10.8 POWER SERIES: $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} a_n (x-c)^n$

So far most of the series we have examined have consisted of numbers (numerical series), but the most important series contain powers of a variable, and they define functions of that variable.

Definition of Power Series

A **power series** is an expression of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$

where $a_0, a_1, a_2, a_3, \ldots$ are constants, called the coefficients of the series, and x is a variable.

(Note: For $n = 0$ we use the convention for power series that $x^0 = 1$ even when $x = 0$. This convention simply makes it easier for us to represent the series using the summation notation.)

For each value of the variable, the power series is simply a numerical series that may converge or diverge. If the power series does converge, the value of the function is the sum of the series, and the domain of the function is the set of x values for which the series converges. Power series are particularly important in mathematics and applications because many important functions such as sin(x), cos(x), e^x, and ln(x) can be represented and approximated by power series.

The following are examples of power series:

$$f(x) = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n \quad (a_n = 1 \text{ for all n})$$

$$g(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \left(a_n = \frac{1}{n!} \text{ for all n and with the definition that } 0! = 1\right)$$

$$h(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \quad \left(a_n = \frac{(-1)^n}{(2n+1)!} \text{ for all n.}\right)$$

Power series look like long (very long) polynomials, and in many ways they behave like polynomials.

This section focuses on what a power series is and on determining where a given power series converges. Section 10.9 looks at the arithmetic (sums, differences, products) and calculus (derivative and integrals) of power series. Section 10.10 examines how to represent particular functions such as sin(x) and e^x and others as power series.

Finding Where a Power Series Converges

The power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ always converges at $x = 0$: $f(0) = \sum_{n=0}^{\infty} a_n (0)^n = a_0$. To find which other values of x make a power series converge, we could try x values one–by–one, but that is very inefficient and time consuming. Instead, the Ratio Test allows us to get answers for lots of x values all at once. (Since we are using a_n to represent the coefficient of the n^{th} term $a_n x^n$, we let $c_n = a_n x^n$ represent the n^{th} term and we use the ratio $\frac{c_{n+1}}{c_n}$.)

Example 1: Find all of the values of x for which the power series $\sum_{n=0}^{\infty} (2n+1) \cdot x^n$ converges.

Solution: $c_n = (2n+1) \cdot x^n$ so $c_{n+1} = (2(n+1)+1) \cdot x^{n+1} = (2n+3) \cdot x^{n+1}$. Then, using the Ratio Test,

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{(2n+3) \cdot x^{n+1}}{(2n+1) \cdot x^n} \right| = \left| \frac{2n+3}{2n+1} \cdot \frac{x^{n+1}}{x^n} \right| = \left| \frac{2n+3}{2n+1} \cdot x \right| \longrightarrow |x| = L .$$

From the Ratio Test we know the series converges if $L < 1$: if $|x| < 1$ or, in other words, $-1 < x < 1$. We also know the series diverges if $L > 1$ so the series diverges if $|x| > 1$: if $x > 1$ or $x < -1$. Finally, we need to check the two remaining values of x: the **endpoints** $x = -1$ and $x = 1$.

When $x = 1$, $\sum_{n=0}^{\infty} (2n+1) \cdot x^n = \sum_{n=0}^{\infty} (2n+1) \cdot 1^n = \sum_{n=0}^{\infty} (2n+1)$ which diverges since the terms do not approach 0. When $x = -1$, $\sum_{n=0}^{\infty} (2n+1) \cdot x^n = \sum_{n=0}^{\infty} (2n+1) \cdot (-1)^n$ which also diverges because the terms do not approach 0.

The power series $\sum_{n=0}^{\infty} (2n+1) \cdot x^n$ converges if and only if $-1 < x < 1$. In other words, the series converges when x is in the interval $(-1, 1)$ and it diverges when x is outside the interval $(-1, 1)$.

Example 2: Find all of the values of x for which the power series $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$ converges.

Solution: $c_n = \frac{x^n}{n \cdot 3^n}$ so $c_{n+1} = \frac{x^{n+1}}{(n+1) \cdot 3^{n+1}}$. Using the Ratio Test,

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{\frac{x^{n+1}}{(n+1) \cdot 3^{n+1}}}{\frac{x^n}{n \cdot 3^n}} \right| = \left| \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{x^{n+1}}{x^n} \right| = \left| \frac{n}{n+1} \cdot \frac{1}{3} \cdot x \right| \longrightarrow \left| \frac{x}{3} \right| = L .$$

Solving $| \frac{x}{3} | < 1$, we have $-1 < \frac{x}{3} < 1$ or $-3 < x < 3$. The series converges for $-3 < x < 3$ and it diverges for $x < -3$ and for $x > 3$.

Finally, we need to check the two remaining values of x: the **endpoints** $x = -3$ and $x = 3$.

When $x = -3$, **Error! Not a valid embedded object.** $= \sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the

Alternating Series Test.

When $x = 3$, $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which diverges.

In summary, the power series $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$ converges if $-3 \le x < 3$, if x is in the interval $[-3, 3)$.

Note: The Ratio Test is very powerful for determining where a power series converges: put $c_n = a_n x^n$, calculate the limit of the ratio $\left| \frac{c_{n+1}}{c_n} \right|$, and then solve the resulting absolute value inequality for x.

Typically, we also need to check the endpoints of the interval by replacing x with the two endpoint values and then determining if the resulting numerical series converge or diverge at these endpoints. The Ratio Test does not help with the endpoints.

Practice 1: Find all of the values of x for which the series $\sum_{n=1}^{\infty} \frac{5^n \cdot x^n}{n}$ converges.

Interval of Convergence, Radius of Convergence

In each of the previous examples, the values of x for which the power series converge form an interval. The next theorem says that always happens.

Interval of Convergence Theorem for Power Series

The values of x for which the power series $\sum_{n=0}^{\infty} a_n x^n$ converges form an interval.

(i) If this power series converges for $x = c$, then the series converges for all satisfying $|x| < |c|$.

(ii) If this power series diverges for $x = d$, then the series diverges for all satisfying $|x| > |d|$.

A proof of the Interval of Convergence Theorem is given after the Practice Answers.

Meaning of the Interval of Convergence Theorem: If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for a value x = c, then the series also converges for all values of x closer to the origin than c. If the power series diverges for a value x = d, then the power series diverges for all values of x farther from the origin than d.

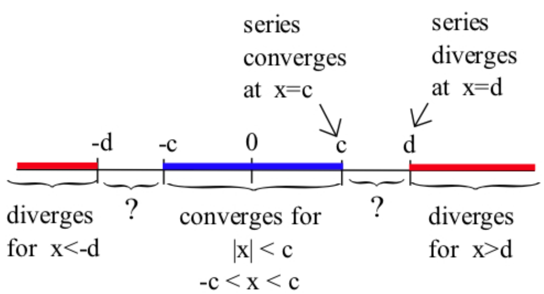


Fig. 1

This guarantees that the values where the power series converges form an interval, from –|c| to |c|. This Theorem does **not** tell us about the convergence of the power series at the endpoints of the interval — we need to check those two points individually. This Theorem also does **not** tell us about the convergence of the power series for values of x with |c| < |x| < |d|. See Fig. 1.

Example 3: Suppose we know that a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at x = 4 and diverges at x = 9.

What can we conclude (converge or diverge or not enough information) about the series when x = 2, –3, –4, 5, –6, 8, –9, 10, –11?

Solution: We know the power series converges at x = 4 (c = 4) so we can conclude that the series converges for x = 2 and x = –3 since |2|<|4| and |–3|<|4|.

We know the power series diverges at x = 9 (d = 9) so we can conclude that the series diverges for x = 10 and x = –11 since |10|>|9| and |–11|>|9|.

The remaining values of x (–4, 5, –6, 8, and –9) do not satisfy |x| < 4 or |x| > 9 so the series may converge or may diverge — we don't have enough information.

Fig. 2 shows the regions where convergence of this power series is guaranteed, where divergence is guaranteed, and where we don't have enough information.

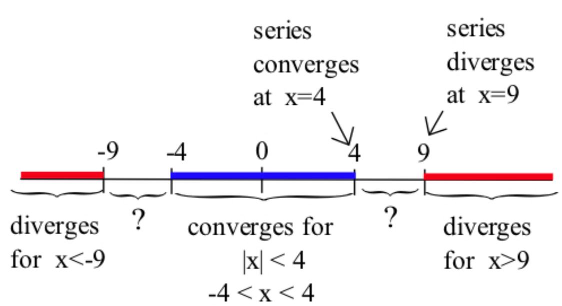


Fig. 2

Practice 2: Suppose we know that a power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 3$ and diverges at $x = -7$.

What can we say about the convergence of the series for $x = -1, 2, -3, 4, -6, 7, -8$, and 17?

Sketch the regions of known convergence and known divergence.

Definition: The **interval of convergence** of a power series $\sum_{n=0}^{\infty} a_n x^n$ is the interval of values of x for which the series converges.

From Example 1 we know the interval of convergence of $\sum_{n=0}^{\infty} (2n+1) \cdot x^n$ is $(-1, 1)$.

From Example 2 we know the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$ is $[-3, 3)$.

Definition:

For each power series $\sum_{n=0}^{\infty} a_n x^n$ there is a number R, called the **radius of convergence**, so that the series converges for $|x| < R$ and diverges when $|x| > R$. (The series may converge or may diverge at $|x| = R$.) (Fig. 3)

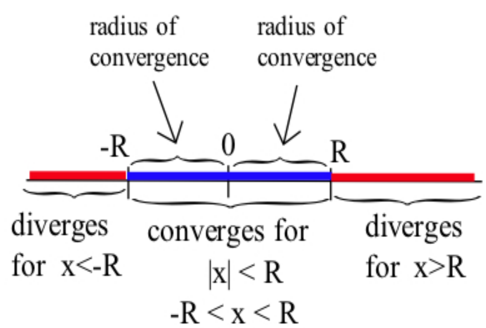


Fig. 3

The radius of convergence is half of the length of the interval of convergence.

Example 4: What is the radius of convergence of each series in Examples 1 and 2?

Solution: The power series $\sum_{n=0}^{\infty} (2n+1) \cdot x^n$ converges if $-1 < x < 1$ so $R = 1$.

The power series $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$ converges if $-3 \le x < 3$, so $R = 3$.

The convergence or divergence of a power series at the endpoints of the interval of convergence does not affect the value of the radius of convergence R, and the value of R does not tell us about the convergence of the power series at the endpoints of the interval of convergence, at $x = R$ and $x = -R$.

Practice 3: What is the radius of convergence of the series in Practice 1?

Summary

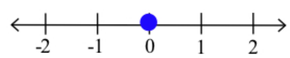
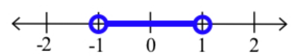
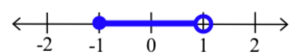
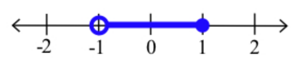
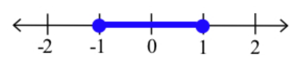
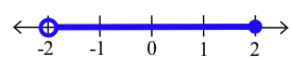
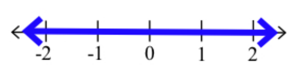
For a power series $\sum_{n=0}^{\infty} a_n x^n$ exactly one of these three situations occurs:

(i) the series converges only for $x = 0$. (Then we say the radius of convergence is 0.)

(ii) the series converges for all x with $|x| < R$ and diverges for all x with $|x| > R$. (Then we say the radius of convergence is R.)

(iii) the series converges for all values of x. (Then we say the radius of convergence is infinite.)

The following list shows the intervals and radii of convergence for several power series. Four of the series in the list have the same radius of convergence, $R = 1$, but slightly different intervals of convergence. (The $\sum$ below simply means a series whose starting index is some finite value of n and whose upper index is ∞.)

Series	Radius of Convergence	Interval of Convergence	Series converges for x Values in the Shaded Intervals
$\sum n!\cdot x^n$	$R = 0$	$\{ 0 \}$, a single point	
$\sum x^n$	$R = 1$	$(-1, 1)$	
$\sum \frac{x^n}{n}$	$R = 1$	$[-1, 1)$	
$\sum (-1)^n \frac{x^n}{n}$	$R = 1$	$(-1, 1]$	
$\sum \frac{x^n}{n^2}$	$R = 1$	$[-1, 1]$	
$\sum \frac{x^n}{2^n}$	$R = 2$	$(-2, 2)$	
$\sum \frac{x^n}{n!}$	$R = \infty$	$(-\infty, \infty)$	

Power Series Centered at c

Sometimes it is useful to "shift" a power series. These shifted power series contain powers of "$x - c$" instead of powers of "x," but many of the properties we have examined still hold.

Definition: A **power series centered at c** is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots + a_n(x-c)^n + \ldots$$

where $a_0, a_1, a_2, a_3, \ldots$ are constants, called the coefficients of the series, and x is a variable.

Note: As usual for power series, for $n = 0$ we use the convention that $(x-c)^0 = 1$, even when $x = c$.

A power series centered at c always converges for $x = c$, and the interval of convergence is an interval **centered at c**. The radius of convergence is half the length of the interval of convergence (Fig. 4).

If a power series centered at c converges for a value of x, then the series converges for all values closer to c. If a power series centered at c diverges for a value of x, then the series diverges for all values farther from c.

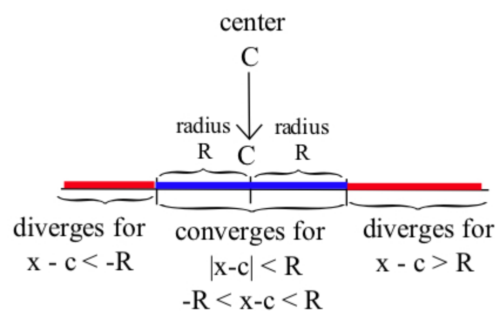


Fig. 4

Example 5: Suppose we know that a power series $\sum_{n=0}^{\infty} a_n(x-4)^n$ converges at $x = 6$ and diverges at $x = 0$. What can we conclude (converge or diverge or not enough information) about the series when $x = 3, 9, -1, 2$, and 7?

Solution: We know the power series converges at $x = 6$ so we can conclude that the series converges for values of x **closer to 4** than $|6-4| = 2$ units: the series converges at $x = 3$.
We know the power series diverges at $x = 0$ so we can conclude that the series diverges for values of x **farther from 4** than $|0-4| = 4$ units: the series diverges at $x = 9$ and -1.
The remaining values of x (2 and 7) do not satisfy $|x-4| < 2$ or $|x-4| > 4$ so the series may converge or may diverge.

Fig. 5 shows the regions where convergence of this power series is guaranteed, where divergence is guaranteed, and where we don't have enough information.

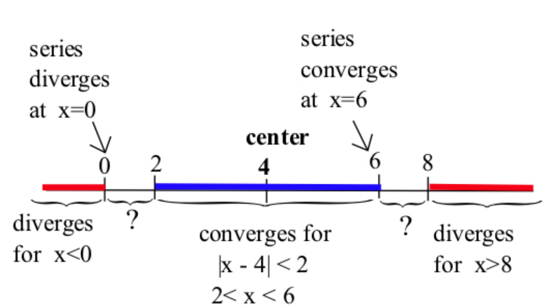


Fig. 5

Practice 4: Suppose we know that a power series $\sum_{n=0}^{\infty} a_n(x+5)^n$ converges at $x = -1$ and diverges at $x = 1$. What can we conclude about the series when $x = -2, -9, 0, -11$, and 3?

The Ratio Test is still our primary tool for finding an interval of convergence of a power series, even if the power series is centered at c rather than at 0.

Example 6: Find the interval and radius of convergence of $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 2^n}$.

Solution: $c_n = \frac{(x-5)^n}{n \cdot 2^n}$ so $c_{n+1} = \frac{(x-5)^{n+1}}{(n+1) \cdot 2^{n+1}}$. The ratio $\left| \frac{c_{n+1}}{c_n} \right|$ for the Ratio Test appears to be messy, but if we group the similar pieces algebraically, the ratio simplifies nicely:

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{\frac{(x-5)^{n+1}}{(n+1) \cdot 2^{n+1}}}{\frac{(x-5)^n}{n \cdot 2^n}} \right| = \left| \frac{n}{n+1} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{(x-5)^{n+1}}{(x-5)^n} \right| = \left| \frac{n}{n+1} \cdot \frac{1}{2} \cdot (x-5) \right| \rightarrow \left| \frac{(x-5)}{2} \right| = L .$$

Solving $\left| \frac{(x-5)}{2} \right| < 1$, we have $-1 < \frac{(x-5)}{2} < 1$ so $-2 < x-5 < 2$ and $3 < x < 7$. The series converges for $3 < x < 7$, and it diverges for $x < 3$ and for $x > 7$.

Finally, we need to check the two remaining values of x: the **endpoints** $x = 3$ and $x = 7$.

When $x = 3$, $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alternating Series Test.

When $x = 7$, $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which diverges.

The power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 2^n}$ converges if $3 \le x < 7$. The interval of convergence is $[3, 7)$, and the radius of convergence is $R = \frac{1}{2}$(length of the interval of convergence) $= \frac{1}{2}(7-3) = 2$.

Practice 5: Find the interval and radius of convergence of $\sum_{n=0}^{\infty} \frac{n \cdot (x-3)^n}{5^n}$.

A power series looks like a very long polynomial. However, a regular polynomial with a finite number of terms is defined at every value of x, but a power series may diverge for many values of the variable x. As we continue to work with power series we need to be alert to where the power series converges (and behaves like a finite polynomial) and where the power series diverges. We need to know the interval of convergence of the power series, and, typically, we use the Ratio Test to find that interval.

PROBLEMS

In problems 1 – 24, (a) find all values of x for which each given power series converges, and (b) graph the interval of convergence for the series on a number line.

1. $\sum_{n=1}^{\infty} x^n$
2. $\sum_{n=1}^{\infty} (x-3)^n$
3. $\sum_{n=1}^{\infty} (x+2)^n$
4. $\sum_{n=1}^{\infty} (x+5)^n$
5. $\sum_{n=3}^{\infty} \frac{x^n}{n}$
6. $\sum_{n=3}^{\infty} \frac{(x-2)^n}{n}$
7. $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n}$
8. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$
9. $\sum_{n=1}^{\infty} \frac{(x-7)^{2n+1}}{n^2}$
10. $\sum_{n=1}^{\infty} \frac{(x+1)^{2n}}{n^3}$
11. $\sum_{n=2}^{\infty} (2x)^n$
12. $\sum_{n=2}^{\infty} (5x)^n$
13. $\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^{2n+1}$
14. $\sum_{n=1}^{\infty} n\left(\frac{x}{4}\right)^{2n+1}$
15. $\sum_{n=1}^{\infty} (2x-6)^n$
16. $\sum_{n=1}^{\infty} (3x+1)^n$
17. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
18. $\sum_{n=0}^{\infty} \frac{(x-5)^n}{n!}$
19. $\sum_{n=1}^{\infty} \frac{n^2 \cdot x^n}{3^n}$
20. $\sum_{n=1}^{\infty} \frac{n^5 \cdot x^n}{3^n}$
21. $\sum_{n=1}^{\infty} n! \cdot x^n$
22. $\sum_{n=1}^{\infty} n! \cdot (x+2)^n$
23. $\sum_{n=3}^{\infty} n! \cdot (x-7)^n$
24. $\sum_{n=0}^{\infty} \frac{3^n \cdot x^n}{n!}$

In problems 25 – 34, the letters "a" and "b" represent positive constants. Find all values of x for which each given power series converges.

25. $\sum_{n=1}^{\infty} (x-a)^n$
26. $\sum_{n=1}^{\infty} (x+b)^n$
27. $\sum_{n=1}^{\infty} \frac{(x-a)^n}{n}$
28. $\sum_{n=1}^{\infty} \frac{(x-a)^n}{n^2}$
29. $\sum_{n=1}^{\infty} (ax)^n$
30. $\sum_{n=1}^{\infty} \left(\frac{x}{a}\right)^n$
31. $\sum_{n=1}^{\infty} (ax-b)^n$
32. $\sum_{n=1}^{\infty} (ax+b)^n$

33. A friend claimed that the interval of convergence for a power series of the form $\sum_{n=1}^{\infty} a_n \cdot (x-4)^n$ is the interval $(1, 9)$. Without checking your friend's work, how can you be certain that your friend is wrong?

34. Which of the following intervals are possible intervals of convergence for the power series in problem 33? $(2, 6)$, $(0, 4)$, $x = 0$, $[1, 7]$, $(-1, 9]$, $x = 4$, $[3, 5)$, $[-4, 4)$, $x = 3$.

35. Which of the following intervals are possible intervals of convergence for $\sum_{n=1}^{\infty} a_n \cdot (x-7)^n$?

$(3, 10)$, $(5, 9)$, $x = 0$, $[1, 13]$, $(-1, 15]$, $x = 4$, $[3, 11)$, $[0, 14)$, $x = 7$.

36. Fill in each blank with a number so the resulting interval could be the interval of convergence for the power series $\sum_{n=1}^{\infty} a_n \cdot (x-3)^n$: $(0, \_\_\_)$, $(\_\_\_, 7)$, $[1, \_\_\_]$, $(\_\_\_, 15]$, $[\_\_\_, 11)$, $[0, \_\_\_)$, $x = \_\_\_$.

37. Fill in each blank with a number so the resulting interval could be the interval of convergence for the power series $\sum_{n=1}^{\infty} a_n \cdot (x-1)^n$: $(0, \_\_\_)$, $(\_\_\_, 7)$, $[1, \_\_\_]$, $(\_\_\_, 5]$, $[\_\_\_, 11)$, $[0, \_\_\_)$, $x = \_\_\_$.

In problems 38 – 45, use the patterns you noticed in the earlier problems and examples to build a power series with the given intervals of convergence. (There are many possible correct answers — find one.)

38. $(-5, 5)$ 39. $[-3, 3)$ 40. $[-2, 2]$ 41. $(0, 6)$

42. $[0, 8)$ 43. $[2, 8)$ 44. $[3, 7]$ 45. $x = 3$

In problems 46 – 59, find the interval of convergence for each series. For x in the interval of convergence, find the sum of the series as a function of x. (Hint: You know how to find the sum of a geometric series.)

46. $\sum_{n=0}^{\infty} x^n$ 47. $\sum_{n=0}^{\infty} (x-3)^n$ 48. $\sum_{n=0}^{\infty} (2x)^n$ 49. $\sum_{n=0}^{\infty} (3x)^n$

50. $\sum_{n=0}^{\infty} x^{2n}$ 51. $\sum_{n=0}^{\infty} x^{3n}$ 52. $\sum_{n=0}^{\infty} \left(\frac{x-6}{2}\right)^n$ 53. $\sum_{n=0}^{\infty} \left(\frac{x-6}{5}\right)^n$

54. $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ 55. $\sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$ 56. $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^{2n}$ 57. $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^{3n}$

58. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\sin(x)\right)^n$ 59. $\sum_{n=0}^{\infty} \left(\frac{1}{3}\cos(x)\right)^n$

Practice Answers

Practice 1: $c_n = \frac{5^n \cdot x^n}{n}$ so $c_{n+1} = \frac{5^{n+1} \cdot x^{n+1}}{n+1}$. Then, using the Ratio Test,

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{\frac{5^{n+1} x^{n+1}}{n+1}}{\frac{5^n x^n}{n}} \right| = \left| \frac{n}{n+1} \cdot \frac{5^{n+1}}{5^n} \cdot \frac{x^{n+1}}{x^n} \right| = \left| \frac{n}{n+1} \cdot 5 \cdot x \right| \longrightarrow |5x| = L .$$

Solving $|5x| < 1$, we have $-1 < 5x < 1$ or $-1/5 < x < 1/5$. The series converges for $-1/5 < x < 1/5$ and it diverges for $x < -1/5$ and for $x > 1/5$.

Finally, we need to check the two remaining values of x: the **endpoints** $x = -1/5$ and $x = 1/5$.

When $x = -1/5$, $\sum_{n=1}^{\infty} \frac{5^n x^n}{n} = \sum_{n=1}^{\infty} \frac{5^n (-1/5)^n}{n} = \sum_{k=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alternating Series Test.

When $x = 1/5$, $\sum_{n=1}^{\infty} \frac{5^n x^n}{n} = \sum_{n=1}^{\infty} \frac{5^n (1/5)^n}{n} = \sum_{k=1}^{\infty} \frac{1}{n}$, the harmonic series, which diverges.

In summary, the power series $\sum_{n=1}^{\infty} \frac{5^n x^n}{n}$ converges if $-1/5 \le x < 1/5$, if x is in the interval $[-1/5, 1/5)$.

Practice 2: The series converges at $x = -1$ and $x = 2$.

The convergence is unknown at

$x = -3$ (endpoint?), 4, –6, and 7 (endpoint?)

The series diverges at $x = -8$ and $x = 17$.

The regions of known convergence and known divergence are shown in Fig. 6.

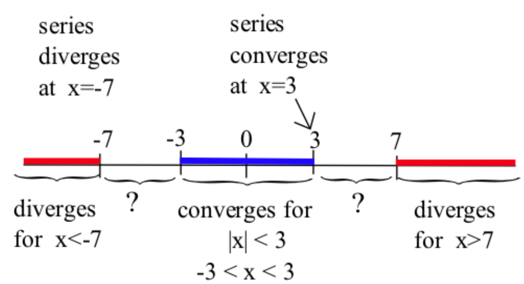


Fig. 6

Practice 3: The radius of convergence is **1/5** .

Practice 4: The series converges at $x = -2$.

The convergence is unknown at

$x = -9$ (endpoint?), 0, and –11 (endpoint?)

The series diverges at $x = 3$.

The regions of known convergence and known divergence are shown in Fig. 7.

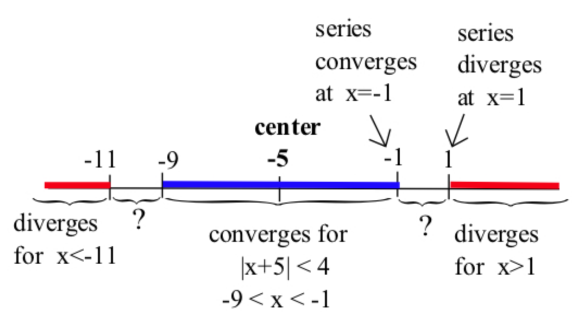


Fig. 7

Practice 5: $c_n = \frac{n \cdot (x-3)^n}{5^n}$ so $c_{n+1} = \frac{(n+1) \cdot (x-3)^{n+1}}{5^{n+1}}$. Then, using the Ratio Test,

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{\frac{(n+1)\cdot(x-3)^{n+1}}{5^{n+1}}}{\frac{n\cdot(x-3)^n}{5^n}} \right| = \left| \frac{n+1}{n} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{(x-3)^{n+1}}{(x-3)^n} \right| = \left| \frac{n+1}{n} \cdot \frac{1}{5} \cdot (x-3) \right| \to \left| \frac{x-3}{5} \right| = L .$$

Solving $\left| \frac{x-3}{5} \right| < 1$, we have $-1 < \frac{x-3}{5} < 1$ so $-5 < x - 3 < 8$. The series converges for $-2 < x < 8$ and it diverges for $x < -2$ and for $x > 8$. The radius of convergence is 5.

To find the interval of convergence we still need to check the endpoints $x = -2$ and $x = 8$:

When $x = -2$, $\sum_{n=0}^{\infty} \frac{n \cdot (x-3)^n}{5^n} = \sum_{n=1}^{\infty} \frac{n \cdot (-5)^n}{5^n} = \sum_{n=1}^{\infty} n \cdot (-1)^n$ which diverges by the n^{th} Term Test.

When $x = 8$, $\sum_{n=0}^{\infty} \frac{n \cdot (x-3)^n}{5^n} = \sum_{n=1}^{\infty} \frac{n \cdot (5)^n}{5^n} = \sum_{n=1}^{\infty} n$ which diverges by the n^{th} Term Test.

The interval of convergence is $-2 < x < 8$ or $(-2, 8)$.

Appendix: Proof of the Interval of Convergence Theorem

(i) Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c$: $\sum_{n=0}^{\infty} a_n c^n$ converges.

Then the terms of the series $a_n c^n$ must approach 0, $\lim_{n\to\infty} a_n c^n = 0$, so there is a number N so that if $n \ge N$, then $| a_n c^n | < 1$ and $|a_n| < \frac{1}{|c^n|}$ for all $n > N$.

If $|x| < |c|$, then $\sum_{n=0}^{\infty} | a_n x^n | = \{ |a_0| + |a_1 x| + ... + |a_{N-1} x^{N-1}| \} + \{ |a_N x^N| + |a_{N+1} x^{N+1}| + ... \}$.

The first piece, $|a_0| + |a_1 x| + ... + |a_{N-1} x^{N-1}|$, consists of a finite number of terms so it is a finite number.

The second piece, $|a_N x^N| + |a_{N+1} x^{N+1}| + ...$ is less than a convergent geometric series:

$$|a_N x^N| + |a_{N+1} x^{N+1}| + |a_{N+2} x^{N+2}| + |a_{N+3} x^{N+3}| + ...$$

$$< \frac{1}{|c^N|}|x^N| + \frac{1}{|c^{N+1}|}|x^{N+1}| + \frac{1}{|c^{N+2}|}|x^{N+2}| + \frac{1}{|c^{N+3}|}|x^{N+3}| + ...$$

$$= \left|\frac{x}{c}\right|^N + \left|\frac{x}{c}\right|^{N+1} + \left|\frac{x}{c}\right|^{N+2} + \left|\frac{x}{c}\right|^{N+3} + \ ...$$ which converges since $|x| < |c|$ and $\left| \frac{x}{c} \right| < 1$.

If $|x| < |c|$, then $\sum_{n=0}^{\infty} | a_n x^n |$ converges so $\sum_{n=0}^{\infty} a_n x^n$ converges.

(ii) This part follows from part (i) of the Theorem. Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x = d$: $\sum_{n=0}^{\infty} a_n d^n$ diverges. If the series converges for some x_0 with $|x_0| > |d|$, we can put $c = x_0$ and conclude from part (i) that the series must converge at $x = d$ because $|c| = |x_0| > |d|$. This contradicts the fact that the series diverges at $x = d$, so the series cannot converge for any x_0 with $|x_0| > |d|$.

If $\sum_{n=0}^{\infty} a_n d^n$ diverges and $|x| > |d|$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ diverges.