10.9 REPRESENTING FUNCTIONS AS POWER SERIES

Power series define functions, but how are these power series functions related to functions we know about such as sin(x), cos(x), e^x, and ln(x)? How can we represent common functions as power series, and why would we want to do so? The next two sections provide partial answers to these questions. In this section we start with a function defined by a geometric series and show how we can obtain power series representations for several related functions. And we look at a few ways in which power series representations of functions are used. The next section examines a more general method for obtaining power series representations for functions.

The foundation for the examples in this section is a power series whose sum we know. The power series $\sum_{n=0}^{\infty} x^n$ is also a geometric series, with the common ratio $r = x$, and, for $|x| < 1$, we know the sum of the series is $\frac{1}{1-x}$.

Geometric Series Formula

For $|x| < 1$, $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}$.

One simple but powerful method of obtaining power series for related functions is to replace each "x" with a function of x.

Substitution in Power Series

Suppose f(x) is defined by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

that converges for $-R < x < R$.

Then $f(x^p) = \sum_{n=0}^{\infty} a_n \{x^p\}^n = a_0 + a_1 x^p + a_2\{x^p\}^2 + a_3\{x^p\}^3 + \ldots + a_n\{x^p\}^n + \ldots$

$$= a_0 + a_1 x^p + a_2 x^{2p} + a_3 x^{3p} + \ldots + a_n x^{np} + \ldots$$

converges for $-R < x^p < R$, and

$$f(x-c) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots + a_n(x-c)^n + \ldots$$

converges for $-R < x - c < R$.

We can use this substitution method to obtain power series for some functions related to the Geometric Series Formula $\frac{1}{1-x}$.

Example 1: Find power series for (a) $\frac{1}{1-x^2}$, (b) $\frac{1}{1+x}$, and (c) $\frac{x}{1-x}$.

Solution:

(a) Substituting "x^2" for "x" in the Geometric Series Formula, we get

$$\frac{1}{1-x^2} = 1 + \{x^2\} + \{x^2\}^2 + \{x^2\}^3 + \{x^2\}^4 + ... = \sum_{n=0}^{\infty} (x^2)^n$$

$$= 1 + x^2 + x^4 + x^6 + x^8 + ... = \sum_{n=0}^{\infty} x^{2n} \quad \text{for } -1 < x < 1.$$

(b) Substituting "–x" for "x" in the Geometric Series Formula,

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + \{-x\} + \{-x\}^2 + \{-x\}^3 + \{-x\}^4 + ... = \sum_{n=0}^{\infty} (-x)^n$$

$$= 1 - x + x^2 - x^3 + x^4 + ... = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{for } -1 < x < 1.$$

(c) We need to recognize that $\frac{x}{1-x}$ is a product, $\frac{x}{1-x} = x \cdot \frac{1}{1-x}$. Then

$$x \cdot \frac{1}{1-x} = x \{ 1 + x + x^2 + x^3 + x^4 + ... \}$$

$$= x + x^2 + x^3 + x^4 + x^5 + ... = \sum_{n=0}^{\infty} x^{n+1} \quad \text{or, equivalently,} \quad \sum_{n=1}^{\infty} x^n \quad \text{for } -1 < x < 1.$$

Practice 1: Find power series for (a) $\frac{1}{1-x^3}$, (b) $\frac{1}{1+x^2}$, and (c) $\frac{5x}{1+x}$.

One of the features of polynomials that makes them very easy to differentiate and integrate is that we can differentiate and integrate them term–by–term. The same result is true for power series.

Term–by–Term Differentiation and Integration of Power Series

Suppose f(x) is defined by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

that converges for $-R < x < R$.

Then,

(a) the derivative of f is given by the power series obtained by term–by–term differentiation of f:

$$f'(x) = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n \cdot a_n x^{n-1} + \ldots$$

(b) an antiderivative of f is given by the power series obtained by term–by–term integration of f:

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} = C + a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} + \ldots + a_n \frac{x^{n+1}}{n+1} + \ldots$$

The power series for the derivative and antiderivative of f converge for $-R < x < R$.
(The power series for f, f ' and the antiderivative of f may differ in whether or not they converge at the **endpoints** of the interval of convergence, but they all converge for $-R < x < R$.)

The proof of this result is rather long and technical and is omitted.

Like the previous substitution method, term–by–term differentiation and integration can be used to obtain power series for some functions related to the Geometric Series Formula $\frac{1}{1-x}$.

Example 2: Find power series for (a) ln(1 – x) , and (b) arctan(x) .

Solution: These two are more challenging than the previous examples because we need to recognize that these two functions are integrals of functions whose power series we already know.

(a) $$\ln(1-x) = \int \frac{-1}{1-x}\,dx = -\int \{ 1 + x + x^2 + x^3 + x^4 + \ldots \}\,dx$$

$$= -\left\{ x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots \right\} + C = C - \sum_{n=1}^{\infty} \frac{x^n}{n} .$$

We can find the value of C by using the fact that ln(1) = 0:

for x = 0, $0 = \ln(1-0) = C - \sum_{n=1}^{\infty} \frac{0^n}{n} = C$ so $C = 0$ and

$$\ln(1-x) = -\left\{ x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots \right\} = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } -1 \le x < 1.$$

(b) $$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \int \left\{ 1 - x^2 + x^4 - x^6 + x^8 - \ldots \right\} dx$$

$$= C + \left\{ x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots \right\} = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} .$$

We can find the value of C by using the fact that $\arctan(0) = 0$:

for x = 0, $0 = \arctan(0) = C + \sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{2n+1} = C$ so $C = 0$ and

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } -1 \le x \le 1.$$

Practice 2: Find a power series for $\ln(1+x)$.

Power series can also be used to help us evaluate definite integrals.

Example 3: Use the power series for $\arctan(x)$ to represent the definite integral $\int_0^{0.5} \arctan(x)\,dx$ as a numerical series. Then approximate the value of the integral by calculating the sum of the first four terms of the numerical series.

Solution: $$\int_0^{0.5} \arctan(x)\,dx = \int_0^{0.5} \left\{ x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots \right\} dx$$

$$= \frac{x^2}{2} - \frac{x^4}{4\cdot 3} + \frac{x^6}{6\cdot 5} - \frac{x^8}{8\cdot 7} + \frac{x^{10}}{10\cdot 9} - \ldots \Bigg|_0^{0.5}$$

$$= \left\{ \frac{1}{2}(0.5)^2 - \frac{1}{12}(0.5)^4 + \frac{1}{30}(0.5)^6 - \frac{1}{56}(0.5)^8 + \frac{1}{90}(0.5)^{10} - \ldots \right\} - \left\{ 0 \right\} .$$

The sum of the first four terms is approximately 0.120243 . Since the numerical series is an alternating series, we know that the fourth partial sum, 0.120243 , is within the value of the next term, $\frac{1}{90}(0.5)^{10} \approx 0.000011$, of the exact value of the sum. The exact value of the definite integral is between $0.120243 - 0.000011 = 0.120232$ and $0.120243 + 0.000011 = 0.120254$.

Practice 3: Use the power series for $x^2 \cdot \ln(1+x)$ to represent the definite integral $\int_0^{0.2} x^2 \cdot \ln(1+x)\,dx$ as a numerical series. Then approximate the value of the integral by calculating the sum of the first three terms of the numerical series.

All of the power series used in this section have followed from the Geometric Series Formula, and their main purpose here was to illustrate some uses of substitution and term–by–term differentiation and integration to obtain power series for related functions. Many functions, however, are not related to a geometric series, and the next section discusses a method for representing them using power series. Once we can represent these new functions using power series, we can then use substitution and term–by–term differentiation and integration to obtain power series for functions related to them.

The following table collects some of the power series representations we have obtained in this section.

Table of Series Based on $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n \text{ with interval of convergence } (-1, 1).$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1-(-x)}$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \ldots = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-(x^2)}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1-(-x^2)}$$

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + x^{12} + \ldots = \sum_{n=0}^{\infty} x^{3n} = \frac{1}{1-(x^3)}$$

$$\ln(1-x) = -\left\{ x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots \right\} = -\sum_{n=1}^{\infty} \frac{x^n}{n} = \int \frac{-1}{1-x}\,dx$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \int \frac{1}{1+x}\,dx$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \int \frac{1}{1+x^2}\,dx$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots = \sum_{n=1}^{\infty} n \cdot x^{n-1} = \mathbf{D}\left(\frac{1}{1-x}\right)$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots = \sum_{n=1}^{\infty} (-1)^n n \cdot x^{n-1} = \mathbf{D}\left(\frac{-1}{1+x}\right)$$

Fig. 1 shows the graphs of arctan(x) and the first few polynomials $x, x - \frac{x^3}{3}, x - \frac{x^3}{3} + \frac{x^5}{5}$ that approximate arctan(x). This type of approximation will be discussed a little in Section 10.10 and a lot in Section 10.11.

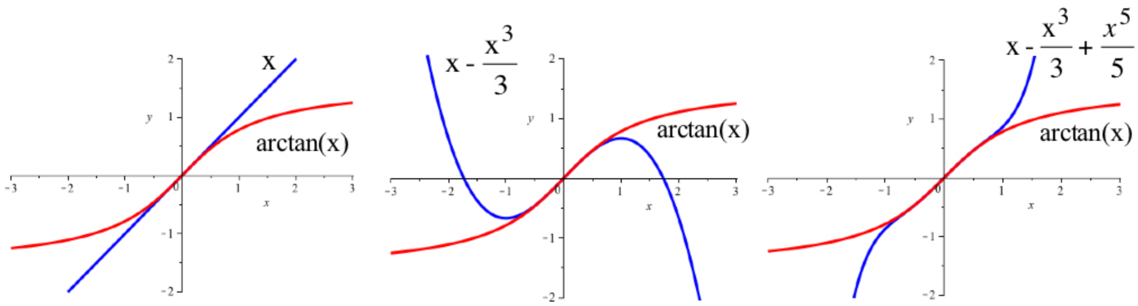

Fig. 1

PROBLEMS

In problems 1 – 14, use the substitution method and a known power series to find power series for the given functions.

1. $\frac{1}{1-x^4}$
2. $\frac{1}{1-x^5}$
3. $\frac{1}{1+x^4}$
4. $\frac{1}{1+x^5}$
5. $\frac{1}{5+x} = \frac{1}{5} \cdot \frac{1}{1+(x/5)}$
6. $\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1-(x/3)}$
7. $\frac{x^2}{1+x^3}$
8. $\frac{x}{1+x^4}$
9. $\ln(1+x^2)$
10. $\ln(1+x^3)$
11. $\arctan(x^2)$
12. $\arctan(x^3)$
13. $\frac{1}{(1-x^2)^2}$
14. $\frac{1}{(1+x^2)^2}$

In problems 15–21, represent each definite integral as a numerical series. Calculate the sum of the first three terms for each series.

15. $\int_0^{0.5} \frac{1}{1-x^3}\,dx$

16. $\int_0^{0.5} \frac{1}{1+x^3}\,dx$

17. $\int_0^{0.6} \ln(1+x)\,dx$

18. $\int_0^{0.5} x^2 \cdot \arctan(x)\,dx$

19. $\int_0^{0.3} \frac{1}{(1-x)^2}\,dx$

20. $\int_0^{0.7} \frac{x^3}{(1-x)^2}\,dx$

In problems 21 – 27, use the Table of Series to help represent each function as a power series. Then calculate each limit.

21. $\lim_{x\to 0} \frac{\arctan(x)}{x}$

22. $\lim_{x\to 0} \frac{\ln(1-x)}{2x}$

23. $\lim_{x\to 0} \frac{\ln(1+x)}{2x}$

24. $\lim_{x\to 0} \frac{\arctan(x^2)}{x}$

25. $\lim_{x\to 0} \frac{\ln(1-x^2)}{3x}$

26. $\lim_{x\to 0} \frac{\ln(1+x^2)}{3x}$

In problems 27 – 32, use the Table of Series or the substitution method to determine a power series for each function and then determine the interval of convergence of each power series.

27. $\frac{1}{1+x}$

28. $\frac{1}{1-x^2}$

29. $\ln(1-x)$

30. $\ln(1+x)$

31. $\arctan(x)$

32. $\arctan(x^2)$

The series given below for $\sin(x)$ and e^x are derived in Section 10.10. Use these given series and the methods of this section to answer problems 33 – 42.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

33. Find a power series for $\sin(x^2)$

34. Find a power series for $\sin(2x)$

35. Find a power series for $e^{(-x^2)}$

36. Find a power series for $e^{(2x)}$

37. Find a power series for $\cos(x)$

38. Find a power series for $\cos(x^2)$

39. Represent the integral as a numerical series: $\int_0^1 \sin(x^2)\,dx$

40. Represent the integral as a numerical series: $\int_0^1 e^{(-x^2)}\,dx$

41. $\lim_{x\to 0} \frac{\sin(x)}{x}$

42. $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$

Practice Answers

Practice 1: Geometric Series Formula: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

(a) Replacing "x" with "x^3" we have $\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}$ for $|x^3| < 1$ or $|x| < 1$.

(b) Replacing "x" with "$-x^2$" we have $\frac{1}{1-(-x^2)} = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

for $|-x| < 1$ or $|x| < 1$.

(c) Using the result of part (b), $\frac{5x}{1+x} = 5x \frac{1}{1+x} = 5x \cdot \sum_{n=0}^{\infty} (-1)^n x^n = 5 \cdot \sum_{n=0}^{\infty} (-1)^n x^{n+1}$

for $|-x| < 1$ or $|x| < 1$.

Practice 2: $\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx = \int 1 - x + x^2 - x^3 + x^4 - \ldots\,dx$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots + C$. Putting $x = 0$, we get $C = 0$. Then

$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ or, equivalently, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

Practice 3: Using the result of Practice 2,

$x^2 \cdot \ln(1+x) = x^2 \cdot \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+2}}{n}$. Then

$$\int x^2 \cdot \ln(1+x)\,dx = \int \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+2}}{n}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{x^{n+3}}{n+3} \Bigg|_0^{0.2}$$

$$= \left\{ \frac{1}{4}(0.2)^4 - \frac{1}{10}(0.2)^5 + \frac{1}{18}(0.2)^6 - \frac{1}{28}(0.2)^7 + \ldots \right\} - \{ 0 \} .$$

$s_3 = \frac{1}{4}(0.2)^4 - \frac{1}{10}(0.2)^5 + \frac{1}{18}(0.2)^6 \approx 0.0003716$ with an error less than $|a_4| = \frac{1}{28}(0.2)^7 \approx 4.57 \cdot 10^{-7}$

Appendix: Partial Sums of Power Series using MAPLE

MAPLE commands for $\sum_{n=1}^{100} \frac{(0.7)^n}{n}$, $\sum_{n=1}^{100} \frac{(0.7)^n}{n^2}$, $\sum_{n=0}^{100} \frac{(0.7)^n}{n!}$, $\sum_{n=0}^{100} (-1)^n \frac{(0.7)^n}{1+\sqrt{n}}$

```
> sum((.7)^n/n , n=1..100) ;   (then press ENTER key)
> sum((.7)^n/n^2 , n=1..100) ;
> sum((.7)^n/n! , n=0..100) ;
> sum((–1)^n*(.7)^n/(1+sqrt(n)) , n=0..100) ;
```