11.4 DOT PRODUCT

In the previous sections we looked at the meaning of vectors in two and three dimensions, but the only operations we used were addition and subtraction of vectors and multiplication by a scalar. Some of the applications of 2–dimensional vectors used the angles that the vectors made with the coordinate axes and with each other, but, so far, in three dimensions we have not used angles. This section addresses both of those situations. It introduces a way to multiply two vectors, in two and three dimensions, called the dot product, and this dot product provides us with a relatively easy way to determine angles between vectors. Section 11.5 introduces a different method of multiplying two vectors, the cross product, in three dimensions that has other useful applications.

Since we will soon have three different types of multiplications for a vector (scalar, dot, and cross), it is important that you distinguish among them and call each multiplication operation by its full name.

Definition: Dot Product

Two dimensions: The **dot product** of $\mathbf{A} = \langle a_1, a_2 \rangle$ and $\mathbf{B} = \langle b_1, b_2 \rangle$ is $\mathbf{A} \bullet \mathbf{B} = a_1 b_1 + a_2 b_2$.

Three dimensions: The **dot product** of $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$ is $\mathbf{A} \bullet \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Both vectors in the dot product must have the same number of components, and the result of the dot product $\mathbf{U} \bullet \mathbf{V}$ is a scalar.

Example 1: For $\mathbf{A} = \langle 4, 1, 8 \rangle$ and $\mathbf{B} = \langle 2, -4, 4 \rangle$, calculate $\mathbf{A} \bullet \mathbf{B}$, $\mathbf{A} \bullet \mathbf{A}$, $\mathbf{B} \bullet \mathbf{B}$, and $(\mathbf{A}-\mathbf{B}) \bullet (\mathbf{A}+2\mathbf{B})$.

Solution: $\mathbf{A} \bullet \mathbf{B} = \langle 4, 1, 8 \rangle \bullet \langle 2, -4, 4 \rangle = (4)(2) + (1)(-4) + (8)(4) = 8 - 4 + 32 = 36$.

$\mathbf{A} \bullet \mathbf{A} = \langle 4, 1, 8 \rangle \bullet \langle 4, 1, 8 \rangle = (4)(4) + (1)(1) + (8)(8) = 81$.

$\mathbf{B} \bullet \mathbf{B} = \langle 2, -4, 4 \rangle \bullet \langle 2, -4, 4 \rangle = (2)(2) + (-4)(-4) + (4)(4) = 36$.

You should notice that $\mathbf{A} \bullet \mathbf{A} = |\mathbf{A}|^2$ and $\mathbf{B} \bullet \mathbf{B} = |\mathbf{B}|^2$.

Finally, $\mathbf{A}-\mathbf{B} = \langle 2, 5, 4 \rangle$ and $\mathbf{A} + 2\mathbf{B} = \langle 8, -7, 16 \rangle$

so $(\mathbf{A}-\mathbf{B}) \bullet (\mathbf{A}+2\mathbf{B}) = \langle 2, 5, 4 \rangle \bullet \langle 8, -7, 16 \rangle = (2)(8) + (5)(-7) + (4)(16) = 45$.

Practice 1: For $\mathbf{U} = \langle 2, 6, -3 \rangle$ and $\mathbf{V} = \langle -1, 2, 2 \rangle$, calculate $\mathbf{U} \bullet \mathbf{V}$, $\mathbf{U} \bullet \mathbf{U}$, $\mathbf{V} \bullet \mathbf{V}$, $\mathbf{U} \bullet (\mathbf{U}+\mathbf{V})$, and $\mathbf{U} \bullet \mathbf{U} + \mathbf{U} \bullet \mathbf{V}$. Does $\mathbf{U} \bullet \mathbf{U} = |\mathbf{U}|^2$? Does $\mathbf{U} \bullet \mathbf{V} = \mathbf{V} \bullet \mathbf{U}$?

As you might have noticed in Example 1 and Practice 1, the dot product seems to have some of the properties of ordinary multiplication of numbers.

Properties of the Dot Product:

(1) $\mathbf{A}\bullet\mathbf{A} = |\mathbf{A}|^2$

(2) $\mathbf{A}\bullet\mathbf{B} = \mathbf{B}\bullet\mathbf{A}$

(3) $k(\mathbf{A}\bullet\mathbf{B}) = (k\mathbf{A})\bullet B = \mathbf{A}\bullet(k\mathbf{B})$

(4) $\mathbf{A}\bullet(\mathbf{B} + \mathbf{C}) = \mathbf{A}\bullet\mathbf{B} + \mathbf{A}\bullet\mathbf{C}$

All of these properties can be proved using the definition of the dot product.

Proof of (1): If $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ then $\mathbf{A}\bullet\mathbf{A} = (a_1)^2 + (a_2)^2 + (a_3)^2$ and

$$|\mathbf{A}|^2 = \left(\sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}\,\right)^2 = (a_1)^2 + (a_2)^2 + (a_3)^2 \text{ so } \mathbf{A}\bullet\mathbf{A} = |\mathbf{A}|^2 .$$

Proof of (3): $k(\mathbf{A}\bullet\mathbf{B}) = k(a_1b_1 + a_2b_2 + a_3b_3) = ka_1b_1 + ka_2b_2 + ka_3b_3$.

$(k\mathbf{A})\bullet B = \langle ka_1, ka_2, ka_3 \rangle\bullet\langle b_1, b_2, b_3 \rangle = ka_1b_1 + ka_2b_2 + ka_3b_3$. And

$\mathbf{A}\bullet(k\mathbf{B}) = \langle a_1, a_2, a_3 \rangle\bullet\langle kb_1, kb_2, kb_3 \rangle = a_1(kb_1) + a_2(kb_2) + a_3(kb_3)$

$= ka_1b_1 + ka_2b_2 + ka_3b_3$ so $k(\mathbf{A}\bullet\mathbf{B}) = (k\mathbf{A})\bullet B = \mathbf{A}\bullet(k\mathbf{B})$

Practice 2: Prove Property (2) for 3–dimensional vectors.

The next result about dot products is very important, and much of the usefulness of dot products follows from it. It enables us to easily determine the angle between two vectors in two or three (or more) dimensions.

Angle Property of Dot Products

$\mathbf{A}\bullet\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos(\theta)$ where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$. Equivalently,

if $\mathbf{A}$ and $\mathbf{B}$ are nonzero vectors, then the angle θ between $\mathbf{A}$ and $\mathbf{B}$ satisfies $\cos(\theta) = \dfrac{\mathbf{A}\bullet\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|}$.

Proof of the Angle Property: The proof uses the Law of Cosines and several of the properties of the dot product.

The vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{A}-\mathbf{B}$ can be arranged to form a triangle (Fig. 1) with the angle θ between $\mathbf{A}$ and $\mathbf{B}$. Applying the Law of Cosines to this triangle, we have

$$|\mathbf{A} - \mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos(\theta) .$$

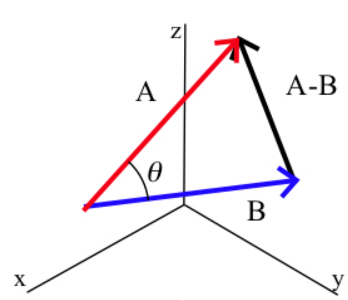

Fig. 1

We can also use the properties of the dot product to expand $| \mathbf{A} - \mathbf{B} |^2$ in a different way: $| \mathbf{A} - \mathbf{B} |^2 = (\mathbf{A} - \mathbf{B})\bullet(\mathbf{A} - \mathbf{B})$

$$= \mathbf{A}\bullet\mathbf{A} - \mathbf{A}\bullet\mathbf{B} - \mathbf{B}\bullet\mathbf{A} + \mathbf{B}\bullet\mathbf{B} = |\mathbf{A}|^2 - 2\mathbf{A}\bullet\mathbf{B} + |\mathbf{B}|^2 .$$

From these two representations for $| \mathbf{A} - \mathbf{B} |^2$ we have that

$$|\mathbf{A}|^2 - 2\mathbf{A}\bullet\mathbf{B} + |\mathbf{B}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos(\theta) \quad \text{so}$$

$$- 2\mathbf{A}\bullet\mathbf{B} = - 2|\mathbf{A}||\mathbf{B}| \cos(\theta) \quad \text{and} \quad \mathbf{A}\bullet\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos(\theta) .$$

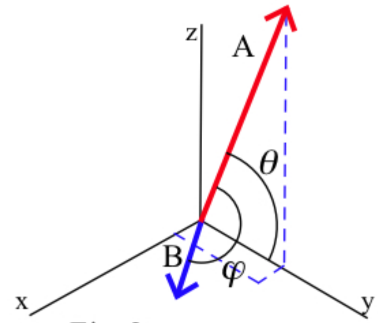

Fig. 2

Example 2: Let $\mathbf{A} = \langle 2, 5, 14 \rangle$ and $\mathbf{B} = \langle 2, 1, -2 \rangle$. Find the angles (a) between $\mathbf{A}$ and $\mathbf{B}$ and (b) between $\mathbf{A}$ and the positive y–axis (Fig. 2).

Solution: (a) $|\mathbf{A}| = \sqrt{4+25+196} = 15$, $|\mathbf{B}| = \sqrt{4+1+4} = 3$, $\mathbf{A}\bullet\mathbf{B} = 4+5-28 = -19$, and

$$\cos(\varphi) = \frac{\mathbf{A}\bullet\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|} = \frac{-19}{(15)(3)} \approx -0.4222 \quad \text{so} \quad \varphi \approx 2.01 \quad \text{or about } 115.0°.$$

(b) The basis vector $\mathbf{j} = \langle 0, 1, 0 \rangle$ points along the positive y–axis so the angle between $\mathbf{A}$ and the positive y–axis is the same as the angle between A and $\mathbf{j}$.

$|\mathbf{A}| = 15$, $|\mathbf{j}| = 1$, and $\mathbf{A}\bullet\mathbf{j} = (2)(0) + (5)(1) + (14)(0) = 5$ so

$$\cos(\theta) = \frac{\mathbf{A}\bullet\mathbf{j}}{|\mathbf{A}|\,|\mathbf{j}|} = \frac{5}{(15)(1)} \approx 0.333 \quad \text{so} \quad \theta \approx 1.23 \quad \text{or about } 70.5°.$$

Practice 3: Let $\mathbf{A} = \langle 2, -6, 3 \rangle$, $\mathbf{B} = \langle 4, 8, -1 \rangle$ and $\mathbf{C} = \langle 3, 0, -4 \rangle$ and determine the angles between the vectors (a) $\mathbf{A}$ and $\mathbf{B}$, (b) $\mathbf{A}$ and C, (c) $\mathbf{B}$ and the negative x–axis, and (d) $\mathbf{C}$ and the positive y–axis.

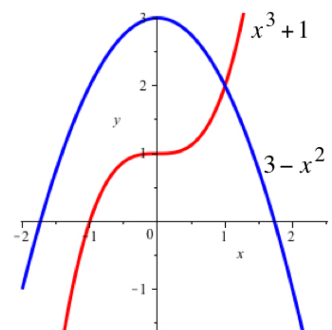

Fig. 3

Example 3: Find the angle of intersection of the graphs of $f(x) = x^3 + 1$ and $g(x) = 3 - x^2$ at the point (1, 2). (The angle of intersection is the angle between tangent vectors to the graphs at the point.) Fig. 3.

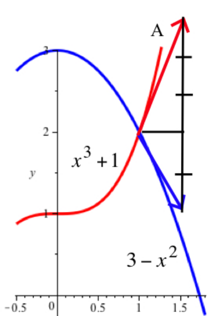

Fig. 4

Solution: $f '(x) = 3x^2$, so the slope of f at (1,2) is $f '(1) = 3(1)^2 = 3$: $\frac{\text{rise}}{\text{run}} = \frac{3}{1}$ and a vector, $\mathbf{A} = 1\mathbf{i} + 3\mathbf{j}$,with this slope is shown in Fig. 4. Similarly, $g '(x) = -2x$, so the slope of g at (1,2) is $g '(1) = -2(1) = -2$: $\frac{\text{rise}}{\text{run}} = \frac{-2}{1}$ and vector $\mathbf{B} = 1\mathbf{i} - 2\mathbf{j}$ has the same slope. Then $|\mathbf{A}| = \sqrt{10}$, $|\mathbf{B}| = \sqrt{5}$, and $\mathbf{A}\bullet\mathbf{B} = -5$, so

$$\cos(\theta) = \frac{-5}{\sqrt{10}\sqrt{5}} \approx -0.707 . \quad \text{Then } \theta = \arccos(-0.707) \approx 2.356 \ (\text{or } 135^{o}).$$

Note: If $y = f(x)$, then a tangent vector to the graph of f at the point $(x_0, f(x_0))$ is $\mathbf{T} = \langle 1, f '(x_0) \rangle$.

If a curve is given parametrically by $x(t)$ and $y(t)$,

then a tangent vector to the curve when $t = t_o$ is $\mathbf{T} = x '(t_o)\mathbf{i} + y '(t_o)\mathbf{j} = \langle x '(t_o), y '(t_o) \rangle$.

Some books define the dot product $\mathbf{A}\bullet\mathbf{B}$ as $\mathbf{A}\bullet\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos(\theta)$ and then derive the definition we gave, $\mathbf{A}\bullet\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$, as a property. Either way, the pattern $\mathbf{A}\bullet\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$ is typically used to compute the dot product, and the angle property $\mathbf{A}\bullet\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos(\theta)$ is typically used to help us see what the dot product measures and to help us derive and simplify some vector algorithms.

Criteria for A and B to be Perpendicular

Let **A** and **B** be nonzero vectors. **A** and **B** are perpendicular if and only if $\mathbf{A}\bullet\mathbf{B} = 0$.

Proof: If **A** and **B** are perpendicular, then $\theta = \pm\pi/2$ so $\mathbf{A}\bullet\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos(\theta) = |\mathbf{A}|\,|\mathbf{B}|\,(0) = 0$.
If $0 = \mathbf{A}\bullet\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos(\theta)$, then $\cos(\theta) = 0$ so $\theta = \pm\pi/2\ (\pm 2\pi n)$ and **A** and **B** are perpendicular.

Example 4: (a) Find a vector perpendicular to $\mathbf{V} = \langle 1, 2, 3 \rangle$.
(b) Find a vector perpendicular to the line $4x + 3y = 24$.

Solution: (a) We need a nonzero vector **N** so that $\mathbf{N}\bullet\mathbf{V} = 0$. If $\mathbf{N} = \langle a, b, c \rangle$, then we want to find values for a, b, and c so that $a + 2b + 3c = 0$, and there are lots of values for a, b, and c that work: $\mathbf{N} = \langle 0, -3, 2 \rangle, \langle 3, 0, -1 \rangle, \langle 2, -1, 0 \rangle, \langle 1, 1, -1 \rangle, \langle 1, -2, 1 \rangle$ and lots of others all give $\mathbf{N}\bullet\mathbf{V} = 0$. Fig. 5 illustrates why a single vector in three dimensions can have perpendicular vectors that point in an infinite number of different directions.

(b) The points $P = (6, 0)$ and $Q = (0, 8)$ both lie on the line, so the vector **V** from P to Q is parallel to the line, and $\mathbf{V} = \langle 0-6, 8-0 \rangle = \langle -6, 8 \rangle$. If **N** is perpendicular to **V**, then **N** is perpendicular to the line, and $\mathbf{N} = \langle 4, 3 \rangle$ is perpendicular to **V** since $\mathbf{N}\bullet\mathbf{V} = 0$. The vector $\mathbf{N} = \langle 4, 3 \rangle$ is perpendicular to the line $4x + 3y = 24$. Every scalar multiple $k\mathbf{N}$ with $k \neq 0$ is also perpendicular to the line.

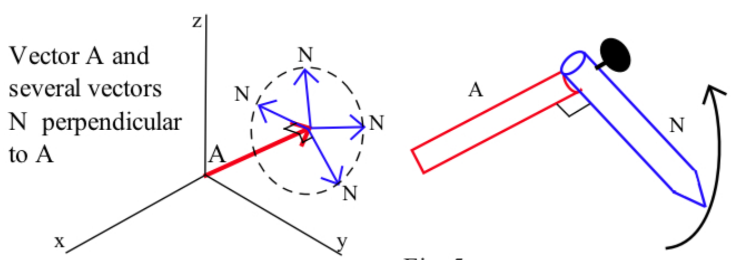

Fig. 5

Practice 4: Find a vector **N** perpendicular to the line $-5x + 2y = 30$. Is $\mathbf{N} = \langle -5, 2 \rangle$ perpendicular to $-5x + 2y = 30$?

You might have noticed a pattern in the vectors perpendicular to the lines in Example 4 and Practice 4. The vector $\langle 4, 3 \rangle$ is perpendicular to the line $4x + 3y = 24$, and the vector $\langle -5, 2 \rangle$ is perpendicular to the line $-5x + 2y = 30$. The next result says this pattern is not an accident.

Finding a Vector Perpendicular to a Line

The vector $\mathbf{N} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by = c$.

Problem 66 asks you to prove this result.

Projection of a Vector onto a Vector

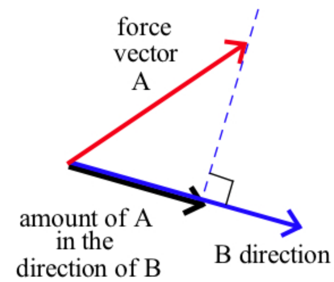

Fig. 6

The length of a force vector tells us the amount of force in the direction of the vector, but sometimes we want to know the size of the force in another direction (Fig. 6). One of the examples in two dimensions (Fig. 7) involved finding the amount of "horizontal force" obtained when we pulled on a box at an angle to the horizontal. Similar questions can also be asked if one of the directions is not horizontal (Fig. 8) and in three or more dimensions (Fig. 9). Since we now have a method for determining the angle between two vectors in three dimensions, the solutions are relatively straightforward. The vector representing the amount of a vector **A** in the direction of a vector **B** is called the "projection of **A** onto **B**" and is denoted as $\mathbf{Proj_B\ A}$.

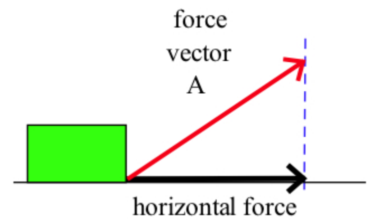

Fig. 7

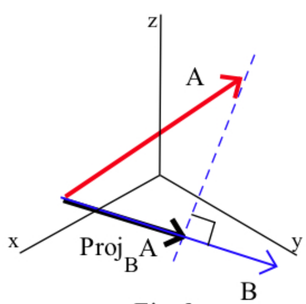

Fig. 9

Visualizing the projection of A onto B : Fig. 10 shows several geometric examples of the projection of a vector **A** onto a vector **B**. We arrange **A** and **B** to have the same starting point, draw a (dotted) line through the head of **A** and perpendicular to **B**, and form the projection of **A** onto **B** as the vector from the starting point of **B** to the point where the dotted line intersects **B** (or an extension of **B**). The projection of **A** onto **B** is a vector along the line of **B** — the direction of the projection of **A** onto **B** is either the same direction as **B**, $\mathbf{B}/|\mathbf{B}|$, or opposite the direction of **B**, $-\mathbf{B}/|\mathbf{B}|$.

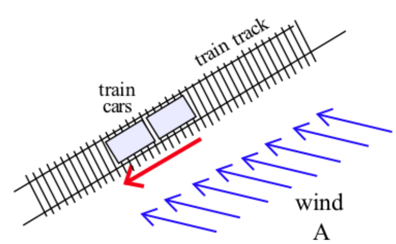

Fig. 8

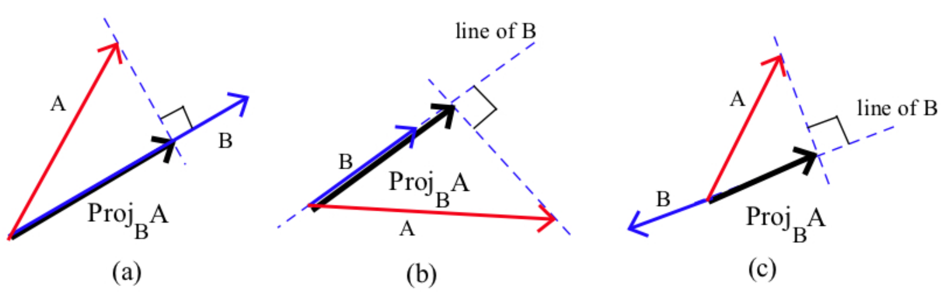

Fig. 10

Fig. 11 shows some examples of the projection of **B** onto **A**, $\mathbf{Proj_A\ B}$.

The resulting projection vector is always along the line of **A** .

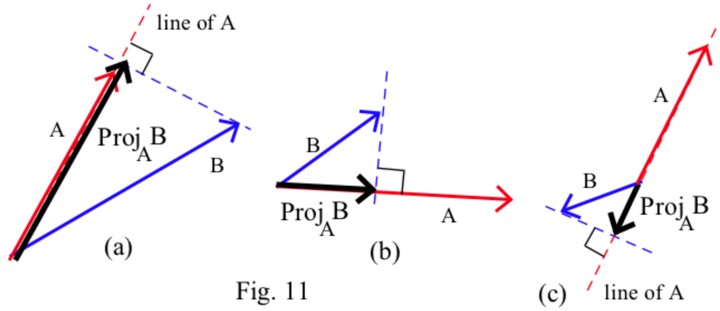

Fig. 11

Once we understand the geometric meaning of "the projection of **A** onto **B**," trigonometry enables us to determine the vector projection of **A** onto **B** : $\mathbf{Proj_B\ A}$ (Fig. 12), and its magnitude, $|\ \mathbf{Proj_B\ A}\ |$, called the scalar projection.

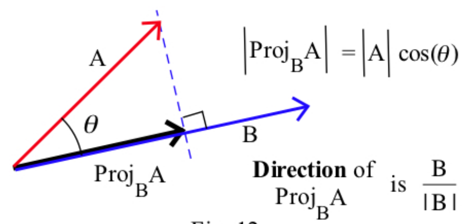

Fig. 12

Definitions: Vector Projection, Scalar Projection

The **vector projection of A onto B** is the

(magnitude of the projection) times (the direction of **B**): $\mathbf{Proj_B\ A} = \left(|\mathbf{A}| \cos(\theta) \right) \left(\frac{\mathbf{B}}{|\mathbf{B}|} \right)$

The **scalar projection of A onto B** is $|\mathbf{A}| \cos(\theta)$ where θ is the angle between **A** and **B** .

Note: "Projection of **A** onto **B**" usually means Vector Projection.

We can use properties of the dot product to simplify the calculation of projections.

Since θ is the angle between **A** and B, then $\cos(\theta) = \frac{\mathbf{A \bullet B}}{|\mathbf{A}||\mathbf{B}|}$ so the scalar projection of **A** onto **B** is $|\mathbf{A}| \cos(\theta) = |\mathbf{A}| \frac{\mathbf{A \bullet B}}{|\mathbf{A}||\mathbf{B}|} = \frac{\mathbf{A \bullet B}}{|\mathbf{B}|}$. Putting this result into the definition of the vector projection of **A** onto **B**, we get

$$\left(|\mathbf{A}| \cos(\theta) \right) \left(\frac{\mathbf{B}}{|\mathbf{B}|} \right) = \frac{\mathbf{A \bullet B}}{|\mathbf{B}|} \left(\frac{\mathbf{B}}{|\mathbf{B}|} \right) = \frac{\mathbf{A \bullet B}}{|\mathbf{B}|^2} \mathbf{B} .$$

Calculating Scalar and Vector Projections

Vector projection of **A** onto **B** is $\mathbf{Proj_B\ A} = \left(\frac{\mathbf{A \bullet B}}{|\mathbf{B}|^2} \right) \mathbf{B}$ (a vector).

Scalar projection of **A** onto **B** = $\frac{\mathbf{A \bullet B}}{|\mathbf{B}|}$ (a scalar) = magnitude of $\mathbf{Proj_B\ A}$.

Example 5: For $\mathbf{A} = \langle 6, -2, 3 \rangle$ and $\mathbf{B} = \langle 4, 8, -1 \rangle$, calculate scalar and vector projections of

(a) $\mathbf{A}$ onto $\mathbf{B}$, (b) $\mathbf{B}$ onto $\mathbf{A}$, (c) $\mathbf{A}$ onto the positive x–axis,

(d) $\mathbf{A}$ onto the positive y–axis, and (e) $\mathbf{A}$ onto the positive z–axis.

Solution: $|\mathbf{A}| = 7$, $|\mathbf{B}| = 9$, and $\mathbf{A}\bullet\mathbf{B} = 24 - 16 - 3 = 5$.

(a) The scalar projection of **A** onto **B** is $\dfrac{\mathbf{A}\bullet\mathbf{B}}{|\mathbf{B}|} = \dfrac{5}{9}$.

The vector projection of **A** onto **B** is $\mathbf{Proj_B\, A} = \left(\dfrac{\mathbf{A}\bullet\mathbf{B}}{|\mathbf{B}|^2}\right)\mathbf{B} = \left(\dfrac{5}{81}\right)\langle 4, 8, -1 \rangle = \langle \dfrac{20}{81}, \dfrac{40}{81}, \dfrac{-5}{81} \rangle$.

(b) Scalar projection of **B** onto **A** is $\dfrac{\mathbf{A}\bullet\mathbf{B}}{|\mathbf{A}|} = \dfrac{5}{7}$. Vector projection is $\left(\dfrac{\mathbf{A}\bullet\mathbf{B}}{|\mathbf{A}|^2}\right)\mathbf{A} = \langle \dfrac{30}{49}, \dfrac{-10}{49}, \dfrac{15}{49} \rangle$.

(c) $\mathbf{i}$ has the same direction as the positive x–axis. Scalar projection of **A** onto $\mathbf{i}$ is $\dfrac{\mathbf{A}\bullet\mathbf{i}}{|\mathbf{i}|} = 6$.

Vector projection of **A** onto $\mathbf{i}$ is $\mathbf{Proj_i\, A} = \left(\dfrac{\mathbf{A}\bullet\mathbf{i}}{|\mathbf{i}|^2}\right)\mathbf{i} = \langle 6, 0, 0 \rangle$.

(d) and (e) The scalar projections of $\mathbf{A}$ onto $\mathbf{j}$ and $\mathbf{k}$ are -2 and 3, respectively.

The vector projections of $\mathbf{A}$ onto $\mathbf{j}$ and $\mathbf{k}$ are $\langle 0, -2, 0 \rangle$ and $\langle 0, 0, 3 \rangle$, respectively.

Practice 5: For $\mathbf{U} = \langle 9, -2, 6 \rangle$ and $\mathbf{V} = \langle 1, 2, -2 \rangle$, calculate the scalar and vector projections of

(a) $\mathbf{U}$ onto $\mathbf{V}$, (b) $\mathbf{V}$ onto $\mathbf{U} + \mathbf{V}$, and (c) $\mathbf{V}$ onto the positive y–axis.

Applications

Projections are useful in a number of situations. The two examples given here illustrate only two of a variety of those uses. In the first example below, we use the geometric meaning of projection to derive a formula for the distance from a point to a line. In the second example, we illustrate how projections can be used to calculate work.

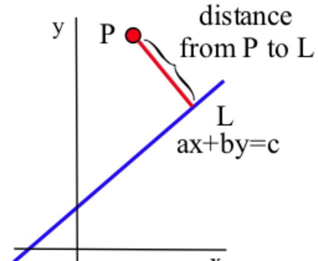

Fig. 13

Example 6: Find the distance from the point $P = (p, q)$ to a line $ax + by = c$. (Fig. 13)

Solution: Using vectors and projections, finding this distance can be broken into several simple steps (Fig. 14). The algorithm:

(1) Find a vector $\mathbf{N}$ perpendicular to the line: $\mathbf{N} = \langle a, b \rangle$.(Fig. 14a)

(2) Find a point Q on the line and call it (x_0, y_0) . (Fig. 14b)

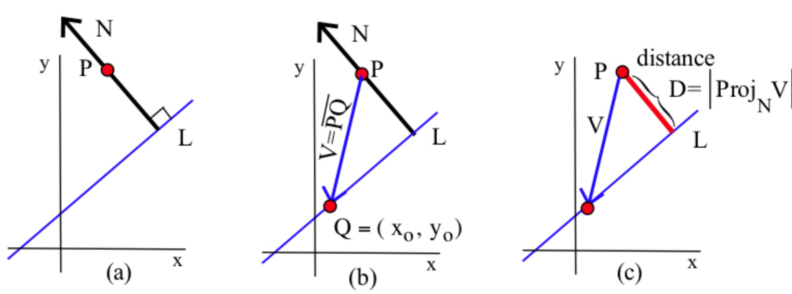

Fig. 14

(3) Form the vector **V** from P to Q: $\mathbf{V} = \langle p - x_0, q - y_0 \rangle$.

(4) Find the absolute value of the scalar projection of **V** onto **N**: $\left| \frac{\mathbf{V} \cdot \mathbf{N}}{|\mathbf{N}|} \right|$ (Fig. 14c)

The formula: The distance from P to the line is this absolute value of the scalar projection:

$$\text{distance} = \left| \frac{\mathbf{V} \cdot \mathbf{N}}{|\mathbf{N}|} \right| = \left| \frac{a(p - x_0) + b(q - y_0)}{\sqrt{a^2 + b^2}} \right|$$

$$= \frac{| ap + bq - (ax_0 + by_0) |}{\sqrt{a^2 + b^2}} = \frac{| ap + bq - c |}{\sqrt{a^2 + b^2}} \quad \text{since } ax_0 + by_0 = c.$$

Practice 6: Step through the algorithm in Example 6 to find the distance from the point $P = (2, 7)$ to the line $3x - 4y = 20$.

The projection of a force vector onto a vector with a different direction tells us the amount of the force in that other direction, a useful result to know for solving work problems.

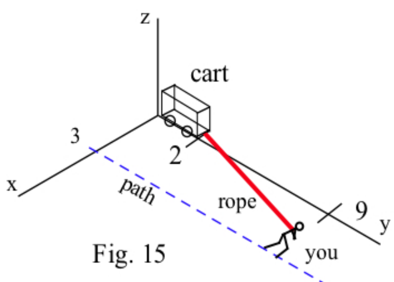

Fig. 15

Example 7: A cart moves on a track located on the y–axis, and you are pulling on a rope with a force of 70 pounds. Find the amount of work that you do in moving the cart from the point $P = (0, 2, 0)$ to the point $Q = (0, 9, 0)$ if you have the rope over your shoulder 4 feet above the ground and walk 12 feet in front of the cart along a path 3 feet to the side of the y–axis (Fig. 15).

Solution: Work = (distance moved)(force in the direction of the movement)

First, we can convert the given information into a vector form: the force vector has a magnitude of 70 pounds in the direction $\left\langle \frac{3}{13}, \frac{12}{13}, \frac{4}{13} \right\rangle$ so the force vector is $\mathbf{F} = \left\langle \frac{210}{13}, \frac{840}{13}, \frac{280}{13} \right\rangle$. The movement of the cart is in the direction of the vector from P to Q, $\mathbf{D} = Q - P = \langle 0, 7, 0 \rangle$, so the amount of work is

work = (distance moved)(magnitude of the force in the direction of the movement)

= (length of PQ)(scalar projection of F onto direction of PQ)

$$= (|\mathbf{D}|)\left(\frac{\mathbf{F} \cdot \mathbf{D}}{|\mathbf{D}|}\right) = \mathbf{F} \cdot \mathbf{D} = \left\langle \frac{210}{13}, \frac{840}{13}, \frac{280}{13} \right\rangle \cdot \langle 0, 7, 0 \rangle = \frac{5880}{13} \approx 452.3 \text{ foot–pounds.}$$

Work

If a constant force vector **F** moves an object from a point P to a point Q,

then the amount of work done is Work = **F•D** where **D** is the displacement vector from P to Q.

Practice 7: In Example 7, determine the amount of work done if you walk 6 feet to the side of the y–axis. (All of the other distances are the same as in Example 7.)

In Chapter 13, we discuss how to determine the amount of work done if the force is variable or if the object moves along a curved path.

Beyond Three Dimensions

If **A** and **B** have the same number of components, then we can define their dot product, the angle between them, and the projection of one onto the other with the same patterns as for two and three dimensions.

Definitions:

If $\mathbf{A} = \langle a_1, a_2, \dots, a_n \rangle$ and $\mathbf{B} = \langle b_1, b_2, \dots, b_n \rangle$ are nonzero vectors in n–dimensional space,

then $\mathbf{A}\bullet\mathbf{B} = a_1b_1 + a_2b_2 + \dots + a_nb_n$,

the angle θ between **A** and **B** satisfies $\cos(\theta) = \dfrac{\mathbf{A}\bullet\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$, and

the vector projection of A onto B is $\mathbf{Proj_B\,A} = \left(\dfrac{\mathbf{A}\bullet\mathbf{B}}{|\mathbf{B}|^2}\right)\mathbf{B}$.

Now, even though we may not visualize 4 or 5–dimensional vectors, we can calculate the dot product and the angle between two vectors.

Practice 8: The psychological profiles for you and a friend were $\mathbf{Y} = \langle 5, 1, -7, 5 \rangle$ and $\mathbf{F} = \langle 6, 10, 8, 5 \rangle$ for the four personality categories measured by the profile. Should you say you and your friend are "very alike" ($\theta < 30°$), "somewhat alike" ($30° \le \theta < 60°$), "different" ($60° \le \theta \le 120°$), "somewhat opposite" ($120° < \theta \le 150°$). or "very opposite" ($150° < \theta \le 180°$)? What about you and and another friend with the profile $\mathbf{A} = \langle 10, -4, -10, 3 \rangle$?

This section has involved very little "calculus," but the ideas of dot products and projections of vectors are very powerful and useful, and they will be used often as we develop "vector calculus" in later chapters.

PROBLEMS

1. $\mathbf{A} = \langle 1, 2, 3 \rangle$, $\mathbf{B} = \langle -2, 4, -1 \rangle$. Calculate $\mathbf{A}\bullet\mathbf{B}$, $\mathbf{B}\bullet\mathbf{A}$, $\mathbf{A}\bullet\mathbf{A}$, $\mathbf{A}\bullet(\mathbf{B}+\mathbf{A})$, and $(2\mathbf{A}+3\mathbf{B})\bullet(\mathbf{A}-2\mathbf{B})$.

2. $\mathbf{A} = \langle 6, -1, 2 \rangle$, $\mathbf{B} = \langle 2, 4, -3 \rangle$. Calculate $\mathbf{A}\bullet\mathbf{B}$, $\mathbf{B}\bullet\mathbf{A}$, $\mathbf{A}\bullet\mathbf{A}$, $\mathbf{A}\bullet(\mathbf{B}+\mathbf{A})$, and $(2\mathbf{A}+3\mathbf{B})\bullet(\mathbf{A}-2\mathbf{B})$.

3. $\mathbf{U} = \langle 6, -1, 2 \rangle$, $\mathbf{V} = \langle 2, 4, -3 \rangle$. Calculate $\mathbf{U \cdot V}$, $\mathbf{U \cdot U}$, $\mathbf{U \cdot i}$, $\mathbf{U \cdot j}$, $\mathbf{U \cdot k}$, and $(\mathbf{V} + \mathbf{i}) \cdot \mathbf{U}$.
4. $\mathbf{U} = \langle -3, 3, 2 \rangle$, $\mathbf{V} = \langle 2, 4, -3 \rangle$. Calculate $\mathbf{U \cdot V}$, $\mathbf{U \cdot U}$, $\mathbf{V \cdot i}$, $\mathbf{V \cdot j}$, $\mathbf{V \cdot k}$, $(\mathbf{V} + \mathbf{k}) \cdot \mathbf{U}$.

5. $\mathbf{S} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{T} = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, $\mathbf{U} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Calculate $\mathbf{S \cdot T}$, $\mathbf{T \cdot U}$, $\mathbf{T \cdot T}$, $(\mathbf{S} + \mathbf{T}) \cdot (\mathbf{S} - \mathbf{T})$, and $(\mathbf{S \cdot T})\, \mathbf{U}$.

6. $\mathbf{S} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{T} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{U} = 2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. Calculate $\mathbf{S \cdot T}$, $\mathbf{S \cdot U}$, $\mathbf{S \cdot S}$, $(\mathbf{T} + \mathbf{U}) \cdot (\mathbf{T} - \mathbf{U})$, and $\mathbf{S}(\mathbf{T \cdot U})$.

In problems 7 – 18, calculate the angle between the given vectors. Also calculate the angles between the first vector and each of the coordinate axes.

7. $\mathbf{A} = \langle 1, 2, 3 \rangle$, $\mathbf{B} = \langle -2, 4, -1 \rangle$.
8. $\mathbf{A} = \langle 6, -1, 2 \rangle$, $\mathbf{B} = \langle 2, 4, -3 \rangle$.
9. $\mathbf{U} = \langle 6, -1, 2 \rangle$, $\mathbf{V} = \langle 2, 4, -3 \rangle$.
10. $\mathbf{U} = \langle -3, 3, 2 \rangle$, $\mathbf{V} = \langle 2, 4, -3 \rangle$.
11. $\mathbf{S} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{T} = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$.
12. $\mathbf{S} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{T} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.
13. $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{B} = -5\mathbf{i} + 0\mathbf{j} + 2\mathbf{k}$.
14. $\mathbf{A} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{B} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.
15. $\mathbf{A} = \langle 5, -2, 0 \rangle$, $\mathbf{B} = \langle -3, 4, 0 \rangle$.
16. $\mathbf{A} = \langle 5, 0, 0 \rangle$, $\mathbf{B} = \langle 0, 4, -3 \rangle$.
17. $\mathbf{U} = \langle 1, 0, 3 \rangle$, $\mathbf{V} = \langle -2, 0, 1 \rangle$.
18. $\mathbf{U} = \langle 0, 1, 2 \rangle$, $\mathbf{V} = \langle 2, 4, 0 \rangle$.

In problems 19 – 28, determine the angle of intersection of the graphs of the given functions at the given point (i.e., determine the angle between the vectors tangent to the functions at the given point).

19. $f(x) = x^2 + 3x - 2$, $g(x) = 3x - 1$ at $(1, 2)$
20. $f(x) = x^2 + 3x - 2$, $g(x) = 3 - x^2$ at $(1, 2)$
21. $f(x) = e^x$, $g(x) = \cos(x)$ at $(0, 1)$
22. $f(x) = \sin(x)$, $g(x) = \cos(x)$ at $(\pi/4, \frac{\sqrt{2}}{2})$
23. $f(x) = 1 + \arctan(3x)$, $g(x) = \ln(e + x)$ at $(0, 1)$
24. $f(x) = \sin(x^2)$, $g(x) = 1 - \cos(x^2)$ at $(0, 0)$
25. f: $x = 3t$, $y = t^2 + t - 1$; g: $x = 2t + 1$, $y = 2t - 1$ at the point when $t = 1$
26. f: $x = \sin(t)$, $y = \cos(t)$; g: $x = t^3$, $y = 2t + 1$ at the point when $t = 0$
27. f: $x = e^t + 3$, $y = 2 + \cos(5t)$; g: $x = t^2 + 3t + 4$, $y = 3 + \ln(1 - t)$ at the point when $t = 0$
28. f: $x = 2 + \cos(5t)$, $y = e^t + 3$; g: $x = 3 + \ln(1 - t)$, $y = t^2 + 3t + 4$ at the point when $t = 0$

In problems 29 – 48, find a vector $\mathbf{N}$ that is perpendicular to the given vector or line. (Typically there are several correct answers.)

29. $\mathbf{A} = \langle 1, -2, 0 \rangle$
30. $\mathbf{B} = \langle -5, 0, 3 \rangle$
31. $\mathbf{C} = \langle 7, 3 \rangle$
32. $\mathbf{D} = \langle 7, -3 \rangle$
33. $\mathbf{E} = \langle 2, -1, 3 \rangle$
34. $\mathbf{S} = \langle 1, 2, 5 \rangle$

35. $\mathbf{T} = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ 36. $\mathbf{U} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ 37. $\mathbf{V} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

38. $\mathbf{W} = -6\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ 39. $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j}$ 40. $\mathbf{B} = 3\mathbf{i} + 2\mathbf{j}$

41. $\mathbf{C} = 3\mathbf{i} - 2\mathbf{j}$ 42. $x + y = 6$ 43. $3x + 2y = 6$

44. $5x - 3y = 30$ 45. $x - 4y = 8$ 46. $5x + y = 10$

47. $y = 3$ 48. $x = 2$

In problems 49 – 52 sketch $\mathbf{Proj_B\, A}$.

49. **A** and **B** in Fig. 16. 50. **A** and **B** in Fig. 17. 51. **A** and **B** in Fig. 18. 52. **A** and **B** in Fig. 19.

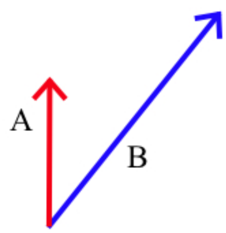

Fig. 16

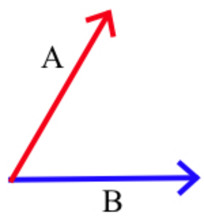

Fig. 17

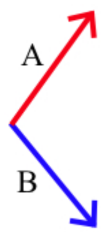

Fig. 18

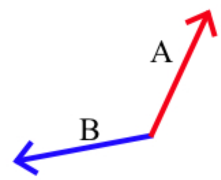

Fig. 19

In problems 53 – 56 sketch $\mathbf{Proj_A\, B}$.

53. **A** and **B** in Fig. 16. 54. **A** and **B** in Fig. 17. 55. **A** and **B** in Fig. 18. 56. **A** and **B** in Fig. 19.

In problems 57 – 63, calculate $\mathbf{Proj_B\, A}$ and $\mathbf{Proj_A\, B}$.

57. $\mathbf{A} = \langle 1, -2, 0 \rangle$, $\mathbf{B} = \langle -5, 0, 3 \rangle$ 58. $\mathbf{A} = \langle 1, 2, 3 \rangle$, $\mathbf{B} = \langle -2, 4, -1 \rangle$

59. $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{B} = -5\mathbf{i} + 0\mathbf{j} + 2\mathbf{k}$ 60. $\mathbf{A} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{B} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

61. $\mathbf{A} = \langle 5, 0, 0 \rangle$, $\mathbf{B} = \langle 0, 4, -3 \rangle$ 62. $\mathbf{A} = \langle 2, -1, 3 \rangle$, $\mathbf{B} = \langle 1, 2, 5 \rangle$

63. $\mathbf{A} = \langle 1, -2, 3 \rangle$, $\mathbf{B} = \mathbf{j}$

64. Suppose **A** and **B** have the same length. Which has the larger magnitude: $\mathbf{Proj_B\, A}$ or $\mathbf{Proj_A\, B}$? Justify your answer.

65. Suppose $|\mathbf{A}| = 3\,|\mathbf{B}|$. Which has the larger magnitude: $\mathbf{Proj_B\, A}$ or $\mathbf{Proj_A\, B}$? Justify your answer.

66. Prove that the vector $\mathbf{N} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by = c$.
(Suggestion: Pick any two points $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ on the line. Then the vector **V** with starting point P and ending point Q has the same direction as the line, and $\mathbf{V} = \langle x_1 - x_0, y_1 - y0 \rangle$. Now show that **N** is perpendicular to **V** .)

In problems 67 – 72, calculate the distance from the given point to the given line using (a) the algorithm of Example 6, and (b) the formula found in Example 6.

67. (1, 3), y = 7 – x
68. (–2, 1), 3x – 2y = 6
69. (5, 3), 3y = 3x + 7
70. (–3, –4), y = 3x + 2
71. (0, 0), 4x + 3y = 7
72. (0, 0), ax + by = c

73. Fig. 20 shows the position of a road and a house. How close is the road to the house (minimum distance)?

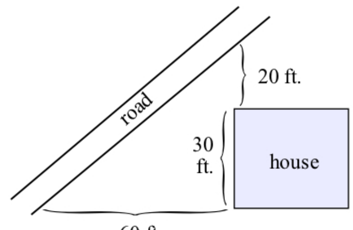

Fig. 20

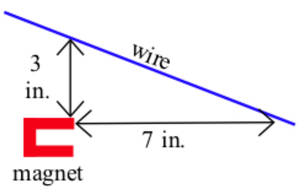

Fig. 21

74. In Fig. 21, how close is the wire to the magnet?

75. A person is standing directly below the electrical transmission wires in Fig. 22. Assuming the wires are so taut that they follow a straight line, how close do they come to the person's head?

76. The **direction cosines** of a vector $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ are the cosines of the angles $\mathbf{A}$ makes with each of the coordinate vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$: if A makes angles θ_x, θ_y, and θ_z with the x, y, and z axes, respectively, then the direction cosines of $\mathbf{A}$ are $\cos(\theta_x)$, $\cos(\theta_y)$, and $\cos(\theta_z)$.
Show that $\cos(\theta_x) = \dfrac{a_1}{|\mathbf{A}|}$, $\cos(\theta_y) = \dfrac{a_2}{|\mathbf{A}|}$, $\cos(\theta_z) = \dfrac{a_3}{|\mathbf{A}|}$,
and that $\cos^2(\theta_x) + \cos^2(\theta_y) + \cos^2(\theta_z) = 1$ for every nonzero vector $\mathbf{A}$.

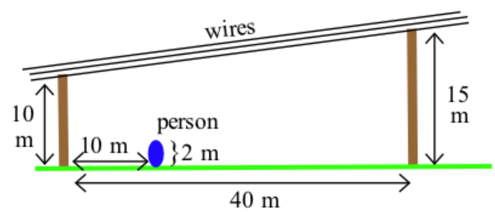

Fig. 22

77. A car moves on a track located on the y–axis, and you are pulling on a rope with a force of 50 pounds. Find the amount of work that you do in moving the cart from the origin to the point Q = (0, 10, 0) if you have the rope over your shoulder 4 feet above the ground and walk 10 feet in front of the cart along a path 5 feet to the side of the y–axis.

78. Redo Problem 77 assuming that you are now pulling on the rope with a force of 100 pounds.

79. A wind blowing parallel to the y–axis exerts a force of 8 pounds on a kite. How much work does the wind do in moving the kite in a straight line from the point (20, 30, 40) to the point (50, 90, 150).

80. Redo problem 79 assuming that the wind is blowing parallel to the x–axis.

81. How much work does the person in Fig. 23 do moving the box 20 feet along the ground?

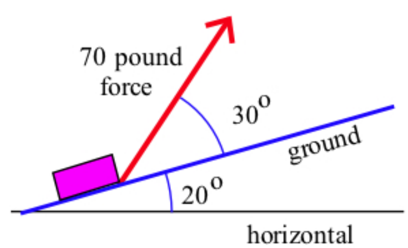

Fig. 23

Beyond Three Dimensions

83. $\mathbf{A} = \langle 1, 2, 3, 4 \rangle$, $\mathbf{B} = \langle -2, 4, -1, 3 \rangle$. Calculate $\mathbf{A} \bullet \mathbf{B}$, $\mathbf{B} \bullet \mathbf{A}$, and the angle between $\mathbf{A}$ and $\mathbf{B}$.

84. $\mathbf{A} = \langle -3, 4, 5, -1 \rangle$, $\mathbf{B} = \langle -2, 4, -2, -4 \rangle$. Calculate $\mathbf{A} \bullet \mathbf{B}$, $\mathbf{B} \bullet \mathbf{A}$, and the angle between $\mathbf{A}$ and $\mathbf{B}$.

In problems 85 – 88, a particular personality profile assigns a number between –1 and + 1 to each person on each of five personality characteristics. Using the categories ("very alike," "somewhat alike," , etc.) of Practice 8, determine the correct category for the pair of personality profiles given in each problem.

85. $\mathbf{A} = \langle 1, -0.2, 0.3, -0.6, 0.4 \rangle$, $\mathbf{B} = \langle -0.5, 0.3, 0.3, -0.1, 0.2 \rangle$

86. $\mathbf{A} = \langle 0.2, -0.3, -0.3, -0.7, 0.6 \rangle$, $\mathbf{B} = \langle 0.4, -0.2, -0.4, -0.5, 0.8 \rangle$

87. $\mathbf{A} = \langle 0.1, 0.2, 0.3, 0.4, 0.5 \rangle$, $\mathbf{B} = \langle 0.3, 0.4, 0.5, 0.6, 0.7 \rangle$

88. $\mathbf{A} = \langle 0.8, 0.3, -0.5, 0.2, 0.6 \rangle$, $\mathbf{B} = \langle -0.2, -0.4, -0.6, -0.4, 0.1 \rangle$

89. Prove the Parallelogram Law for vectors: $| \mathbf{A} + \mathbf{B} |^2 + | \mathbf{A} - \mathbf{B} |^2 \quad 2| \mathbf{A} |^2 + 2| \mathbf{B} |^2$

Practice Answers

Practice 1: $\mathbf{U} \bullet \mathbf{V} = (2)(-1) + (6)(2) + (-3)(2) = 4$. $\mathbf{U} \bullet \mathbf{U} = (2)(2) + (6)(6) + (-3)(-3) = 49$.

$\mathbf{V} \bullet \mathbf{V} = (-1)(-1) + (2)(2) + (2)(2) = 9$.

$\mathbf{U} \bullet (\mathbf{U}+\mathbf{V}) = \langle 2, 6, -3 \rangle \langle 1, 8, -1 \rangle = 53$. $\mathbf{U} \bullet \mathbf{U} + \mathbf{U} \bullet \mathbf{V} = 49 + 4 = 53$.

$|\mathbf{U}|^2 = (\sqrt{2^2 + 6^2 + (-3)^2})2 = 2^2 + 6^2 + (-3)^2 = 49 = \mathbf{U} \bullet \mathbf{U}$.

$\mathbf{V} \bullet \mathbf{U} = (-1)(2) + (2)(6) + (2)(-3) = 4$ so $\mathbf{U} \bullet \mathbf{V} = \mathbf{V} \bullet \mathbf{U}$.

Practice 2: $\mathbf{A} \bullet \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = b_1 a_1 + b_2 a_2 + b_3 a_3 = \mathbf{B} \bullet \mathbf{A}$.

Practice 3: $|\mathbf{A}| = \sqrt{4+36+9} = 7$, $|\mathbf{B}| = \sqrt{16+64+1} = 9$, and $|\mathbf{C}| = \sqrt{9+0+16} = 5$.

(a) $\cos(\theta) = \dfrac{\mathbf{A} \bullet \mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|} = \dfrac{8-48-3}{(7)(9)} = \dfrac{-43}{63} \approx -0.683$ so $\theta \approx 2.32$ (radians) or $133.04°$.

(b) $\cos(\theta) = \dfrac{\mathbf{A} \bullet \mathbf{C}}{|\mathbf{A}|\,|\mathbf{C}|} = \dfrac{6+0-12}{(7)(5)} = \dfrac{-6}{35} \approx -0.171$ so $\theta \approx 1.74$ or $99.87°$.

(c) $\cos(\theta) = \dfrac{\mathbf{B} \bullet (-\mathbf{i})}{|\mathbf{B}|\,|-\mathbf{i}|} = \dfrac{-4+0+0}{(9)(1)} = \dfrac{-4}{9} \approx -0.444$ so $\theta \approx 2.03$ or $116.36°$.

(d) $\cos(\theta) = \dfrac{\mathbf{C}\cdot\mathbf{j}}{|\mathbf{C}|\,|\mathbf{j}|} = \dfrac{0+0+0}{(5)(1)} = 0$ so $\theta = \pi/2$ or $90°$. $\mathbf{C}$ and $\mathbf{j}$ are perpendicular.

Practice 4: First we need to find two points on the line and then use those two points to form a vector parallel to the line: $P = (-6, 0)$ and $Q = (0, 15)$ are on the line, and $\mathbf{V} = \langle 0-(-6), 15-0 \rangle = \langle 6, 15 \rangle$ is parallel to the line. $\mathbf{N}\cdot\mathbf{V} = \langle -5, 2 \rangle\cdot\langle 6, 15 \rangle = -30 + 30 = 0$ so $\mathbf{N} = \langle -5, 2 \rangle$ is perpendicular to $-5x + 2y = 30$. Every scalar multiple $k\mathbf{N}$ with $k \neq 0$ is also perpendicular to the line.

Practice 5: (a) $|\mathbf{U}| = 11$, $|\mathbf{V}| = 3$, $\mathbf{U}\cdot\mathbf{V} = -7$. scalar projection of $\mathbf{U}$ onto $\mathbf{V}$ is $\dfrac{\mathbf{U}\cdot\mathbf{V}}{|\mathbf{V}|} = \dfrac{-7}{3}$.

Vector projection of $\mathbf{U}$ onto $\mathbf{V}$ is $\left(\dfrac{\mathbf{U}\cdot\mathbf{V}}{|\mathbf{V}|^2}\right)\mathbf{V} = \left\langle \dfrac{-7}{9}, \dfrac{-14}{9}, \dfrac{14}{9} \right\rangle$.

(b) $\mathbf{U} + \mathbf{V} = \langle 10, 0, 4 \rangle$, $|\mathbf{U} + \mathbf{V}| = \sqrt{116}$, $\mathbf{V}\cdot(\mathbf{U} + \mathbf{V}) = 10+0-8= 2$.

Scalar projection of $\mathbf{V}$ onto $\mathbf{U} + \mathbf{V}$ is $\dfrac{\mathbf{V}\cdot(\mathbf{U} + \mathbf{V})}{|\mathbf{U} + \mathbf{V}|} = \dfrac{2}{\sqrt{116}} \approx 0.19$.

Vector projection of $\mathbf{V}$ onto $\mathbf{U} + \mathbf{V}$ is $\left(\dfrac{\mathbf{V}\cdot(\mathbf{U} + \mathbf{V})}{|\mathbf{U} + \mathbf{V}|^2}\right)(\mathbf{U} + \mathbf{V}) = \left\langle \dfrac{5}{29}, 0, \dfrac{2}{29} \right\rangle$.

(c) Scalar projection of $\mathbf{V}$ onto the positive y–axis is $\dfrac{\mathbf{V}\cdot\mathbf{j}}{|\mathbf{j}|} = 2$.

Vector projection of $\mathbf{V}$ onto $\mathbf{j}$ is $\left(\dfrac{\mathbf{V}\cdot\mathbf{j}}{|\mathbf{j}|^2}\right)\mathbf{j} = \langle 0, 2, 0 \rangle$.

Practice 6: (1) $\mathbf{N} = \langle 3, -4 \rangle$. Take $Q = (0, -5)$ (any other point on the line also works -- try one).

(3) $\mathbf{V}$ is the vector from P to Q: $\mathbf{V} = \langle -2, -12 \rangle$.

(4) scalar projection of $\mathbf{V}$ onto $\mathbf{N}$ is $\dfrac{\mathbf{V}\cdot\mathbf{N}}{|\mathbf{N}|} = \dfrac{-6 + 48}{\sqrt{9 + 16}} = \dfrac{42}{5} = 8.4$, the distance from the point to the line.

We get the same answer, but perhaps less understanding, using the formula:

$$\frac{|\,ap + bq - c\,|}{\sqrt{a^2 + b^2}} = \frac{|\,(3)(2) + (-4)(7) - 20\,|}{\sqrt{9 + 16}} = \frac{|\,6 - 28 - 20\,|}{5} = \frac{42}{5} .$$

Practice 7: The direction of the force vector is $\langle 6, 12, 4 \rangle / |\langle 6, 12, 4 \rangle| = \left\langle \dfrac{6}{14}, \dfrac{12}{14}, \dfrac{4}{14} \right\rangle$ so $\mathbf{F} = 70 \left\langle \dfrac{6}{14}, \dfrac{12}{14}, \dfrac{4}{14} \right\rangle = \langle 30, 60, 20 \rangle$. Then the work done is

Work = $\mathbf{F}\cdot\mathbf{D} = \langle 30, 60, 20 \rangle \cdot \langle 0, 7, 0 \rangle = 0 + (60 \text{ pounds})(7 \text{ feet}) + 0 = 420$ foot–pounds .

Practice 8: $|\mathbf{Y}| = \sqrt{5^2 + 1^2 + (-7)^2 + 5^2} = \sqrt{100} = 10$, $|\mathbf{F}| = 15$, and $|\mathbf{A}| = 15$.

For $\mathbf{Y}$ and $\mathbf{F}$, $\cos(\theta) = \dfrac{\mathbf{Y}\bullet\mathbf{F}}{|\mathbf{Y}||\mathbf{F}|} = \dfrac{30+10-56+25}{(10)(15)} = \dfrac{9}{150}$ so $\theta \approx 86.6°$: "different."

For $\mathbf{Y}$ and $\mathbf{A}$, $\cos(\theta) = \dfrac{\mathbf{Y}\bullet\mathbf{A}}{|\mathbf{Y}||\mathbf{A}|} = \dfrac{50-4+70+15}{(10)(15)} = \dfrac{131}{150}$ so $\theta \approx 29.2°$: "very alike."