11.1 VECTORS IN THE PLANE

Measurements of some quantities such as mass, speed, temperature, and height can be given by a single number, but a single number is not enough to describe measurements of some quantities in the plane such as displacement or velocity. Displacement or velocity not only tell us how much or how fast something has moved but also tell the direction of that movement. For quantities that have both length (magnitude) and direction, we use vectors.

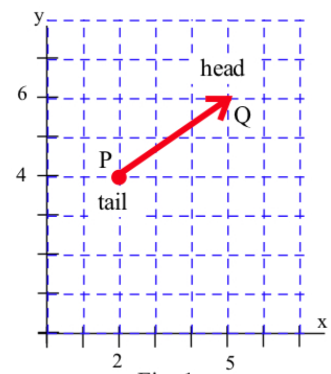
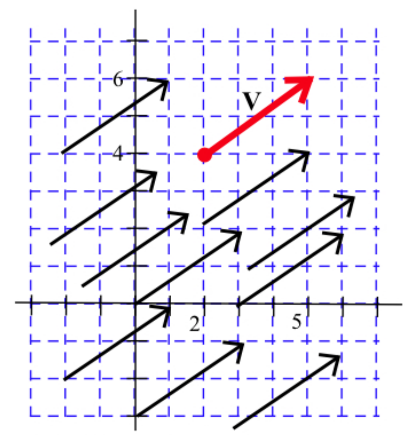


Fig. 1

A **vector** is a quantity that has both a magnitude and a direction, and vectors are represented geometrically as directed line segments (arrows). The vector **V** given by the directed line segment from the starting point P = (2,4) to the point Q = (5,6) is shown in Fig. 1. The starting point is called the **tail** of the vector and the ending point is called the **head** of the vector. Geometrically, two vectors are equal if they have the same length and point in the same direction. Fig. 2 shows a number of vectors that are equal to vector **V** in Fig. 1. The equality of vectors in the plane depends only on their lengths and directions. Equality of vectors does not depend on their locations in the plane.



All of these directed line segments represent the vector V
Fig. 2

A vector in the plane can be represented algebraically as an ordered pair of numbers measuring the horizontal and vertical displacement of the endpoint of the vector from the beginning point of the vector. The numbers in the ordered pair are called the **components** of the vector. Vector **V** in Fig. 1 can be represented as $\mathbf{V} = \langle 5-2\,, 6-4 \rangle = \langle 3, 2 \rangle$, an ordered pair of numbers enclosed by "bent" brackets. All of the vectors in Fig. 2 are also represented algebraically by $\langle 3, 2 \rangle$.

Notation: In our work with vectors, it is important to recognize when we are describing a number or a point or a vector. To help keep those distinctions clear, we use different notations for numbers, points and vectors:

a number: regular lower case letter: $a, b, x, y, x_1, y_1, \ldots$
A number is called a scalar quantity or simply a **scalar**.

a point: regular upper case letter: A, B, ...
ordered pair of numbers enclosed by ( ) : $(2, 3), (a, b), (x_1, y_1), \ldots$

a vector: **bold** upper case letter: **A**, **B**, **U**, **V**, ...
ordered pair of numbers enclosed by $\langle\ \rangle$: $\langle 2, 3 \rangle, \langle a, b \rangle, \langle x_1, y_1 \rangle, \ldots$
a letter with an arrow over it: $\vec{A}, \vec{B}, \vec{U}, \vec{V}, \ldots$

Definition: Equality of Vectors

Geometrically, two vectors are equal if their lengths are equal and their directions are the same.

Algebraically, two vectors are equal if their respective components are equal:

if $\mathbf{U} = \langle a, b \rangle$ and $\mathbf{V} = \langle x, y \rangle$, then $\mathbf{U} = \mathbf{V}$ if and only if $a = x$ and $b = y$.

Example 1: Vectors **U** and **V** are given in Fig. 3.

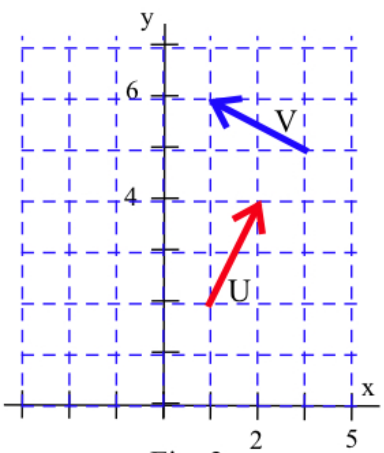


Fig. 3

(a) Represent **U** and **V** using the $\langle \ \rangle$ notation.

(b) Sketch **U** and **V** as line segments starting at the point (0,0).

(c) Sketch **U** and **V** as line segments starting at the point (–1,5).

(d) If (x,y) is the starting point, what is the ending point of the line segment representing the vector **U** ?

Solution:

(a) The components of a vector are the displacements from the starting to the ending points so $\mathbf{U} = \langle 2-1, 4-2 \rangle = \langle 1, 2 \rangle$ and $\mathbf{V} = \langle 1-3, 6-5 \rangle = \langle -2, 1 \rangle$.

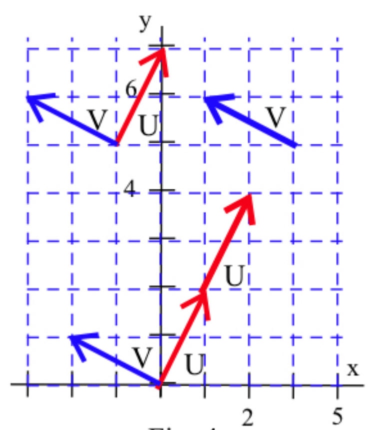


Fig. 4

(b) If **U** starts at (0,0), then the ending point of the line segment is (0+1, 0+2) = (1,2).

The ending point of **V** is (0–2, 0+1) = (–2, 1). See Fig. 4.

(c) The ending point of the line segment for **U** is (–1+1, 5+2) = (0,7). For **V**, the ending point is (–1–2, 5+1) = (–3, 6). See Fig. 4.

(d) If (x,y) is the starting point for $\mathbf{U} = \langle 1, 2 \rangle$ then the ending point is (x+1, y+2).

Practice 1: $\mathbf{A} = \langle 3, 4 \rangle$ and $\mathbf{W} = \langle -2, 3 \rangle$.

(a) Represent **A** and **W** as line segments beginning at the point (0, 0).

(b) Represent **A** and **W** as line segments beginning at the point (2, –4).

(c) If **A** and **W** begin at the point (p, q), at which points do they end?

(d) How long is a line segment representing vector **A**? **W**?

(e) What is the slope of a line segment representing vector **A**? **W**?

(f) Find a vector whose line segment representation is perpendicular to **A**. To **W**.

The magnitude of a vector $\mathbf{V}$, written $|\mathbf{V}|$, is the length of the line segment representing the vector. That length is the distance between the starting point and the ending point of the line segment. The magnitude can be calculated by using the distance formula.

> The **magnitude** or **length** of a vector $\mathbf{V} = \langle a, b \rangle$ is $|\mathbf{V}| = \sqrt{a^2 + b^2}$.

The only vector in the plane with magnitude 0 is $\langle 0, 0 \rangle$, called the **zero vector** and written $\mathbf{0}$ or $\vec{0}$. The zero vector is a line segment of length 0 (a point), and it has no specific direction or slope.

Adding Vectors

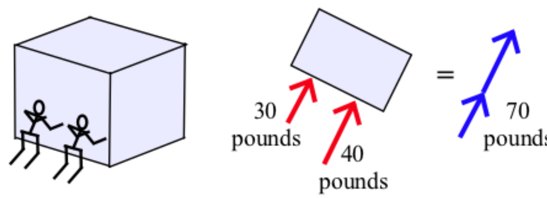


Fig. 5

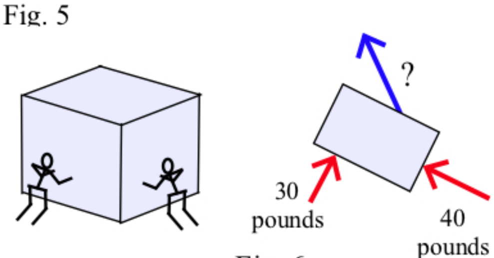


Fig. 6

If two people are pushing a box in the same direction along a line, one with a force of 30 pounds and the other with a force of 40 pounds (Fig. 5), then the result of their efforts is equivalent to a single force of 70 pounds along the same line. However, if the people are pushing in different directions (Fig. 6), the problem of finding the result of their combined effort is slightly more difficult. Vector addition provides a simple solution.

> **Definition**: Vector Addition
>
> If $\mathbf{A} = \langle a_1, a_2 \rangle$ and $\mathbf{B} = \langle b_1, b_2 \rangle$, then $\mathbf{A} + \mathbf{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$.

The result of applying two forces, represented by the vectors $\mathbf{A}$ and $\mathbf{B}$, is equivalent to the single force represented by the vector $\mathbf{A} + \mathbf{B}$. In Fig. 6, the effort of person A can be represented by the vector $\mathbf{A} = \langle 30, 0 \rangle$ and the effort of person B by the vector $\mathbf{B} = \langle 0, 40 \rangle$. Their combined effort is equivalent to a single force vector $\mathbf{C} = \mathbf{A} + \mathbf{B} = \langle 30, 40 \rangle$. (Since $|\mathbf{C}| = 50$ pounds, if the two people cooperated and pushed in the direction of $\mathbf{C}$, they could achieve the same result by each exerting 10 pounds less force.)

Example 2: Let $\mathbf{A} = \langle 3, 5 \rangle$ and $\mathbf{B} = \langle -1, 4 \rangle$. (a) Graph $\mathbf{A}$ and $\mathbf{B}$ each starting at the origin.

(b) Calculate $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and graph it, starting at the origin.

(c) Calculate the magnitudes of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

(d) Find a vector $\mathbf{V}$ so $\mathbf{A} + \mathbf{V} = \langle 4, 2 \rangle$.

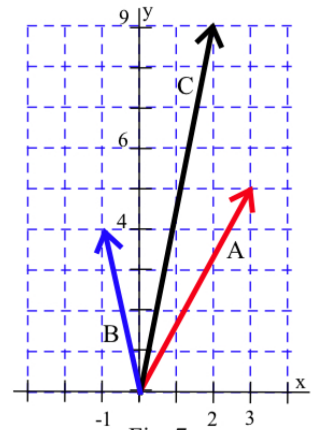


Fig. 7

Solution: (a) The graphs of $\mathbf{A}$ and $\mathbf{B}$ are shown in Fig. 7.

(b) $\mathbf{C} = \langle 3, 5 \rangle + \langle -1, 4 \rangle = \langle 2, 9 \rangle$. The graph of $\mathbf{C}$ is shown in Fig. 7.

(c) $|\mathbf{A}| = \sqrt{3^2 + 5^2} = \sqrt{34} \approx 5.8$, $|\mathbf{B}| = \sqrt{17} \approx 4.1$, and $|\mathbf{C}| = \sqrt{85} \approx 9.2$.

(d) Let $\mathbf{V} = \langle x, y \rangle$. Then $\mathbf{A} + \mathbf{V} = \langle 3+x, 5+y \rangle = \langle 4, 2 \rangle$ so $3+x = 4$ and $x = 1$. Also, $5+y = 2$ so $y = -3$ and $\mathbf{V} = \langle 1, -3 \rangle$

Practice 2: Let $\mathbf{A} = \langle -2, 5 \rangle$ and $\mathbf{B} = \langle 7, -4 \rangle$. (a) Graph $\mathbf{A}$ and $\mathbf{B}$ each starting at the origin.

(b) Calculate $\mathbf{C} = \mathbf{A} + \mathbf{B}$ and graph it, starting at the origin.

(c) Find and graph a vector $\mathbf{V}$ so $\mathbf{V} + \mathbf{C} = 0$.

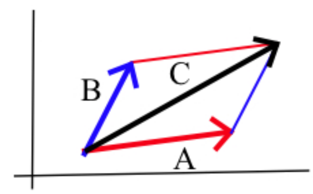


Fig. 8

The parallelogram method and the head–to–tail method are two commonly used methods for adding vectors graphically.

The parallelogram method (Fig. 8):

i) arrange the vectors $\mathbf{A}$ and $\mathbf{B}$ to have a common starting point

ii) use the two given vectors to complete a parallelogram

iii) draw a vector $\mathbf{C}$ from the common starting point of the two original vectors to the opposite corner of the parallelogram: $\mathbf{C} = \mathbf{A} + \mathbf{B}$

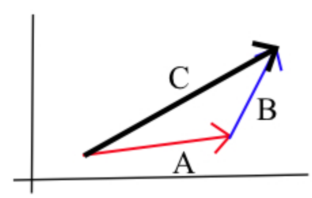


Fig. 9

The head–to–tail method (Fig. 9):

i) position the tail of vector $\mathbf{B}$ at the head of the vector $\mathbf{A}$

ii) draw a vector $\mathbf{C}$ from the tail of $\mathbf{A}$ to the head of $\mathbf{B}$: $\mathbf{C} = \mathbf{A} + \mathbf{B}$

The head–to–tail method is particularly useful when we need to add several vectors together. We can simply string them along head–to–tail, head–to–tail, ... and finally draw their sum as a directed line segment from the tail of the first vector to the head of the last vector (Fig. 10).

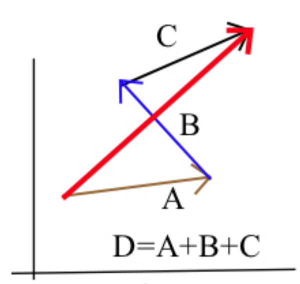


Fig. 10

Practice 3: Draw the vectors $\mathbf{U} = \mathbf{A} + \mathbf{C}$ and $\mathbf{V} = \mathbf{A} + \mathbf{B} + \mathbf{C}$ for $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ given in Fig. 11.

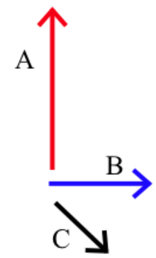


Fig. 11

Scalar Multiplication

We can multiply a vector by a number (a scalar) by multiplying each component by that number.

> **Definition**: Scalar Multiplication
>
> If k is a scalar and $\mathbf{A} = \langle a_1, a_2 \rangle$ is a vector,
> then $k\mathbf{A} = \langle ka_1, ka_2 \rangle$, a vector.

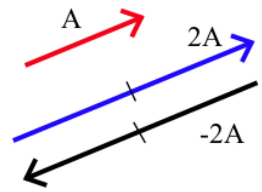


Fig. 12

Multiplying by a scalar k gives a vector that is $|k|$ times as long as the original vector. If k is positive, then $\mathbf{A}$ and $k\mathbf{A}$ have the same direction. If k is negative, then $\mathbf{A}$ and $k\mathbf{A}$ point in opposite directions (Fig. 12).

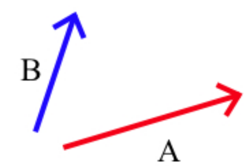


Fig. 13

Example 3: Vectors $\mathbf{A}$ and $\mathbf{B}$ are shown in Fig. 13.

Graph and label $\mathbf{C} = 2\mathbf{A}$, $\mathbf{D} = \frac{1}{2}\mathbf{B}$, $\mathbf{E} = -2\mathbf{B}$, $\mathbf{F} = \mathbf{B} + 2\mathbf{A}$, and $\mathbf{G} = \mathbf{A} + (-1)\mathbf{A}$.

Solution: The vectors are shown in Fig. 14.

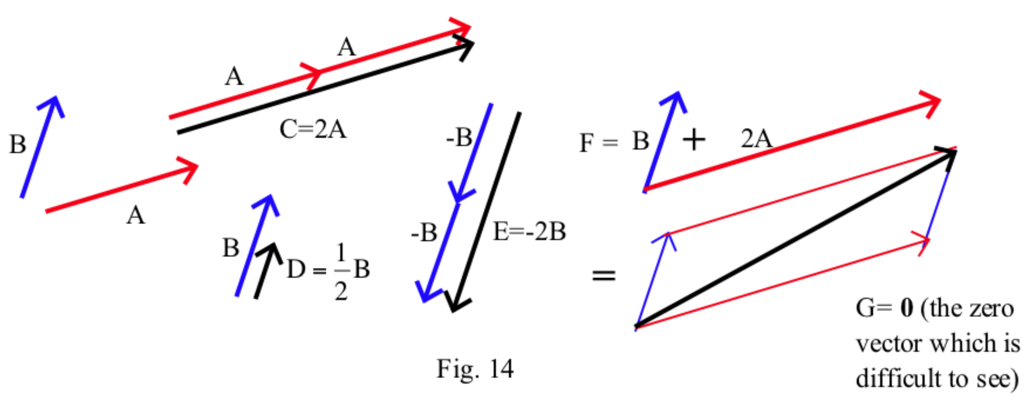


Fig. 14

Practice 4: Vectors $\mathbf{U}$ and $\mathbf{V}$ are shown in Fig. 15.

Graph and label $\mathbf{A} = 2\mathbf{U}$, $\mathbf{B} = -\frac{1}{2}\mathbf{U}$, $\mathbf{C} = (-1)\mathbf{V}$, and $\mathbf{D} = \mathbf{U} + (-1)\mathbf{V}$.

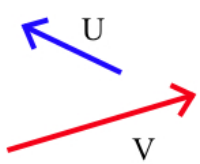


Fig. 15

Subtracting Vectors

We can also subtract vectors algebraically and graphically.

Definition: Vector Subtraction

If $\mathbf{A} = \langle a_1, a_2 \rangle$ and $\mathbf{B} = \langle b_1, b_2 \rangle$, then $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} = \langle a_1 - b_1, a_2 - b_2 \rangle$, a vector.

Graphically, we can construct the line segment representing the vector $\mathbf{A} - \mathbf{B}$ either using the head–to–tail method (Fig. 16) to add $\mathbf{A}$ and $-\mathbf{B}$, or by moving $\mathbf{A}$ and $\mathbf{B}$ so they have a common starting point (Fig. 17) and then drawing the line segment from the head of $\mathbf{B}$ to the head of $\mathbf{A}$.

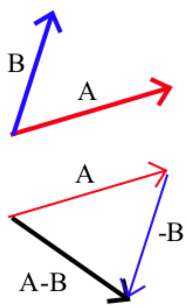


Fig. 16

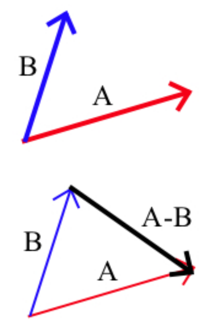


Fig. 17

Practice 5: Vectors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are shown in Fig. 18. Graph $\mathbf{U} = \mathbf{A} - \mathbf{B}$ and $\mathbf{V} = \mathbf{C} - \mathbf{A}$.

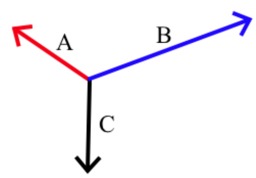


Fig. 18

Algebraic Properties of Vector Operations

Some of the properties of vectors are given below.

If $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are vectors in the plane and x and y are scalars, then

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
3. $\mathbf{A} + \mathbf{0} = \mathbf{A}$ ($\mathbf{0} = \langle 0,0 \rangle$ is called the additive identity vector)
4. $\mathbf{A} + (-1)\mathbf{A} = \mathbf{0}$ ($-\mathbf{A} = (-1)\mathbf{A}$ is called the additive inverse vector of A)
5. $x(\mathbf{A} + \mathbf{B}) = x\mathbf{A} + x\mathbf{B}$
6. $(x + y)\mathbf{A} = x\mathbf{A} + y\mathbf{A}$

These and additional properties of vectors are easily verified using the definitions of vector equality, vector addition and scalar multiplication.

Unit Vector, Direction of a Vector, Standard Basis Vectors

Definitions:

A **unit vector** is a vector whose length is 1.

The **direction** of a nonzero vector $\mathbf{A}$ is the unit vector $\frac{1}{|\mathbf{A}|}\mathbf{A} = \frac{\mathbf{A}}{|\mathbf{A}|}$.

The **standard basis vectors** in the plane are $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. (Fig. 19)

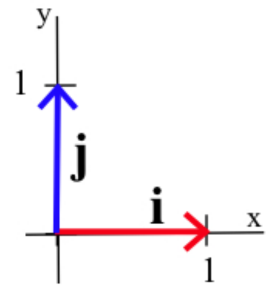


Fig. 19

Every nonzero vector $\mathbf{A}$ is the product of its magnitude and its direction: $\mathbf{A} = |\mathbf{A}| \frac{\mathbf{A}}{|\mathbf{A}|}$.

The standard basis vectors are unit vectors.

Every vector in the plane can be written as a sum of scalar multiples of these two basis vectors:

if $\mathbf{A} = \langle a_1, a_2 \rangle$, then $\mathbf{A} = \langle a_1, 0 \rangle + \langle 0, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$.

Example 4: Let $\mathbf{A} = 5\mathbf{i} + 12\mathbf{j}$ and $\mathbf{B} = 3\mathbf{i} - 4\mathbf{j}$.

(a) Determine the directions of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C} = \mathbf{A} - 3\mathbf{j}$.

(b) Write $3\mathbf{A} + 2\mathbf{B}$ and $4\mathbf{A} - 5\mathbf{B}$ in terms of the standard basis vectors.

Solution: (a) $|\mathbf{A}| = \sqrt{25 + 144} = \sqrt{169} = 13$ so the direction of $\mathbf{A}$ is

$\frac{1}{13}\langle 5, 12 \rangle = \langle \frac{5}{13}, \frac{12}{13} \rangle = \frac{5}{13}\mathbf{i} + \frac{12}{13}\mathbf{j}$. The direction of B is $\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$.

$|\mathbf{C}| = |\, 5\mathbf{i} + 9\mathbf{j}\, | = \sqrt{106}$ so the direction of $\mathbf{C}$ is $\frac{5}{\sqrt{106}}\mathbf{i} + \frac{9}{\sqrt{106}}\mathbf{j}$.

(b) $3\mathbf{A} + 2\mathbf{B} = 3(5\mathbf{i} + 12\mathbf{j}) + 2(3\mathbf{i} - 4\mathbf{j}) = 15\mathbf{i} + 36\mathbf{j} + 6\mathbf{i} - 8\mathbf{j} = 21\mathbf{i} + 28\mathbf{j}$.

$4\mathbf{A} - 5\mathbf{B} = 4(5\mathbf{i} + 12\mathbf{j}) - 5(3\mathbf{i} - 4\mathbf{j}) = 20\mathbf{i} + 48\mathbf{j} - 15\mathbf{i} + 20\mathbf{j} = 5\mathbf{i} + 68\mathbf{j}$.

Practice 6: Let $\mathbf{U} = 7\mathbf{i} + 24\mathbf{j}$ and $\mathbf{V} = 15\mathbf{i} - 8\mathbf{j}$.

Determine the directions of $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W} = \mathbf{U} + 3\mathbf{i}$.

Often in applications a vector is described in terms of a magnitude at an angle to some line. In those situations we typically need to use trigonometry to determine the components of the vector.

Example 5: Suppose you are pulling on the rope with a force of 50 pounds at an angle of 25° to the horizontal ground (Fig. 20). What are the components of this force vector parallel and perpendicular to the ground?

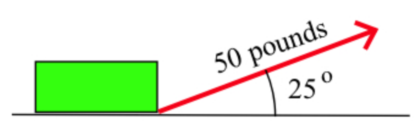


Fig. 20

Solution: The horizontal component (parallel to the ground) is $50\cos(25°) \approx 45.3$ pounds.

The vertical component (perpendicular to the ground) is $50\sin(25°) \approx 21.1$ pounds.

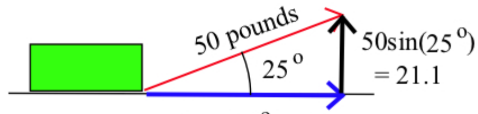


Fig. 21

A force of approximately 45.3 pounds operates to pull the box along the ground (Fig. 21), and a force of approximately 21.1 pounds is operating to lift the box.

The following result from trigonometry is used to find the components of vectors in the plane.

> If a vector $\mathbf{V}$ with magnitude $|\mathbf{V}|$ makes an angle of θ with a horizontal line,
>
> then $\mathbf{V} = |\mathbf{V}|\cos(\theta)\,\mathbf{i} + |\mathbf{V}|\sin(\theta)\,\mathbf{j} = \langle |\mathbf{V}|\cos(\theta), |\mathbf{V}|\sin(\theta) \rangle$.

Practice 7: A horizontal force of 50 pounds is required to move the box in Fig. 22. If you can pull on the rope with a total force of 70 pounds, what is the largest angle that the rope can make with the ground and still move the box?

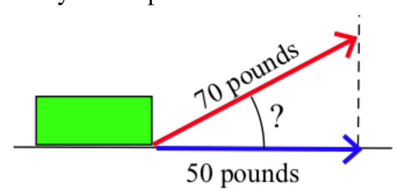


Fig. 22

Additional Applications of Vectors in the Plane

The following applications are more complicated than the previous ones, but they begin to illustrate the range and power of vector methods for solving applied problems. In general, vector methods allow us to work separately with the horizontal and vertical components of a problem, and then put the results together into a complete answer.

Example 6: In water with no current, your boat can travel at 20 knots (nautical miles per hour). Suppose your boat is on the ocean and you want to follow a course to travel due north, but the water current is $\mathbf{W} = -6\mathbf{i} - 8\mathbf{j}$ (Fig. 23). At what angle θ, east of due north, should you steer your boat so the resulting course $\mathbf{R}$ is due north?

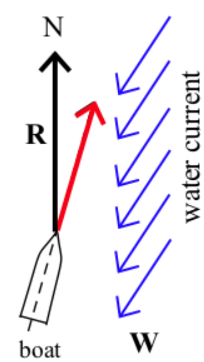


Fig. 23

Solution: First, we should notice that since $\mathbf{R}$ points due north, the $\mathbf{i}$ component $\mathbf{R}$ is 0: $\mathbf{R} = \langle 0, s \rangle$. We also need to recognize that $\mathbf{V}$ makes an angle of $90° - \theta$ with the horizontal so $\mathbf{V} = \langle 20\cos(90° - \theta), 20\sin(90° - \theta) \rangle$. Finally $\mathbf{V} + \mathbf{W} = \mathbf{R}$ so $\langle 20\cos(90° - \theta) - 6, 20\sin(90° - \theta) - 8 \rangle = \langle 0, s \rangle$. Equating the first components of this vector equation, we have $20\cos(90° - \theta) - 6 = 0$ so $\cos(90° - \theta) = 6/20 = 0.3$, $90° - \theta \approx 72.5°$, and $\theta \approx 17.5°$. You should steer your boat approximately $17.5°$ east of due north in order to maintain a course taking you due north. Your speed along this course is $|\mathbf{R}| = s = 20\sin(90° - \theta) - 8 \approx 20\sin(72.5°) - 8 \approx 11.1$ knots.

Practice 8: With the same boat and water current as in Example 6, at what angle θ, east of due north, should you steer your boat so the resulting course **R** is due east? What is your resulting speed due east?

Example 7: A video camera weighing 15 pounds is going to be suspended by two wires from the ceiling of a room as shown in Fig. 24. What is the resulting tension in each wire? (The tension in a wire is the magnitude of the force vector.)

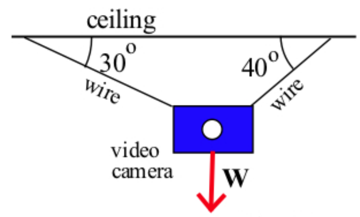


Fig. 24

Solution: The force vector of the camera is straight down so $\mathbf{W} = \langle 0, -15 \rangle$. Let vector **A** be the force vector for the left (30^o) wire, and vector **B** be the force vector for the right (40^o) vector. Vector **A** has magnitude $|\mathbf{A}|$ and can be represented as $\langle -|\mathbf{A}|\cos(30°), |\mathbf{A}|\sin(30°) \rangle$. Similarly, $\mathbf{B} = \langle |\mathbf{B}|\cos(40°), |\mathbf{B}|\sin(40°) \rangle$. Since the system is in equilibrium, the sum of the force vectors is 0 so

$\mathbf{0} = \mathbf{A} + \mathbf{B} + \mathbf{W} = \langle -|\mathbf{A}|\cos(30°) + |\mathbf{B}|\cos(40°) + 0, |\mathbf{A}|\sin(30°) + |\mathbf{B}|\sin(40°) - 15 \rangle$.

From the components of the vector equation we have two equations,

$$0 = -|\mathbf{A}|\cos(30°) + |\mathbf{B}|\cos(40°) + 0 \quad \text{and} \quad 0 = |\mathbf{A}|\sin(30°) + |\mathbf{B}|\sin(40°) - 15 ,$$

that we want to solve for the tensions $|\mathbf{A}|$ and $|\mathbf{B}|$.

From the first, we get $|\mathbf{A}|\cos(30°) = |\mathbf{B}|\cos(40°)$ so $|\mathbf{B}| = |\mathbf{A}|\dfrac{\cos(30°)}{\cos(40°)}$.

Substituting this value for $|\mathbf{B}|$ into the second equation we have

$$0 = |\mathbf{A}|\sin(30°) + |\mathbf{A}|\frac{\cos(30°)}{\cos(40°)}\sin(40°) - 15 = |\mathbf{A}|\left\{\sin(30°) + \cos(30°)\tan(40°)\right\} - 15$$

so $|\mathbf{A}| = \dfrac{15}{\sin(30°) + \cos(30°)\tan(40°)} \approx 12.2$ pounds. Putting this value back into

$|\mathbf{B}| = |\mathbf{A}|\dfrac{\cos(30°)}{\cos(40°)}$, we get $|\mathbf{B}| = (12.2)\dfrac{\cos(30°)}{\cos(40°)} \approx 13.9$ pounds .

Practice 9: What are the tensions in the wires if the angles are changed to 35° and 50° ?

PROBLEMS

In problems 1 – 4, vectors **U** and **V** are given graphically. Sketch the vectors $3\mathbf{U}$, $-2\mathbf{V}$, $\mathbf{U}+\mathbf{V}$, and $\mathbf{U}-\mathbf{V}$.

1. **U** and **V** are given in Fig. 25.
2. **U** and **V** are given in Fig. 26.
3. **U** and **V** are given in Fig. 27.
4. **U** and **V** are given in Fig. 28.

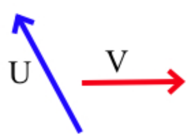


Fig. 25

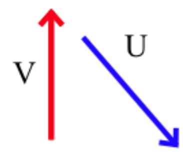


Fig. 26

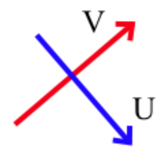


Fig. 27

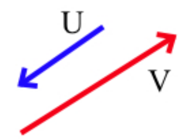


Fig. 28

In problems 5 – 10, vectors **U** and **V** are given.

(a) Sketch the vectors $\mathbf{U}, \mathbf{V}, 3\mathbf{U}, -2\mathbf{V}, \mathbf{U}+\mathbf{V}$, and $\mathbf{U}-\mathbf{V}$.

(b) Calculate $|\mathbf{U}|$ and $|\mathbf{V}|$ and find the directions of **U** and **V**.

(c) Find the slopes of the line segments representing **U** and **V** and their angles with the x–axis.

5. $\mathbf{U}=\langle 1,4 \rangle$ and $\mathbf{V}=\langle 3,2 \rangle$
6. $\mathbf{U}=\langle 2,5 \rangle$ and $\mathbf{V}=\langle 6,1 \rangle$
7. $\mathbf{U}=\langle -2,5 \rangle$ and $\mathbf{V}=\langle 3,-7 \rangle$
8. $\mathbf{U}=\langle -3,4 \rangle$ and $\mathbf{V}=\langle 1,-5 \rangle$
9. $\mathbf{U}=\langle -4,-3 \rangle$ and $\mathbf{V}=\langle 3,-4 \rangle$
10. $\mathbf{U}=\langle -5,2 \rangle$ and $\mathbf{V}=\langle -2,-5 \rangle$

In problems 11 – 14, vectors **A**, **B**, and **C** are given. Calculate $\mathbf{U}=\mathbf{A}+\mathbf{B}-\mathbf{C}$ and $\mathbf{V}=\mathbf{A}-\mathbf{B}+\mathbf{C}$.

11. **A**, **B**, and **C** in Fig. 29.
12. **A**, **B**, and **C** in Fig. 30.
13. $\mathbf{A}=\langle 1,4 \rangle$, $\mathbf{B}=\langle 3,1 \rangle$, $\mathbf{C}=\langle 5,2 \rangle$
14. $\mathbf{A}=\langle -1,5 \rangle$, $\mathbf{B}=\langle 2,0 \rangle$, $\mathbf{C}=\langle 6,-2 \rangle$

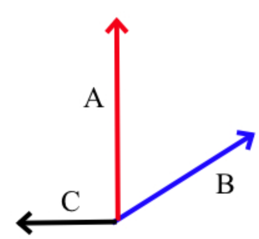


Fig. 29

In problems 15 – 20, find a vector **V** with the given properties.

15. $|\mathbf{V}|=3$ and direction $0.6\mathbf{i}+0.8\mathbf{j}$
16. $|\mathbf{V}|=2$ and direction $0.6\mathbf{i}-0.8\mathbf{j}$
17. $|\mathbf{V}|=5$ and **V** makes an angle of 35^{o} with the positive x–axis.
18. $|\mathbf{V}|=2$ and **V** makes an angle of 150^{o} with the positive x–axis.
19. $|\mathbf{V}|=7$ and **V** has slope 3.
20. $|\mathbf{V}|=7$ and **V** has slope –2.

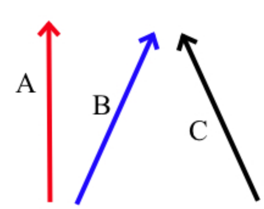


Fig. 30

In problems 21 – 26, find the direction of each function at the given point. That is, find a unit vector, there are two, parallel to the tangent line to the curve at the given point.

21. $f(x) = x^2 + 3x - 2$ at $(1, 2)$

22. $f(x) = \sin(3x) + e^x$ at $(0, 1)$

23. $f(x) = \ln(x^2 + 1)$ at $(0, 0)$

24. $f(x) = \dfrac{2 + \sin(x)}{1 + x}$ at $(0, 2)$

25. $f(x) = \cos^2(3x)$ at $(\pi, 1)$

26. $f(x) = \arctan(x)$ at $(0, 0)$

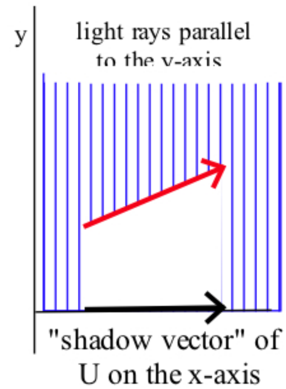


Fig. 31

In problems 27 – 34, a vector $\mathbf{U}$ is shown or given as $\mathbf{U} = \langle a, b \rangle$.

(a) Sketch the "shadow vector" (Fig. 31) of $\mathbf{U}$ on the x–axis. (This is the "projection of **U**" onto the x–axis.)

(b) Sketch the "shadow vector" of $\mathbf{U}$ on the y–axis. (This is the "projection of **U**" onto the y–axis.)

(c) Represent each of these "shadow vectors" in the form $a\mathbf{i} + b\mathbf{j}$.

27. **U** is given in Fig. 32.

28. **U** is given in Fig. 33.

29. **U** is given in Fig. 34.

30. **U** is given in Fig. 35.

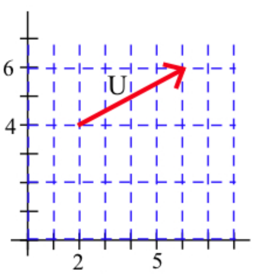


Fig. 32

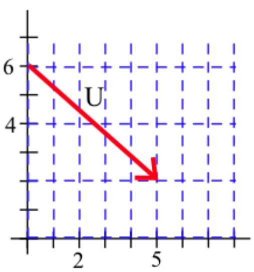


Fig. 33

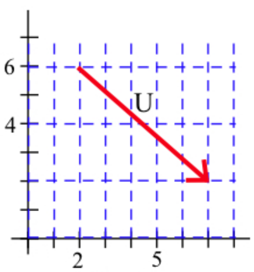


Fig. 34

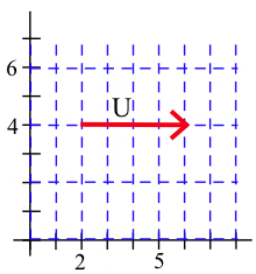


Fig. 35

31. $\mathbf{U} = \langle 1, 4 \rangle$

32. $\mathbf{U} = \langle -2, 3 \rangle$

33. $\mathbf{U} = \langle 5, -2 \rangle$

34. $\mathbf{U} = \langle -1, -3 \rangle$

In problems 35 – 38, vectors $\mathbf{A}$ and $\mathbf{B}$ are given. Find a vector $\mathbf{C}$ so that $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$.

35. **A** and **B** are shown in Fig. 36.

36. **A** and **B** are shown in Fig. 37.

37. $\mathbf{A} = \langle 7, 4 \rangle$ and $\mathbf{B} = \langle -3, 2 \rangle$

38. $\mathbf{A} = \langle -5, -3 \rangle$ and $\mathbf{B} = \langle 2, -4 \rangle$

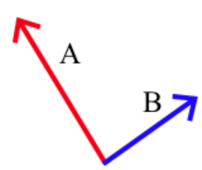


Fig. 36

39. Suppose you are pulling on the rope with a force of 60 pounds at an angle of 65° to the horizontal ground. What are the components of this force vector parallel and perpendicular to the ground?

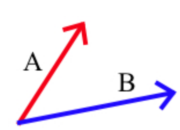


Fig. 37

40. Suppose you are pulling on the rope with a force of 100 pounds at an angle of 15° to the horizontal ground. What are the components of this force vector parallel and perpendicular to the ground?

41. Two ropes are attached to the bumper of a car. Rope A is pulled with a force of 50 pounds at an angle of 45° to the horizontal ground, and rope B is pulled with a force of 70 pounds at an angle of 35° to the horizontal ground. The same effect can be produced by a single rope pulling with what force and at what angle to the ground?

42. Two ropes are attached to the bumper of a car. Rope A is pulled with a force of 60 pounds at an angle of 30° to the horizontal ground, and rope B is pulled with a force of 80 pounds at an angle of 15° to the horizontal ground. The same effect can be produced by a single rope pulling with what force and at what angle to the ground?

43. Rope A is pulled with a force of 100 pounds at an angle of 30° to the horizontal ground, rope B is pulled with a force of 90 pounds at an angle of 25° to the horizontal, and rope C is pulled with a force of 80 pounds at an angle of 15° to the horizontal. The same effect can be produced pulling on a single rope with what force and at what angle to the ground?

44. A plane is flying due east at 200 miles per hour when it encounters a wind $\mathbf{W} = 30\mathbf{i} + 40\mathbf{j}$.
   (a) What is the path of the plane in this wind if the pilot keeps it pointed due east?
   (b) What direction should the pilot point the plane in order to fly due east?

45. A boat is moving due north at 18 miles per hour when it encounters a current $\mathbf{C} = -3\mathbf{i} + 4\mathbf{j}$.
   (a) What is the path of the boat in this current if the boater keeps it pointed due north?
   (b) What direction should the boater steer in order to go due north?

46. Suppose you leave home and hike 10 miles due north, then 8 miles in the direction 40° east of north, and then 6 miles due east. (a) How far are you from home? (b) What direction should you hike in order to return home?

47. A 60 foot rope is attached to the tops of two poles that are 50 feet apart, and a 100 pound person is going hand–over–hand from one end of the rope to the other (Fig. 38).
   (a) What is the tension in each part of the rope when the person is half way?
   (b) What is the tension in the rope when the person is 10 feet (horizontally) from the start?

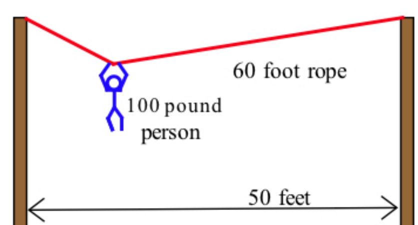


Fig. 38

Practice Answers

Practice 1: (a) and (b) See Fig. 39.

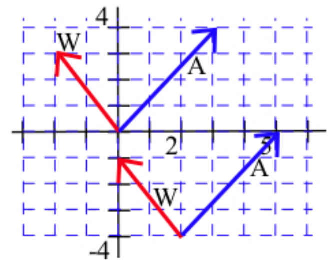


Fig. 39

(c) **A** ends at the point (p+3, q+4). **W** ends at the point (p–2, q+3).

(d) We can use the Pythagorean distance formula to determine the length of the hypotenuse.

For **A**, length $= \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

For **W**, length $= \sqrt{(-2)^2 + 3^2} = \sqrt{13} \approx 3.6$.

(e) Slope = rise/run. For **A**, slope = 4/3. For **W**, slope = 3/(–2).

(f) Two lines are perpendicular if one slope is the negative reciprocal of the other. If **B** is perpendicular to **A**, then the slope of **B** is –1/(slope of **A**) = –3/4. The vector $\mathbf{B} = \langle 4, -3 \rangle$ is one vector that has the slope we want. $\mathbf{B} = \langle -4, 3 \rangle$, and $\mathbf{B} = \langle 2, -3/2 \rangle$ are two other vectors with the same slope, and there are lots of others. $\mathbf{U} = \langle 3, 2 \rangle$ has slope 2/3 and is perpendicular to **W**.

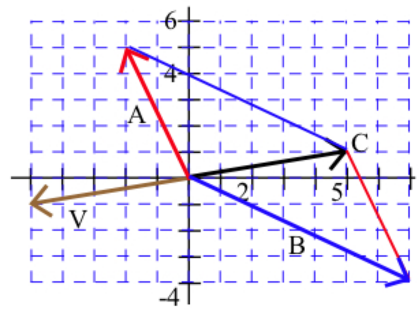


Fig. 40

Practice 2: (a) **A** and **B** are shown in Fig. 40.

(b) $\mathbf{C} = \langle -2, 5 \rangle + \langle 7, -4 \rangle = \langle 5, 1 \rangle$. The graph of **C** is shown in Fig. 40.

(c) $\mathbf{V} = \langle -5, -1 \rangle$. The graph of **V** is shown in Fig. 40.

Practice 3: See Fig. 41.

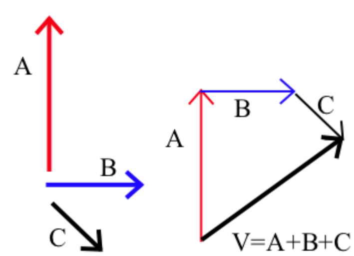


Fig. 41

Practice 4: See Fig. 42.

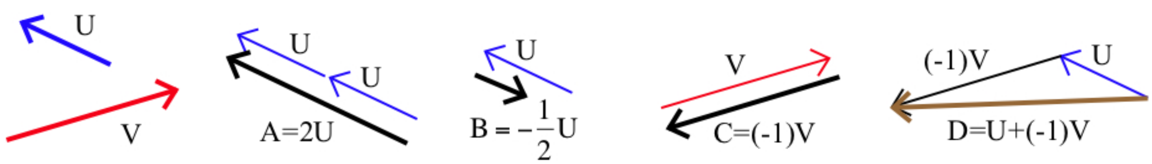


Fig. 42

Practice 5: See Fig. 43.

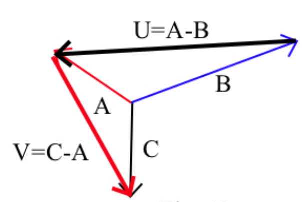


Fig. 43

Practice 6: $|\mathbf{U}| = \sqrt{(7)^2 + (24)^2} = \sqrt{625} = 25$, so the direction of $\mathbf{U}$ is

$$\frac{\mathbf{U}}{|\mathbf{U}|} = \frac{7\mathbf{i} + 24\mathbf{j}}{25} = \frac{7}{25}\mathbf{i} + \frac{24}{25}\mathbf{j} .$$

$|\mathbf{V}| = \sqrt{(15)^2 + (-8)^2} = \sqrt{289} = 17$, so the direction of $\mathbf{V}$ is $\dfrac{\mathbf{V}}{|\mathbf{V}|} = \dfrac{15\mathbf{i} - 8\mathbf{j}}{17} = \dfrac{15}{17}\mathbf{i} - \dfrac{8}{17}\mathbf{j}$.

$\mathbf{W} = \mathbf{U} + 3\mathbf{i} = (7\mathbf{i} + 24\mathbf{j}) + 3\mathbf{i} = 10\mathbf{i} + 24\mathbf{j}$, so $|\mathrm{W}|\ \sqrt{(10)^2 + (24)^2} = \sqrt{676} = 26$.

The direction of $\mathbf{W}$ is $\dfrac{10\mathbf{i} + 24\mathbf{j}}{26} = \dfrac{5}{13}\mathbf{i} + \dfrac{12}{13}\mathbf{j}$.

Practice 7: $|\mathbf{V}| = 70$ pounds, and you want the horizontal component, $|\mathbf{V}|\cos(\theta)$, to be 50 pounds, so $70\cos(\theta) = 50$ and $\theta = \arccos(5/7) \approx 44.4°$.

Practice 8: $\mathbf{R} = \langle s, 0 \rangle$. $\mathbf{V}$ makes an angle of $90° - \theta$ with the horizontal so

$\mathbf{V} = \langle 20\cos(90° - \theta), 20\sin(90° - \theta) \rangle$.

Also, $\mathbf{V} + \mathbf{W} = \mathbf{R}$ so $\langle 20\cos(90° - \theta) - 6, 20\sin(90° - \theta) - 8 \rangle = \langle s, 0 \rangle$.

Equating the second components of this vector equation, we have

$20\sin(90° - \theta) - 8 = 0$ so $\sin(90° - \theta) = 8/20 = 0.8$, $90° - \theta \approx 23.6°$, and $\theta \approx 66.4°$.

You should steer your boat approximately 66.4° east of due north in order to maintain a course taking you due north. Your speed due east is

$|\mathbf{R}| = s = 20\cos(90° - \theta) - 6 \approx 20\cos(90° - 66.4°) - 6 = 12.3$ knots.

Practice 9: The method of solution is the same as Example 7.

$\mathbf{W} = \langle 0, -15 \rangle$, $\mathbf{A} = \langle -|\mathbf{A}|\cos(35°), |\mathbf{A}|\sin(35°) \rangle$, $\mathbf{B} = \langle |\mathbf{B}|\cos(50°), |\mathbf{B}|\sin(50°) \rangle$, and

$\mathbf{0} = \mathbf{A} + \mathbf{B} + \mathbf{W} = \langle -|\mathbf{A}|\cos(35°) + |\mathbf{B}|\cos(50°) + 0, |\mathbf{A}|\sin(35°) + |\mathbf{B}|\sin(50°) - 15 \rangle$.

Imitating the algebraic steps of Example 7, we get

$$|\mathbf{A}| = \frac{15}{\sin(35°) + \cos(35°)\tan(50°)} \approx 9.68 \text{ pounds and}$$

$$|\mathbf{B}| = |\mathbf{A}|\frac{\cos(35°)}{\cos(50°)} \approx 12.33 \text{ pounds.}$$