Sometimes we know the value of n, and we want to know how close s_n is to S. Sometimes we know how close we want s_n to be to S, and we want to find a value of n to ensure that level of closeness. The next two Examples illustrate these two different uses of the Estimation Bound.

Example 3: How close is $\sum_{n=1}^{4} (-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} \approx 0.79861$ to the sum $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^2}$?

Solution: $a_n = \frac{1}{n^2}$ and $s_4 = \sum_{n=1}^{4} (-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} \approx 0.79861$ so, by the Estimation Bound, we can conclude that $| S - s_4 | < a_5 = \frac{1}{25} = 0.04$: 0.79861 is within 0.04 of the exact value S. Then $0.79861 - 0.04 < S < 0.79861 + 0.04$ and $0.7\mathbf{5}861 < S < 0.8\mathbf{3}861$.

Similarly, $s_9 = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots + \frac{1}{81} \approx 0.82796$ is within $a_{10} = \frac{1}{100} = 0.01$ of the exact value of S, and $s_{99} \approx 0.822517$ is within $a_{100} = \frac{1}{100^2} = 0.0001$ of the exact value of S. Then $0.822517 - 0.0001 < S < 0.822517 + 0.0001$ and $0.822\mathbf{4}17 < S < 0.822\mathbf{6}17$.

Practice 3: Evaluate s_4 and s_9 for the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^3}$ and determine bounds for $| S - s_4 |$ and $| S - s_9 |$.

Example 4: Find the number of terms needed so that s_n is within 0.001 of the exact value of $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n!}$ and evaluate s_n.

Solution: We know $| S - s_n | < a_{n+1}$ so we want to find n so that $a_{n+1} \le 0.001 = \frac{1}{1000}$. With a little numerical experimentation on a calculator, we see that $6! = 720$ and $1/720$ is not small enough, but

$7! = 5{,}040 > 1{,}000$ so $\frac{1}{7!} = \frac{1}{5040} \approx 0.000198 < 0.001$. Since $n+1 = 7$, $s_6 \approx 0.631944$ is the first partial sum guaranteed to be within 0.001 of S. In fact, s_6 is within $\frac{1}{5040} \approx 0.000198$ of S, so $0.631746 < S < 0.632142$.

Practice 4: Find the number of terms needed so s_n is within 0.001 of the value of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3 + 5}$ and evaluate s_n.

The Estimation Bound guarantees that s_n is **within** a_{n+1} of S. In fact, s_n is often much closer than a_{n+1} to S. The value a_{n+1} is an **upper bound** on how far s_n can be from S.

Note 1: The first **finite** number of terms do not affect the convergence or divergence of a series (they do effect the sum S) so we can use the Alternating Series Test and the Estimation Bound if the terms of a series "eventually" satisfy the conditions of the hypotheses of these results. By "eventually" we mean there is a value M so that for $n > M$ the series is an alternating series.

Note 2: If a series has some positive terms and some negative terms and if those terms do NOT "eventually" alternate signs, then we can NOT use the Alternating Series Test – it simply does not apply to such series.

PROBLEMS

In problems 1 – 6 you are given the values of the first four **terms** of a series. (a) Calculate and graph the first four partial sums for each series. (b) Which of the series are not alternating series?

1. $1, -0.8, 0.6, -0.4$
2. $-1, 1.5, -0.7, 1$
3. $-1, 2, -3, 4$
4. $2, -1, -0.5, 0.3$
5. $-1, -0.6, 0.4, 0.2$
6. $2, -1, 0.5, -0.3$

In problems 7 – 12 you are given the values of the first five **partial sums** of a series. Which of the series are not alternating series. Why?

7. $2, 1, 3, 2, 4$
8. $2, 1, 1.8, 1.4, 1.6$
9. $2, 3, 2.1, 2.9, 2.8$
10. $-3, -1, -2.5, -1.5, -2$
11. $-1, 1, -0.8, -0.6, -0.4$
12. $-2.3, -1.6, -1.4, -1.8, -1.7$

13. Fig. 6 shows the graphs of the partial sums of three series. Which is/are not the partial sums of alternating series? Why?

14. Fig. 7 shows the graphs of the partial sums of three series. Which is/are not the partial sums of alternating series? Why?

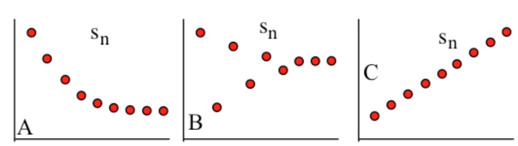

Fig. 6

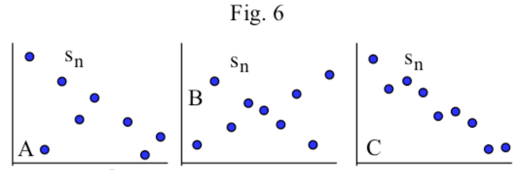

Fig. 7

15. Fig. 8 shows the graphs of the partial sums of three series. Which is/are not the partial sums of alternating series? Why?

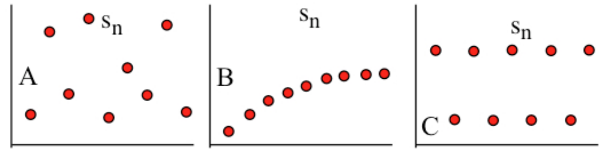

Fig. 8

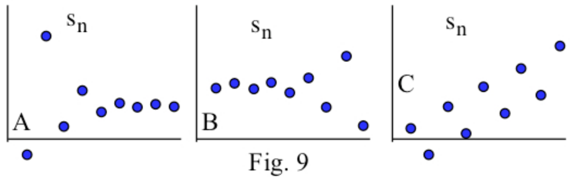

Fig. 9

16. Fig. 9 shows the graphs of the partial sums of three series. Which is/are not the partial sums of alternating series? Why?

In problems 17 – 31 determine whether the given series converge or diverge.

17. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+5}$

18. $\sum_{n=2}^{\infty} (-1)^{n} \frac{1}{\ln(n)}$

19. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+3}$

20. $\sum_{n=5}^{\infty} (-0.99)^{n+1}$

21. $\sum_{n=1}^{\infty} (-1)^{n} \frac{|n+3|}{|n+7|}$

22. $\sum_{n=1}^{\infty} (-1)^{n+1} \sin(\frac{1}{n})$

23. $\sum_{n=4}^{\infty} \frac{\cos(n\pi)}{n}$

24. $\sum_{n=3}^{\infty} \frac{\sin(n\pi)}{n}$

25. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n)}{n}$

26. $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{\ln(n)}{\ln(n^3)}$

27. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln(n^3)}{\ln(n^{10})}$

28. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5}{\sqrt{n+7}}$

29. $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{1+(-3)^n}$

30. $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{(-2)^{n+1}}{1+3^n}$

31. $\sum_{n=1}^{\infty} \cos(n\pi) \cdot \sin(\pi/n)$

In problems 32 – 40, (a) calculate s_4 for each series and determine an upper bound for how far s_4 is from the exact value S of the infinite series. Then (b) use s_4 find lower and upper bounds on the value of S so that $\{\text{lower bound}\} < S < \{\text{upper bound}\}$.

32. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n+6}$

33. $\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{\ln(n+1)}$

34. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\sqrt{n+21}}$

35. $\sum_{n=1}^{\infty} (-0.8)^{n+1}$

36. $\sum_{n=1}^{\infty} (-\frac{1}{3})^{n}$

37. $\sum_{n=1}^{\infty} (-1)^{n+1} \sin(\frac{1}{n})$

38. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

39. $\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n^3}$

40. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+\ln(n)}$

In problems 41 – 50 find the number of terms needed to guarantee that s_n is within the specified distance D of the exact value S of the sum of the given series.

41. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+6}$, D = 0.01

42. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n+1)}$, D = 0.01

43. $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}}$, D = 0.01

44. $\sum_{n=1}^{\infty} (-0.8)^{n+1}$, D = 0.003

45. $\sum_{n=1}^{\infty} (-\frac{1}{3})^{n}$, D = 0.002

46. $\sum_{n=1}^{\infty} (-1)^{n+1} \sin(\frac{1}{n})$, D = 0.06

47. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$, D = 0.001

48. $\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n^3}$, D = 0.0001

49. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+\ln(n)}$, D = 0.04

Problems 50 – 60 ask you to use the series S(x), C(x), and E(x) given below. For each problem, (a) substitute the given value for x in the series, (b) evaluate s_3, the sum of the first three terms of the series, (c) determine an upper bound on the error $|\text{actual sum} - s_3| = |S - s_3|$.
(The partial sums of S(x), C(x), and E(x) approximate sin(x), cos(x), and e^x, respectively.)

$$S(x) = x - \frac{x^3}{2\cdot3} + \frac{x^5}{2\cdot3\cdot4\cdot5} - \frac{x^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

$$C(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2\cdot3\cdot4} - \frac{x^6}{2\cdot3\cdot4\cdot5\cdot6} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

$$E(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{2\cdot3} + \frac{x^4}{2\cdot3\cdot4} + \ldots + \frac{x^n}{n!} + \ldots$$

50. x = 0.5 in S(x)
51. x = 0.3 in S(x)
52. x = 1 in S(x)
53. x = 0.1 in S(x)
54. x = –0.3 in S(x)
55. x = 1 in C(x)
56. x = 0.5 in C(x)
57. x = –0.2 in C(x)
58. x = –0.3 in C(x)
59. x = –1 in E(x)
60. x = –0.5 in E(x)
61. x = –0.2 in E(x)

Practice Answers

Practice 1: (a) (i) $a_n = \frac{1}{n^2} > 0$ for all n. (ii) $n^2 < (n+1)^2$ so $\frac{1}{n^2} > \frac{1}{(n+1)^2}$ and $a_n > a_{n+1}$.

(iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converges.

(b) (i) $a_n = \frac{3}{\ln(n)} > 0$ for all $n \geq 2$. (ii) $\ln(n) < \ln(n+1)$ so $\frac{3}{\ln(n)} > \frac{3}{\ln(n+1)}$ and $a_n > a_{n+1}$.

(iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{\ln(n)} = 0$. The series $\sum_{n=1}^{\infty} (-1)^{n} \frac{3}{\ln(n)}$ converges.

Practice 2: (a) $|(-1)^{n+1} n| = |n| \longrightarrow \infty \neq 0$ so the terms $(-1)^{n+1} n$ do NOT approach 0, and the **series diverges** by the nth Term Test for Divergence (Section 10.2).

(b) (i) $a_n = \dfrac{1}{\sqrt{2n+1}} > 0$ for all $n \geq 1$.

(ii) $2n+1 < 2(n+1)+1$ so $\sqrt{2n+1} < \sqrt{2(n+1)+1}$ and $\dfrac{1}{\sqrt{2n+1}} > \dfrac{1}{\sqrt{2(n+1)+1}}$ so $a_n > a_{n+1}$.

(iii) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{2n+1}} = 0$. Therefore, by the Alternating Series Test,

the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{2n+1}}$ converges.

Practice 3: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$: $s_4 = 1 - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} \approx 0.896412$.

$|S - s_4| < |a_5| = |(-1)^6 \frac{1}{5^3}| = \frac{1}{125} = 0.008$ so $s_4 - 0.008 < S < s_4 + 0.008$

and $0.888412 < S < 0.904412$.

$s_9 \approx 0.9021164$. $|S - s_9| < |a_{10}| = |(-1)^{11} \frac{1}{10^3}| = \frac{1}{1000} = 0.001$ so

$s_9 - 0.001 < S < s_9 + 0.001$ and $0.90\mathbf{1}116 < S < 0.90\mathbf{3}116$.

Practice 4: We need to find a value for n so $|a_{n+1}| < 0.001$.

$|a_{n+1}| = \left| (-1)^{n+2} \dfrac{1}{(n+1)^3 + 5} \right| = \dfrac{1}{(n+1)^3 + 5}$

n	$\lvert a_{n+1} \rvert$
1	$\frac{1}{2^3+5} = \frac{1}{13} \approx 0.0769$
2	$\frac{1}{3^3+5} = \frac{1}{32} \approx 0.03125$
8	$\frac{1}{9^3+5} = \frac{1}{734} \approx 0.00136$
9	$\mathbf{\frac{1}{10^3+5} = \frac{1}{1005} \approx 0.000995 < 0.001}$

Since $|a_{10}| < 0.001$ we can be sure that s_9 is within 0.001 of S.

$s_9 = \displaystyle\sum_{n=1}^{9} (-1)^{n+1} \frac{1}{n^3+5} = \frac{1}{6} - \frac{1}{13} + \frac{1}{32} - \ldots + \frac{1}{1005} \approx 0.112156571145$ so

$0.11\mathbf{1}156571145 < S < 0.11\mathbf{3}156571145$.